

**CONVEX STRUCTURES AND THEORY OF THE
INVESTMENTS.**

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The paper is devoted to the application of the main conceptions of convex analysis to the theory of investments. Convex cones and sets associated with investments and rate of interest are defined. Extreme elements of these objects are prescribed. The set of extreme problems of investments is set up.

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1. Introduction.

The development of modern theory of finance requires attraction of a wide arsenal of mathematical methods. The deep appendices of stochastic methods in the theory of finance are realized in [1, 2].

The theory of the investment projects is one of major sections of financial mathematics. Bases of this theory see in [3,4,5,6].

The basic concepts of the convex analysis are applied to the theory of the investment projects in the present paper. The income is used as criterion of comparison of the projects. Expediency of use of this parameter is proved including in [7]. The used approach allows to formulate a lot of the important extreme tasks. Some tasks of such type are considered in [8,9].

2. Investment projects.

DEFINITION 1. The investment project (flow of payments) is the vector $C=(c_0, c_1, \dots, c_n, \dots)$, satisfying to the following conditions:

1. There exists an index k such that $c_i=0$ at $i > k$.
2. If $b(C)=\min\{i: c_i \neq 0\}$ then $c_{b(C)} < 0$.
3. If $e(C)=\max\{i: c_i \neq 0\}$ then $c_{e(C)} > 0$.
4.
$$\sum_{i=0}^{\infty} C_i \geq 0.$$

The financial sense of this concept is the following: c_i is the size of payment in i moment of time (a year, for example). The positive meanings of c_i correspond to payments to the investor, negative to the contributions of the investor to the project. Numbers $b(C)$ and $e(C)$ are accordingly actual beginning and ending of the project C . Accordingly $e(C) - b(C)$ is the term of performance of the project. Let's notice, that zero vector satisfies to definition 1. The zero project corresponds to him.

The set P of the investment projects is a subset of linear space C_0 of sequences satisfying to a condition 1 of definition 1. The structure of this subset is described by the following theorem.

THEOREM 1. The set of the investment projects P is a convex subcone of the space C_0 .

PROOF. It is necessary to check up, that if $C, D \in P$ and $\lambda \geq 0$, then $C + D \in P$ and $\lambda C \in P$. Let's check up the first statement, the second one is obvious.

If one of vectors is zero then the statement is obvious. Let vectors C, D are nonzero. The condition 1 for vector $C+D$ is carried out. As

$$\sum_{i=0}^{\infty} (c_i + d_i) = \sum_{i=0}^{\infty} c_i + \sum_{i=0}^{\infty} d_i \geq 0, \text{ then condition 4 is also carried out.}$$

Let's check up a condition 2. Let $b(C) < b(D)$. Then $b(C+D) = b(C)$, and $(C+D)_{b(C+D)} = C_{b(C)} < 0$. If $b(C) = b(D)$ then $b(C+D) = b(C) = b(D)$ and $(C+D)_{b(C+D)} = C_{b(C)} + D_{b(D)} < 0$. The condition 2 is proved.

The check of a condition 3 is like the previous one.

Convex operations (addition and multiplication by non-negative numbers) in a cone P have the following financial sense: the addition means formation of a portfolio from two projects, multiplication to non-negative number - formation of the multiple project. This sense of the operations will be used further.

The major role at the analysis of convex sets is played by extreme elements. Let's remind that x is an extreme point of a convex cone K , if from conditions $x = y + z$, $y, z \in K$ the existence of such numbers $\lambda, \mu \geq 0$, that $y = \lambda x$, $z = \mu x$ follows. Thus, the extreme points of a cone K sweep up beams $\{\lambda x: \lambda \geq 0\}$.

Let's describe structure of the set of extreme beams of the cone P . The set of extreme beams of the cone P is designated through $extr P$. The zero element of a cone is always extreme.

THEOREM 2. For inclusion $C \in extr P$, $C \neq 0$ it is necessary and sufficient the existence of numbers $\lambda \in R^+$, $n \in N$ such that $C_i = 0$ at $i \neq n, n+1$; $C_n = -\lambda$; $C_{n+1} = \lambda$.

