

## CLASSIFICATION OF USURIOUS LOANS

S. David Promislow

Department of Mathematics & Statistics  
York University  
NORTH YORK ON CANADA M3J1P3

Telephone: (416)736-5250  
Facsimile: (416)736-5757  
e-mail: promis@mathstat.yorku.ca

### Abstract

The motivation for this work is a provision in the criminal code of Canada which makes it an offense to lend money at an annual effective rate exceeding 60%. A major problem which arises is to interpret this law for transactions which do not possess a unique yield rate. This paper considers an axiomatic approach to the classification of usurious loans. An “impossibility theorem” is proved, which shows that under a certain natural set of axioms there is no general solution to this problem. By relaxing the requirement that all loans be classified, and allowing for ambiguous cases, various solutions are obtained.

*Keywords:* yield rates, internal rate of return.

## 1 Introduction

Anti-usury legislation goes back to ancient times, and remains with us to the present. In the criminal code of Canada, it is an offense to receive money at a “criminal interest rate”, which is defined to be an annual effective rate of interest which exceeds 60%. The actuarial profession is involved, since the act contains a provision which accepts a certificate of a Fellow of the Canadian Institute of Actuaries as evidence of whether or not a given transaction violates the code.

At the time of enactment this was thought to be a relatively simple (if not trivial) actuarial problem. This is indeed true for the usual type of loan where repayments all occur in time *after* the advances. It is well known that there is a unique annual rate of interest applicable to the transaction, and one merely checks whether or not it exceeds the upper limit. In fact, even the numerical procedure of solving a polynomial can be avoided. As pointed out in [1], one need only accumulate at a rate of 60%, the value at maturity of all repayments, less all loan advances. The loan will violate the code if and only if this amount is positive.

A major problem arises however in interpreting the legislation when the loan does not follow this usual pattern of payments. As is well known, when loan advances and repayments alternate in time, the respective roles of the borrower and lender become blurred. The equation that one normally uses to find a yield rate can have many solutions (or perhaps none). It is customary to refer to this as a case of “*multiple yields*”, but as pointed out in [3] among other references, these various solutions can no longer be interpreted as yields. ( We will however continue in this paper to follow this conventional but inaccurate terminology.)

There have been few, if any, criminal prosecutions under this Act, first implemented in 1981. There have been however a number of civil cases, in which one party has sought to nullify a loan on the grounds that it contravened the criminal code. It is therefore a problem of practical significance to arrive at a general method for classifying loans as usurious or not.

It is suggested by some, that for criminal code purposes, these exceptional cases rarely arise and can safely be ignored. However, even aside from the desire of the theoretician to obtain a general answer to the classification problem, there are practical reasons against this suggestion. In the first place they do arise with some frequency. It is common in some industries, such as

construction, for the lender to require an application fee from the prospective borrower at the time the loan is requested. The act treats any exchange of money pursuant to the loan as a payment. In effect, an application fee is considered as a repayment of the loan before it is received, and this typically results in multiple yields. In addition, a lender who knew that the criminal code did not deal adequately with loans that differed from the standard pattern, could artificially create such a situation to escape usury charges. For example, one day after the loan is repaid in full, a small sum is refunded.

It may appear natural to begin an analysis of the general case by looking at the various "yields". Several questions immediately arise. Should the loan be considered usurious if any one of these is above the prescribed limit, or only if some of them are? Should one include negative yields ( i.e. those in the interval  $(-1, 0)$ )? Algorithms based on such a method can easily lead to inconsistencies. We illustrate some of these in Section 4 below.

In this paper we adopt a different approach. Our method is to postulate a set of axioms which we feel that the classification should have, and then try to find solutions satisfying those axioms. Our intuition of course is that any loan with an interest rate exceeding the given maximum is usurious, but that does not provide an answer when there are multiple or nonexistent rates.

This paper will deal exclusively with the case of loans where the amounts and times of payments are fixed. This of course is not always the case, and in many situations one or both of the parties will have options. A prime example is illustrated by credit cards. The card user, in effect the borrower, has considerable leeway in how they repay the loan. The analysis of the optional payment case presents many additional difficulties however, and will not be attempted in the present work.