PROOF.

1. We shall check up that the elements described in the theorem are extreme in the cone P . Let sequence C is characterized by numbers $\lambda \in R^+$, $n \in N$ and $C = D + E$, where $D, E \in P$. From property 4 of definitions it 1 follows that

$$\sum_{i=0}^{\infty} d_i = 0, \quad \sum_{i=0}^{\infty} e_i = 0. \text{ Let's check up that } d_i = 0 \text{ at } i \neq n, n+1. \text{ Let, for example,}$$

at some $i < n$ $d_i \neq 0$. Let's choose minimal i_0 from such i . Then i_0 has the same property for a sequence E also. But it is impossible as these members of sequences are negative on property 2 of definitions 1 and their sum is equal to 0. Similarly at $i > n + 1$.

It follows from here that $d_n + d_{n+1} = 0$, and $d_n \leq 0$, that is $D = (-d_n/\lambda)C$. Similarly $E = (d_n/\lambda)C$, that is $C \in extr P$.

2. Let now vector $C \in P$ has not the properties described in the theorem.

2a. Let $\sum_{i=0}^{\infty} C_i > 0$. By definition of the investment project $c_{e(C)} > 0$. At

rather small $\varepsilon > 0$ vectors $(c_0, c_1, \dots, c_{e(C) \pm \varepsilon}, \dots) / 2 \in P$ and their sum is equal to C . As they are disproportionate C then $C \notin \text{extr} P$.

2b. Let $\sum_{i=0}^{\infty} C_i = 0$ and no less than three of c_i are different from zero.

Let's choose any two such indexes. We shall designate them through $p < q$.

At rather small positive ε vectors $D = (c_0, \dots, c_p + \varepsilon, \dots, c_q - \varepsilon, \dots) / 2$ and $E = (c_0, \dots, c_p - \varepsilon, \dots, c_q + \varepsilon, \dots) / 2$ are the investment projects. $C = D + E$ and the vectors D and E are not proportional to C , that is $C \notin \text{extr} P$.

2b. Let now only two components c_p, c_q ($q - p > 1$) are different from zero, $c_p < 0$, $c_q = -c_p = y$. We shall consider the following investment projects D and E : for the project D only p and $p+1$ components are different from zero and $d_p = -y$, $d_{p+1} = y$; for the project E only p and $p+1$ components are different from zero and $e_{p+1} = -y$, $e_p = y$. As in previous case, $C = D + E$ and by that $C \notin \text{extr} P$.

All elements of the cone P distinct from zero described in the theorem are considered in items 2a-2b. The theorem is proved.

In space C_0 it is possible to introduce the norm which transforms C_0 into the linear normalized (incomplete) space. It allows to estimate affinity of the investment projects.

The most widespread norms are following:

$$\|C\|_0 = \max \lambda_i |c_i|;$$

$$\|C\|_p = \left(\sum_{i=0}^{\infty} \lambda_i |c_i|^p \right)^{1/p}, \text{ where } p \geq 1. \text{ Thus the sum actually is finite.}$$

Here parameter $\lambda_i \in [0, 1]$ reflects the importance of this or the other moment of time - the more is λ_i the more important is the appropriate moment of time.

Which norm to use is determined by a concrete situation. Distance between flows C_1 and C_2 (at the chosen norm) is determined by the formula $\|C_1 - C_2\|$.

In some cases any element can be approached (in this or that norm) by linear combination of extreme elements with non-negative coefficients in convex cones in the linear normalized spaces [10]. In our case it not so. At any such linear combination the sum of components is equal to 0, though for the project this condition can be not satisfied. In this case it is explained by nonclosure of a cone.

The following variant of the theorem about representability for a cone is true.

THEOREM 3. Any element $C \in P$ can be represented as $(0, 0, \dots, 0, t, 0, \dots)$ + $\sum_{i=1}^n C_i$, where $t \geq 0$, $C_i \in \text{exp}P$.

PROOF. First of all we shall present C as $C' + C''$ where a component with number $e(C)$ of a vector C' equals to $\sum c_i$ (c_i are components of a vector C). It is obvious that $C'' \in P$ and the sum of components of vector C'' equals to 0.

Let's check up that a vector C'' can be presented as the sum of vectors from $\text{exp}P$. For this purpose we shall carry on an induction on number of components of C'' unequal to zero.

Let this number is 2. Let's carry on an auxiliary induction on a difference $k = e(C'') - b(C'')$. If $k = 1$ then $C'' \in \text{exp}P$. Let $k > 1$ and for smaller k the statement is true. Let's present $C'' = (0, \dots, p, 0, \dots, 0, -p, 0, \dots)$ as the sum of vectors $(0, \dots, p, -p, 0, \dots) + (0, \dots, p, 0, \dots, 0, -p, 0, \dots)$ where for the second vector the size k is less by one than for C'' .

Let number the nonzero components for vector C'' is $s > 2$ and the statement is true for vectors with smaller appropriate sizes. There are more than one component of the same signs among components of C'' .

If such components are negative we shall choose the last of them. Let its number is v . We shall note that from properties of the investment project $b(C'') < v < e(C'')$. Let's present C'' as the sum $D + E$ where components of a vector D with numbers smaller than v are zeros, sum of components is zero and the components with numbers larger than v coincide with the appropriate components of a vector C'' . It is obvious, that $D, E \in P$ and for them the assumption of an induction is fair.

The case when the vector C'' has more then one positive components is exhausted similarly - as v it is necessary to choose the number of the first such component. The theorem is proved.

3. Bank politics.

Alongside with flows of payments it is necessary to consider the sequence of the bank percentage rates, as the income (or damage) from a flow of payments depends on this sequence.

Let sequence P of the bank percentage rates on intervals of time (actual or expected) has a kind of $(i_0, i_1, \dots, i_n, \dots)$. For our investigations it is convenient to move to values q_n — factors of discount carrying cost of money at the moment n to the zero time moment. The values q_n are calculated as follows:

$q_0=1$, for $n > 1$

$$q_n = [(1+i_0)(1+i_1)\dots(1+i_{n-1})]^{-1} \quad (1)$$

DEFINITION 2. The sequence $Q=(q_0, q_1, \dots, q_n, \dots)$ we shall define as the bank politics.

An indefinitely decreasing geometrical serious I , $q_0, q_1, \dots, q_n, \dots$ corresponds to the constant interest rate. Let's describe conditions of a sequence Q sufficient for Q being a bank politics.

THEOREM 4. A sequence $Q=(q_0, q_1, \dots, q_n, \dots)$ is bank politics for some sequence of the percentage rates if and only if

1. Sequence Q is not increasing;
2. $q_0=1$;
3. $q_n > 0$.

PROOF. The necessity of the formulated conditions is obvious. Let's check up sufficiency. Let's assume $i_0=(1/q_0 - 1)$, $i_n=(q_{n-1}/q_n - 1)$ when $n > 0$. From properties q_n values i_n is non-negative. Giving them sense of the percentage rates, we receive the conclusion of the theorem.

As we are interested only in finite number of components of a vector Q we may consider that $\lim_{n \rightarrow \infty} q_n = 0$.

Let's designate as $\{Q\}$ the set of possible bank politics. Similarly to set P $\{Q\}$ is a subset of the space C_1 of the bounded sequences. Let's describe properties of the set $\{Q\}$.

THEOREM 5.

$\{Q\}$ is a convex subset of C_1 .

PROOF. Let $Q_1, Q_2 \in \{Q\}$, $\lambda \in [0, 1]$. Let's check up that $Q = \lambda Q_1 + (1 - \lambda) Q_2 \in \{Q\}$.