Before getting into the mathematics of the situation we describe the axioms in an intuitive fashion in section 2. In section 3 we set out the mathematical model. In Section 5 we review the work of Teichroew, Montalbano, and Robicek , (often referred to as TRM theory) as developed in [5]. (See also [4] or ([2],Section 5.9)) This theory will play an important role throughout the paper. In Sections 7 and 8 we recast our axioms in rigorous form and state our major conclusions, Theorems 5, 6 and 7. Further discussion and refinements are given in Sections 9 to 11.

## 2 Intuitive form of the axioms

We will adopt a somewhat more general point of view than the motivating example of the Candian Criminal Code. The basic problem is as follows. We are given a maximum allowable rate of interest  $i$  and we want to classify all loans as being either usurious or nonusurious, with respect to that maximum rate. We postulate here a set of axioms or principles governing such a classification. Following are seven such axioms, with an explanatory comment after each.

**Axiom 1:** *Pure loan classification.* A pure loan is usurious or nonusurious, depending on whether its unique yield rate is greater than, or less than or equal to the maximum rate .

A loan is called *pure* with unique yield  $j$  , if  $j$  is a yield rate such that the outstanding balances at all durations prior to maturity are negative. This implies that status of the lender remains as such and does not revert to a borrower status. The rate  $j$  is necessarily the unique yield. The familiar type of loans mentioned in the introduction are of this type.

**Axiom 2:** *Comparison.* A loan which is clearly more favorable to the lender and less favorable to the borrower than a usurious loan should also be usurious. A loan which is clearly less favorable to the lender and more favorable to the borrower than a nonusurious loan should be also nonusurious

The general idea seems self evident. “More favorable to the lender ” would mean for example that higher amounts are repaid for the same advances, or that the same amounts are repaid, but at earlier times

**Axiom 3:** *Decomposition.* If a loan can be decomposed in two separate usurious loans then the original loan should be usurious. If it can be decomposed into two separate nonusurious loans the original loan should be nonusurious.

As an example of what we mean by decomposition, a loan of 4 units to be repaid by 3 units at the end of each year for two years, could be decomposed into two loans of 2 units each, one to be repaid by 3 at the end of one year

and one to be repaid by 3 at the end of two years.

This axiom seems natural. It would be illogical if a lender could avoid making an otherwise usurious loan by writing it up as two separate loans, both classified as nonusurious. From another point of view, it appears illogical for a lender making two nonusurious loans to the same borrower to suddenly find that the composite result is usurious.

**Axiom 4:** *Scale Invariance.* Multiplying all payments by the same positive constant should not change the classification.

One may well dispute this axiom in general. It could be argued that the *size* of the payments should enter into the classification. This indeed was the case with the previous Canadian legislation which dealt only with loans of less than a certain amount. However, if one considers *rate* as the single defining criteria for a loan to be usurious, as present Canadian law does, this axiom follows.

**Axiom 5:** *Definiteness.* No loan should be both usurious and nonusurious.

This seems self evident

**Axiom 6:** *Sensitivity.* If the maximum allowable rate decreases, those loans which were usurious at the old rate should remain such. If this rate increases, those loans which were nonusurious at the old rate should remain such.

Suppose for example it was decided that the 60% rate in Canada allowed for too many questionable loans, and in an effort to tighten up, the maximum was lowered to 50%. It would then be indeed illogical, if a loan which was usurious at the old rate suddenly became nonusurious.

**Axiom 7:** *Completeness.* Any loan is either usurious or nonusurious.

After setting up a mathematical model and formulating these axioms in a more precise manner, we will show that they are inconsistent and there is no way to satisfy them. Suppose however we eliminate Axiom 7. This means that certain loans will be ambiguous and classified as neither usuri-

ous or nonusurious. We are then able to describe various solutions to the classification problem.

### 3 The basic mathematical model

We will be considering financial transactions which involve exchanges between two parties, of fixed amounts of capital at definite times. We will make two simplifying assumptions. First, we assume that there is a maximum duration, and we only consider transactions which are completed before this time. (That is, we do not consider transactions of unbounded length or perpetuities. This restriction could be removed, but it would significantly complicate the mathematical analysis.) Secondly, we assume that a certain period of time has been chosen and all transactions are periodic, that is, made at time  $k$  where  $k$  is a nonnegative integer. This does not involve any real practical restriction. If the times of all payments are rational, we can always achieve this by simply selecting a smaller period.