1. $q_{n+1} = \lambda(q_1)_{n+1} + (1 - \lambda)(q_2)_{n+1} \leq \lambda(q_1)_n + (1 - \lambda)(q_2)_n = q_n$
2. $q_0 = \lambda(q_1)_0 + (1 - \lambda)(q_2)_0 = \lambda + (1 - \lambda) = 1$.
3. $q_n = \lambda(q_1)_n + (1 - \lambda)(q_2)_n > 0$
4. $\lim_{n \rightarrow \infty} q_n = \lambda \lim_{n \rightarrow \infty} (q_1)_n + (1 - \lambda) \lim_{n \rightarrow \infty} (q_2)_n = 0$.

The theorem is proved.

It is useful to expand set $\{Q\}$ by adding to it all sequences with components equal to zero since one of them. Economically it corresponds to sharp inflationary jump.

We shall consider the space C_1 to be equipped with the norm $\|Q\| = \sup |q_i|$. The application of the diagonal method allows to conclude that the following theorem is fair.

THEOREM 6. $\{Q\}$ is the convex closed subset of space C_1 with the introduced norm.

Let's establish properties of set $ext\{Q\}$ of extreme points of set $\{Q\}$. Let's remind that $Q \in ext\{Q\}$ if from condition $Q = (Q_1 + Q_2)/2$ where $Q_1, Q_2 \in \{Q\}$ it follows that $Q = Q_1 = Q_2$.

THEOREM 7. For validity of inclusion $Q \in ext\{Q\}$ the existence of such index n that $q_i = 1$ when $i \leq n$ and $q_i = 0$ when $i > n$ is necessary and sufficient.

PROOF. Let vector Q has the specified property and $Q = (Q_1 + Q_2)/2$ where $Q_1, Q_2 \in \{Q\}$. As the components of vectors Q_1 and Q_2 are non-negative and do not surpass 1, it is obvious that $Q_1 = Q_2 = Q$ that is $Q \in ext\{Q\}$.

Let now vector Q has not the specified property that is has a component $q_i \in (0, 1)$. Let's allocate all components of Q equal to q_i . As the components decrease the appropriate indexes cover some interval (from n_1 up to n_2 , for definiteness). Let Q_1 and Q_2 differ from Q by that their components from n_1 up to n_2 are equal accordingly to $q_i + \varepsilon$ and to $q_i - \varepsilon$. For enough small positive ε $Q_1, Q_2 \in \{Q\}$ and $Q = (Q_1 + Q_2)/2$ where $Q_1 \neq Q_2$. Thus $Q \notin ext\{Q\}$. The theorem is proved.

Thus the set of extreme points of the convex closed set $\{Q\}$ is denumerable. Let's designate an extreme point $\{Q\}$, described in the theorem 7 by S_n . The theorem of representation is fair in metric convex compacts, i.e. any element can be approached by a convex combination of extreme points as precisely as needed. Though the set $\{Q\}$ is not compact, the following statement is true.

THEOREM 8. In the convex closed set $\{Q\}$ any element can be approached by a convex combination of extreme points.

PROOF. Let number $\varepsilon > 0$. Let's record such index n , that $q_n < \varepsilon$. Let's assume $Q' = (1-q_1)S_0 + (q_1-q_2)S_1 + \dots + (q_{n-1}-q_n)S_{n-1} + q_n S_n$. Q' is a convex combination of sequences S_0, S_1, \dots, S_n . Thus $Q-Q' = (0, 0, \dots, 0, q_{n+1}, q_{n+2}, \dots)$. Then by definition of number n we have: $\|Q-Q'\| = \sup\{q_{n+1}, q_{n+2}, \dots\} < \varepsilon$ as it is required.

The problems where the families of bank politics appear will be considered further. In particular, the following situation seems to have practical importance: it is impossible to predict precisely the bank interest rates for each year, but it is possible to predict possible borders of the percentage rates rather reliably. Let's analyze subsets of $\{Q\}$ arising at it.

Let the predicted bank rate in interval n is between i_n' and i_n'' . Then the appropriate discount factors q_n and q_{n+1} satisfy to the relation $1+i_n' \leq q_n/q_{n+1} \leq 1+i_n''$.