Accordingly, we fix a unit of time and a maximum duration  $K$ . A financial transaction from the point of view of a prospective lender is a sequence

$$x = x_0, x_1 \dots x_K$$

where  $x_k$  denotes the net cashflow at time  $k$ , a negative amount representing an advance of funds and a positive one representing a repayment. The set of all such sequences will be known as the *transaction space* and denoted by  $X$ . (So in effect  $X$  is just the vector space  $R^{K+1}$ ) To simplify the notation we will often write elements of  $X$  as  $k$ -dimensional vectors with  $k < K + 1$ , with the understanding that remaining entries are 0. For example  $(-3, 2, 2)$  denotes the transaction which involves a loan of 3 units at time zero, to be repaid by two periodic installments of 2 units. After time two there are no exchanges and all entries are 0.

Following is some notation and terminology used throughout the paper: The sequence  $x \in X$  with  $x_i = 1$  for  $i = k$ , and  $x_i = 0$  for  $x \neq k$  will be denoted by  $\delta_k$ . A set  $A$  in  $X$  is *convex* if given any  $x, y$  in  $A$ ,  $\alpha x + (1 - \alpha)y \in A$ , whenever  $0 \leq \alpha \leq 1$ . A set  $C$  which is convex and such that  $\alpha x \in C$  for all  $x \in C, \alpha > 0$ , is called a *convex cone*. The *convex cone generated by a set  $A$*  is the intersection of all convex cones containing  $A$ . Given subsets  $A, B$  in  $X$  we let  $A + B = \{x : x = a + b, \text{ for some } a \in A, b \in B\}$ , and we similarly

define  $A - B$ . For any set  $A \subseteq X$ , we let  $\tilde{A}$  denote the complement of  $A$  in  $X$ .

There are various versions of the following result. We quote the one most useful to us in this paper.

**The Separating Hyperplane Theorem.** *Given any two disjoint convex subsets of  $X$ ,  $A$  and  $B$ , where  $A$  is open, there is a linear functional  $f$  on  $X$  such that  $f(a) > 0$  for all  $a \in A$ ,  $f(b) \leq 0$ , for all  $b \in B$ .*

## 4 Examples

In this section we give examples of transactions which indicate some of the difficulties involved, and show that a simple calculation of yield rates can lead us astray. We will refer to them in later sections. Similar examples and comments appear in [3]. We consider exchanges between two parties  $A$  and  $B$ , and let  $x$  denote the transaction from  $A$ 's point of view. It follows that  $-x$  will denote the same transaction from  $B$ 's point of view. Consider the time period to be one year.

**Example 1.**  $x = (-1, 5, -6)$ . This transaction has yield rates of 100% and 200%. Is it usurious when the limit is 60%? One might first query as to who is the lender and who is the borrower. It is of course essential to decide this when classifying on the basis of yield rates only, since both  $x$  and  $-x$  have the same yields. In other words, it is usurious to *lend* at a high rate, not to *borrow* at such. In the case of pure transactions we identify *loans* by the requirement that the initial outstanding balances are negative. There is no convenient way to do so in the case when outstanding balances change sign. In effect, this example can be decomposed into the two transactions of  $(-1, 2) + (0, 3, -6)$  or  $(-1, 3) + (0, 2, -6)$ . Each party is both borrowing and lending at these high rates which is the source of the multiple yields. Of course  $A$  is borrowing more. This might lead us to consider  $B$  as a possible usurious lender.

**Example 2.**  $x = (1, -5, 6)$ . This is the same basic transaction as above, but from  $B$ 's viewpoint. This could be interpreted as a loan at 20% per

period, with an application fee of 20% of the loan amount paid one period prior. To assess the effect, one could consider what the lender will do with the application fee. If the lender simply kept it and earned nothing, the effect will be a loan of 4 repaid by 6 one period later, for a periodic yield of 50%. The periodic yield will be above 60% if and only if the lender earns more than 25% on the deposit.

The examples with application fees indicate that we cannot just consider as the lender, the party who makes the first advance, as would be so in the pure case. Some would advocate that the person receiving back more than they advance should be considered the lender. The following example shows that this is not always reasonable.

**Example 3.**  $x = (-1, 2, \dots - (1 + \epsilon))$ . where last entry is at time  $k$ . For large  $k$  and small  $\epsilon$ , there is a yield rate close to 100%. However it would not be logical to designate B, the party who receives more than they pay out, as the lender, and to then claim that B was making a usurious loan. This transaction would seem to an unfavorable one for B, and a favorable one for A, who is paying very little for the the use of B's money for a long period,

In the proposed classification criteria which follow, we will not have the need to identify a particular party as a lender. Each transaction will "speak for itself" and the methods will automatically distinguish between the transactions  $x$  and  $-x$ .

Even for transactions with a unique rate, it may not be easy to classify the loan.

**Example 4.**  $x = (-1, 4, -4)$ . This has a unique yield of 100% but the analysis of  $x$  and  $-x$  is similar to that of Examples 1 and 2.

**Example 5.**  $x = (1, -1000, 1050)$  In this case A is lending money at a modest rate of 5%, and asking a small application fee of 1 per 1000 one period prior. The two yield rates are 5.1% and 99.89%. The first seems like a reasonable measure of the return. What about the second? We will return to this in Section 9.

## 5 Basic aspects of TRM theory

Consider an investment or loan, in which the lender is attempting to calculate the internal rate of return (yield), that is, a rate  $i > -1$  for which the final balance is zero. The principle behind the TRM theory is that when outstanding balances prior to maturity become positive and the status of the lender has reverted to that of a borrower, these positive balances should not accumulate at the presumably higher rate of return which is being sought, but rather they should accumulate at some fixed standard rate (the so called *deposit rate*). This is the rate at which the investor could normally acquire funds or could earn on standard investments. One can think of it as a type of risk-free rate available in the market. We are led then to consider accumulations at two rates of interest.

We first note that it is much more convenient to work in terms of *ratios* rather than *rates*. Throughout the remainder of this paper we use  $r$  in place of  $1 + i$  and  $d$  for  $1$  plus the deposit rate referred to.

Given any  $x$  in  $X$ ,  $d, r \geq 0$ , and  $t = 0, 1, \dots, K$  we define inductively the quantities  $h_{r,d,t}(x)$ , the outstanding balance at time  $t$  with respect to accumulation at investment ratio  $r$  and deposit ratio  $d$ .

$$h_{r,d,0}(x) = x_0$$

$$h_{r,d,t+1}(x) = \begin{cases} r h_{r,d,t}(x) + x_{t+1} & \text{if } h_{r,d,t}(x) < 0 \\ d h_{r,d,t}(x) + x_{t+1} & \text{if } h_{r,d,t}(x) \geq 0 \end{cases}$$

The following statements can all be easily proved by induction directly from the definition. For any  $x$  and  $y$  in  $X$ ,  $d, r > 0$  and  $t$  a nonnegative integer

$$\text{If } d \leq r, \text{ then } h_{r,d,t}(x + y) \geq h_{r,d,t}(x) + h_{r,d,t}(y) \quad (1)$$

$$\text{If } d \geq r, \text{ then } h_{r,d,t}(x + y) \leq h_{r,d,t}(x) + h_{r,d,t}(y) \quad (2)$$

$$\text{If } d \leq f \text{ and } r \geq s, \text{ then } h_{r,d,t}(x) \leq h_{s,f,t}(x) \quad (3)$$

$$h_{r,d,t}(cx) = c h_{r,d,t}(x) \text{ for all } c \geq 0. \quad (4)$$

It follows that for all  $r \geq 0$  and nonnegative integer  $t$

$$h_{r,r,t} \text{ is a linear functional on } X \quad (5)$$

In particular we will let

$$f_r = h_{r,r,K}$$

The conventional definition of *internal rate of return* or *yield rate* of a transaction  $x$  is a number  $i$  such that  $f_{1+i}(x) = 0$ . Of course there may be many such rates or none at all. The TRM theory recovers uniqueness by defining yield relative to a deposit rate. For fixed  $d$ , and any nonzero  $x$  in  $X$  we see from (3) that

$$h_{r,d,K}(x) \text{ is a decreasing function of } r \quad (6)$$

(in fact strictly decreasing) and hence there is at most one value of  $r$  for which it is zero. This will be denoted by  $r_d(x)$ . The quantity  $r_d - 1$  is the *TRM yield* on the transaction relative to the deposit ratio  $d$ . We will let  $r_d = \infty$  if  $h_{d,r,K} > 0$  for all  $r \geq 0$ , and  $r_d = 0$ , if  $h_{d,r,K} \leq 0$  for all  $r \geq 0$ .