Thus the appropriate family $\{Q\}$ of bank politics satisfies to the following condition: there are two sequences $U = \{u_n\}$ and $V = \{v_n\}$ such that $1 \leq u_n \leq v_n$ and bank politics $Q = (q_0, q_1, \dots, q_n, \dots) \in \{Q\}$ if for all natural n

$$u_n \leq q_n/q_{n+1} \leq v_n.$$

Let's designate such set of bank politics as $Q(U, V)$. Let's find the properties of such sets of bank politics.

THEOREM 9. The set $Q(U, V)$ is convex and closed. If such number $d > 1$ that $u_n > d$ for all n exists, then the set $Q(U, V)$ is compact.

REMARK. The economic sense of the condition $u_n > d$ is that the bank interest rate can not fall below some threshold. This condition is rather natural.

PROOF. From definition of set the appropriate bank politics satisfy to system of linear inequalities $u_n q_{n+1} \leq q_n \leq v_n q_{n+1}$. As the set of vectors satisfying to a linear inequality is convex and the crossing of convex sets is convex then $Q(U, V)$ is convex set.

The closure of $Q(U, V)$ is obvious.

Let's prove the compactness of set $Q(U, V)$ under the formulated conditions. Let $Q_1, Q_2, \dots, Q_n, \dots$ is an arbitrary sequence of vectors from $Q(U, V)$. Using limitation of every component it is possible to extract from this sequence such subsequence (we shall designate it the same way) which converges component-wise to some sequence Q_0 .

Let's check up that $\|Q_n - Q_0\| \rightarrow 0$. Let ε be any positive number. Let's choose such index $n(\varepsilon)$ that $(1/d)^{n(\varepsilon)} < \varepsilon$. Let's choose such index $n_1(\varepsilon)$ such that when $n > n_1(\varepsilon)$ $\|(Q_n)_i - (Q_0)_i\| < \varepsilon$ for all $i < n(\varepsilon)$. Thus for $n > n_1(\varepsilon)$

$|(Q_n)_i - (Q_0)_i| < \varepsilon$ for all i . It means the validity of inequality $\|Q_n - Q_0\| < \varepsilon$ as was to be shown.

The following statement describes extreme points of convex set $Q(U, V)$.

THEOREM 10. For validity of including $Q \in Q(U, V)$ it is necessary and sufficient that for any index n one of two relations carries out: $q_n = u_n q_{n+1}$, $q_n = v_n q_{n+1}$.

PROOF. 1. Let bank politics $Q \in Q(U, V)$ is such that for some index n the strict inequalities $u_n q_{n+1} < q_n < v_n q_{n+1}$ are carried out and n is minimal of such indexes. Let's assume $\varepsilon = \min\{q_n/q_{n+1} - u_n, v_n - q_n/q_{n+1}\}$ and define two bank politics as follows. For Q_1 $(q_1)_i / (q_1)_{i+1} = q_i / q_{i+1}$ when $i \neq n$ and $(q_1)_n / (q_1)_{n+1} = q_n / q_{n+1} - \varepsilon$. Similarly for Q_2 $(q_2)_i / (q_2)_{i+1} = q_i / q_{i+1}$ when $i \neq n$ and $(q_2)_n / (q_2)_{n+1} = q_n / q_{n+1} + \varepsilon$. Then $Q_1, Q_2 \in Q(U, V)$ and $Q = (Q_1 + Q_2) / 2$ that is $Q \notin \text{ext } Q(U, V)$.

2. Let now bank politics $Q \in Q(U, V)$ has the described property and in addition $Q = (Q_1 + Q_2) / 2$ where $Q_1 \neq Q$. Let n is the first component for which $(q_1)_n \neq q_n$. Then at least one of the numbers $(q_1)_{n-1} / (q_1)_n$, $(q_2)_{n-1} / (q_2)_n$ does not lay in an interval $[u_{n-1}, v_{n-1}]$, that is at least one of these sequences does not contain in $Q(U, V)$. Thus $Q \notin \text{ext } Q(U, V)$

REMARK. It follows from here that set $\text{ext } Q(U, V)$ is closed and if the sequences U and V differ by infinite number of components is uncountable.