For any  $r > 0$ , we let  $PL_r$  denote the set of all transactions which satisfy

$$h_{r,r,k}(x) \leq 0, k = 0, 1, \dots, K - 1$$

$$h_{r,r,K} = 0$$

Transactions in  $PL_r$  are the *pure loans* with yield  $r - 1$ . A simple induction shows that for  $x$  in  $PL_r$  and  $d \geq 0$

$$r_d(x) = d$$

It is important to note, as shown by Example 4 above, that uniqueness of yield does not imply pureness.

For technical purposes it is convenient to make the following definition

$$PL_0 = \{x \in R^{K+1} : x_i \leq 0, i = 0, 1, \dots, K\} \quad (7)$$

(This special definition is necessary since if  $x_K$  is nonzero, the final balance at time  $K$  will not be 0 when accumulated at  $r = 0$ .)

Note that the zero transaction,  $(x(i) = 0 \text{ for all } i)$  is in  $PL_r$  for all  $r \geq 0$ .

We now can identify those pure loans which are usurious or nonusurious when the maximum ratio is set at  $r$ . These are respectively

$$PU_r = \text{all nonzero elements of } \bigcup_{s > r} PL_s$$

$$PN_r = \bigcup_{s \leq r} PL_s$$

We now wish to define a class  $F$  of transactions, which will be favorable to all rational lenders. Certainly  $\delta_k$  should be in  $F$ , for it means that one receives a positive return with no investment at all. Moreover,  $\delta_j - \delta_k$  where  $j \leq k$  should be in  $F$  since it involves receiving a unit at one time, to be repaid at a future date. Taking all linear combinations of these elements with positive coefficients we arrive at the definition.  $F$  will consist of all  $x \in X$  such that

$$\sum_{i=0}^k x_k \geq 0 \text{ for } k = 0, 1 \dots K$$

## 6 Technical Lemmas

In this section we gather together some technical results which will be used to derive the main conclusions given in the sections to follow.

**Lemma 1** For any  $x \in F$

$$h_{r,d,t}(x) \geq 0 \text{ if } d \geq 1$$

$$h_{r,d,t}(-x) \leq 0 \text{ if } 1 \leq r.$$

*Proof.* We simply invoke (3) together with the obvious fact that for  $x$  in  $F$ , we have  $h_{r,1,t}(x) \geq 0$  for all  $r \geq 1$  and  $h_{1,d,t}(-x) \leq 0$  for all  $d \geq 1$ .  $\square$

**Lemma 2** Suppose the transaction  $x$  satisfies

$$h_{r,r,k}(x) \leq 0, k = 1, 2, \dots, K - 1,$$

$$h_{r,r,K}(x) > 0$$

Then  $x$  is in  $PL_s$  for some  $s > r$ .

*Proof.* Given any  $s > r$ , it follows easily that  $h_{s,s,t}(x) \leq h_{r,r,t}(x)$  for  $t = 1, 2, \dots, K$ . From the fact that the first nonzero entry in  $x$  is necessarily negative, we must have  $h_{s,s,K}(x) < 0$  for sufficiently large  $s$  and the result follows by continuity.  $\square$

The following lemma will be used later to write certain transactions as the sum of two transactions, both of which are in some  $PL$  set (not necessarily for the same rate).