4. Project income and extreme problems.

If C is flow of payments and Q is bank politics the discount income of a flow in the generalized sense (possible damage is also included here) has a kind

$$DO(C, Q) = CQ = \sum_{i=0}^{\infty} c_i q_i.$$

Let's notice that from properties of the flow C it follows that the sum is finite. Similarly the discount income (or damage) for the investor from a flow C

at the moment of time k at bank politics Q is equal to $(CQ)_k = \sum_{i=0}^k c_i q_i$.

$(CQ)_k = CQ$ for rather large k .

Let's introduce continuous analogues of the concepts entered. As the investment project (flow of payments) C we shall understand continuous function $C: [0, \infty) \rightarrow \mathbb{R}$ such that for some x and y $C(t) = 0$ at $t < x$ and $t > y$, $C(t) < 0$ in the right half-neighborhood of a point x , $C(t) > 0$ in the left half-neighborhood of a point y . Sense of this function is the density of money receipt, that is the receipt of money for an interval of time $[t, t + \Delta t]$ is equal to

$c(t)\Delta t + o(\Delta t)$. Numbers x and y are moments of beginning and ending of the project. The set of the projects is similar to discrete case is convex subcone of linear space L_0 of continuous functions with the finite carrier. The space L_0 can be equipped by some norm. The most widespread are the following:

$$\|f\|_C = \sup \lambda(t) |f(t)|,$$

$$\|f\|_p = \left[\int_0^{\infty} \lambda(t) |f(t)|^p dt \right]^{1/p}$$

Let's notice that integral in last formula is actually calculated on a final interval.

The sense of function $\lambda(t)$ is similar to sense of the appropriate sizes in a discrete case.

Continuous analogue of bank politics is the function $Q(t)$ which meaning in a point $t \geq 0$ is equal to discount factor on an interval $[0, t]$. This function is easily expressed through force of interest $\delta(t)$:

$$Q(t) = \exp \left[- \int_0^t \delta(t) dt \right]$$

We shall consider functions $\delta(t)$ to be continuous. The set of bank politics consists of not growing positive functions such that $Q(0) = 1$. Thus, the set of bank politics is a convex subset of space $C[0, \infty)$ of continuous functions.

The discounted generalized income of a flow C at bank politics Q is equal

$$DO(C, Q) = CQ = \int_0^{\infty} C(t) \cdot Q(t) dt.$$

Similarly to discrete case the income of a flow of payments C at bank politics Q in time k can be entered:

$$DO_k(C, Q) = (CQ)_k = \int_0^k C(t) \cdot Q(t) dt.$$

Further we shall mainly analyze discrete case.

Let's define the concept of preference of the projects at bank politics Q . The project C_1 is more preferable than the project C_2 concerning politics Q , if $(C_1 - C_2)Q \geq 0$. Let $\{Q\}$ is any family of bank politics. Let's say that the project C_1 is more preferable than the project C_2 concerning set $\{Q\}$, if C_1 is more preferable than C_2 concerning any politics $Q \in \{Q\}$. It is obvious that the relation of preference of flows concerning set $\{Q\}$ is the relation of the nonstrict incomplete order.

Let's describe the structure of flows, which are more preferable than the given flow C concerning set of bank politics $\{Q\}$. Let $K(Q) = \{D \in P: QD \geq 0 \text{ for}$

$Q \in \{Q\}$. The set $K(Q)$ is a convex cone. The set of flows, which are more preferable than C concerning Q , has a kind $C+K(Q)$. It follows from the theorem 3 that such set of flows should satisfy to simple system of inequalities.

For some problems it is useful to consider set of bank politics $U(C_1, C_2)$ concerning to which the flow C_1 is more preferable than C_2 . This set can be empty, for example when all components of the vector $C_2 - C_1$ are positive.

STATEMENT 1. The set of bank politics $U(C_1, C_2)$ is intersection of $\{Q\}$ with a convex cone and by that is a convex subset of $\{Q\}$.

PROOF. If $Q_1(C_1 - C_2) \geq 0$, $Q_2(C_1 - C_2) \geq 0$, then $(\lambda Q_1 + \mu Q_2)(C_1 - C_2) = \lambda Q_1(C_1 - C_2) + \mu Q_2(C_1 - C_2)$ for any $\lambda, \mu \geq 0$ as it is required.

The description of extreme elements of a cone of the investment projects and sets of bank politics allows to solve some general problems. In particular it is easy to describe criterion that $U(C_1, C_2) = Q$.

STATEMENT 2. For the investment project C_1 to be more preferable than C_2 at any bank politics, it is necessary and sufficient that for any k

$$\sum_{i=0}^k (C_{1i} - C_{2i}) > 0.$$

PROOF. If for any bank politics Q the inequality $C_1 Q \geq C_2 Q$ is carried out, it is necessary and sufficient that this inequality is carried out for extreme points of set of bank politics. Using the description from the theorem 7 we receive the statement needed.

The stated approach allows to state uniformly the statements of some practically important problems. Let's have some examples.

1. Some investment projects C_i , where $i=1, 2, \dots, n$ are offered to the investment company having means at rate of F_0 . The company has the reliable forecast of bank politics Q for the prospective period T of all n projects. The company wishes to choose a part of them so that maximize profit under condition of non-ruining at any moment.

The mathematical statement of this task has the following kind.

It is necessary to find numbers $x_1, x_2, \dots, x_n \in \{0, 1\}$ satisfying to the conditions:

a. $(\sum_{i=1}^n x_i C_i \times Q)_k \leq F_0$ at $k < T$. This is the condition of non-ruining at

the time moment k

b. The size of $(\sum_{i=1}^n x_i C_i \times Q)_T$ is maximal.

This problem refers to a class of tasks of integer linear programming.

Let's have the continuous analogue of this task. For this purpose we shall

designate the function $\int_0^t c_i(t) \cdot q(t) dt$ as $\varphi_i(t)$.

It is necessary to find numbers $x_1, x_2, \dots, x_n \in \{0, 1\}$ satisfying to the conditions:

a. $\sum_{i=1}^n x_i \varphi_i(t) \leq F_0$ for all $t \in [0, \infty)$. Actually this condition is meaningful

only on a segment $[0, T]$ where T is maximum of the moments of projects ending.

6. Size $\sum_{i=1}^n x_i \varphi_i(T)$ is maximal.

It is the problem of integer linear programming with continuum of restrictions.

2. The investor has an opportunity to borrow money and the bank politics Q_1 , connected with borrowing differs from the basic bank politics Q . It is clear from substantial reasons that all components of Q_1 do not surpass appropriate components of Q . Let's formulate the problem of maximizing the profit of the investor in this situation in discrete statement.

Let's construct functions $\varphi^+(Q, Q_1, F_0, R; k)$, where $R \subset \{C_1, C_2, \dots, C_n\}$ by an induction as follows.

$$\varphi^+(Q, Q_1, F_0, R; 0) = F_0;$$

If $k \geq 1$ then

$$\varphi^-(Q, Q_1, F_0, R; k+1) = \varphi^+(Q, Q_1, F_0, R; k) \times q_{k+1}/q_k \text{ when } \varphi^+(Q, Q_1, F_0, R; k) > 0;$$

$$\varphi^-(Q, Q_1, F_0, R; k+1) = \varphi^+(Q, Q_1, F_0, R; k) \times (q)_{k+1}/(q)_k \text{ when } \varphi^+(Q, Q_1, F_0, R; k) \leq 0;$$

$$\varphi^+(Q, Q_1, F_0, R; k) = \varphi^-(Q, Q_1, F_0, R; k) + \sum_{C_i \in R} (c_i)_k$$

Here $Q = (q_1, q_2, \dots, q_k, \dots)$; $Q_1 = ((q_1)_1, (q_1)_2, \dots, (q_1)_k, \dots)$.

It is required to find set $R^* \subset \{C_1, C_2, \dots, C_n\}$ such, that

$$\varphi^+(Q, Q_1, F_0, R^*; T) = \max_R \varphi^+(Q, Q_1, F_0, R; T) \text{ where } T \text{ is maximal of the}$$

moments of ending of the projects C_1, C_2, \dots, C_n

It means substantially that we select such subset of the projects which provides the maximum income at the moment of ending of all projects.

3. Let's suppose that it is allowed for the investor to bring in some changes to the project which can concern both sums and moments of payments. Let we can replace the project C by the project C_1 such that $\|C_1 - C\| \leq d$, where both meaning of norm and size d are assumed to be given. In this case the following task is reasonable at the given bank politics Q :

find a flow of payments C_1 such that $\|C_1 - C\| \leq d$ and value $C_1 Q$ is maximal and positive.

The situations when validity of a flow of payments C_1 becomes infinite are possible in such a general statement. It can be avoided entering additional restriction for the term of the project C_1 .

In the previous considerations the bank politics Q (or bank politics Q and Q_1) was assumed known. Frequently real situation is those: there are bases to believe, that can be realized what - or bank politics from some family. Usually it is supposed (in a discrete case), that this family in our designations has a kind $Q(U, V)$ for some sequences U and V .

At such approach it is possible to formulate some important problems.

4. Let flow of payments C is given. It is favorable for the investor for any $Q \in Q(U, V)$ if $Q(U, V) \subset U(C, 0)$. By convexity of sets $Q(U, V)$ and $U(C, 0)$ it is enough to check up that the extreme points of convex set $Q(U, V)$ which are described in the theorem 10 contain in $U(C, 0)$.

For example, if $U = (1, q, q^2, \dots)$ and $V = (1, 0, 0, \dots)$ (the sense of this condition is that the investor supposes only that the interest rate can not be omitted below some level and except this condition can have any meaning), then conditions as in statement 2 when the project C is always favorable is the

following: for any k $\sum_{i=0}^k C_i q^i \geq 0$.

The following extreme task is reasonable in this connection. Let the investor has opportunity to bring in some (of course, reasonably small) changes in the investment project C and thus the investor has the forecast of set of bank politics of the type $Q(U, V)$. It is necessary to find the project C_1 satisfying to the following conditions:

- a) $Q(U, V) \subset U(C_1, 0)$;
- б) $\|C - C_1\|$ is minimal.

In fact we are describing a choice of such project which is not unprofitable at all possible bank politics and minimally differs from C .

The remark made while stating the problem 3 about possible infinity of performance term of the project C is true here.

5. It is possible to put the problem of choosing of one of two projects C_1, C_2 in conditions of uncertain bank politics similarly to problem 2.

The project C_1 is more preferable than C_2 on family $Q(U, V)$ if

$$Q(U, V) \subset U(C_1, 0) \cap U(C_1, C_2).$$

As previously if the changes in the project C_1 are admitted it is possible to put the following extreme problem:

find a flow \tilde{C}_1 such, that

a) $\{Q\} \subset U(\tilde{C}_1, 0) \cap U(\tilde{C}_1, C_2)$;

b) the norm $\|\tilde{C}_1 - C_1\|$ is minimal.

The norm can be understood differently in these statements. It depends on situation.

6. Let's admit that the investor has an opportunity to correct the offered flow of payments C , the bank politics belongs to the family $Q(U, V)$ (as in problem 3). Then the following statement is reasonable: find a flow C_1 such that $\|C_1 - C\| \leq d$ and size of $\min(C_1 Q)$ maximal and positive at $Q \in Q(U, V)$.

Such statement corresponds to cautious strategy focused on the extreme case.

As before at the analysis of this problem (at minimization) it is possible to consider only extreme points of set $Q(U, V)$.

5. Conclusion.

Not all extreme problems which can be put within the framework of the offered formalism are formulated in this paper. It is also possible to put many other problems. For example, rearrangement of an optimum portfolio, account of unreliability of the projects and some others.

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