**Lemma 3** *Given any  $x$  in  $X$  and  $d, r, \geq 0$  there exists a transaction  $z$  such that*

$$z \in PL_d$$

$$h_{r,d,k}(x+z) \leq 0, k = 0, 1, \dots, K-1$$

$$h_{r,d,K}(x+z) = h_{r,d,K}(x)$$

*Proof.* This is by induction on  $i(x)$ , the number of indices  $i$  in the set  $\{0, \dots, K-1\}$  for which  $h_{r,d,i}(x) \leq 0$ . If  $i(x) = 0$ , we simply take  $z = 0$ . Assume the assertion is true for  $i(x) < m$  and suppose that  $i(x) = m$ . Choose any index  $j$  for which  $h_{d,r,j}(x) > 0$ . Let

$$w = -h_{d,r,j}\delta_j + h_{d,r,j}d\delta_{j+1}$$

Then  $w$  is in  $PL_d$ . Moreover,  $h_{r,d,j}(x+w) = 0$  and  $h_{r,d,k}(x+w) = h_{r,d,k}(x)$  for all  $k \neq j$ . Therefore  $i(x+w) = m-1$  and by the induction hypothesis there exists an element  $v$  in  $PL_d$  such that

$$h_{r,d,k}(x+w+v) \leq 0, k = 0, 1, 2, \dots, K-1$$

$$h_{r,d,K}(x+w+v) = h_{r,d,K}(x+w)$$

From (5) we see that  $PL_d$  is closed under addition, and we can take  $z = w+v$  to establish the conclusion for  $x$ .  $\square$

**Lemma 4** *Fix  $r > 1$ . Let  $f$  be a nonzero linear functional on  $X$  such that*

$$f(PU_r) \geq 0 \text{ and } f(PN_r) \leq 0.$$

*Then  $f$  is a positive multiple of  $f_r$ .*

*Proof.* We first show that  $f(\delta_0) > 0$ . Suppose to the contrary that  $f(\delta_0) \leq 0$ . If for some  $k > 0$ ,  $f(\delta_k) > 0$  then for  $s < r^k$ ,  $x = -\delta_0 + s\delta_k$  is in  $PN_r$ , but

$$f(x) = -f(\delta_0) + sf(\delta_k) > 0$$

contradicting the hypothesis. Similarly if for some  $k > 0$ ,  $f(\delta_k) < 0$ , we can choose  $t$  large enough so that  $t > r^k$  and  $tf(\delta_k) < f(\delta_0)$ . Then  $y = \delta_0 + t\delta_k$  is in  $PU_r$  but

$$f(y) = f(\delta_0) + tf(\delta_k) < 0$$

again contradicting the hypothesis. If for all  $k > 0$ ,  $f(\delta_k) = 0$ , then  $f(\delta_0) < 0$ , and we get a contradiction with  $x$  as above, for any choice of  $k$ .

We now will assume that  $f(\delta_0) = r^K$  and then show that  $f = f_r$ . To do this we need only show that for all positive integers  $k \leq K$  we must have

$$f(\delta_k) = r^{K-k}$$

If in fact  $f(\delta_k) < r^{K-k}$  then for  $\epsilon > 0$ ,  $x = -\delta_0 + (r^k + \epsilon)\delta_k$  is in  $PU_r$ , but for  $\epsilon$  sufficiently small,  $f(x) < 0$ . If  $f(\delta_k) > r^{K-k}$  then  $y = \delta_0 + r^k$  is in  $PN_r$ , but  $f(y) > 0$ , a contradiction  $\square$ .

## 7 $U_r$ and $N_r$ - sets.

Suppose we are given  $r > 1$ . A subset  $A$  of  $X$  will be called a  $U_r$ - set if

1.  $PU_r \subseteq A$
2.  $A$  is a convex cone
3.  $A + F \subseteq A$
4.  $A$  is disjoint from the convex cone generated by  $PN_r$ ,
5.  $A$  is open

A subset  $B$  of  $X$  will be called a  $N_r$ - set if

1.  $PN_r \subseteq B$

**Theorem 5** Let  $r \geq 1$  be fixed.

If  $1 \leq d \leq r$ ,  $U_{r,d}$  is a  $U_r$ -set. Moreover, for any  $U_r$ -set  $A$ ,

$$U_{1,r} \subseteq A \subseteq U_{r,r}$$

If  $r \leq d$ ,  $N_{r,d}$  is a  $N_r$ -set. Moreover, for any  $N_r$ -set  $B$ ,

$$N_{r,\infty} \subseteq B \subseteq N_{r,r}$$

*Proof.* Formulas (1) (2) (4) show that  $U_{r,d}$  and  $N_{r,d}$  are convex cones, and together with Lemma 1 show that  $U_{r,d} + F \subseteq U_{r,d}$  and  $N_{r,d} + F \subseteq N_{d,r}$ . We use (3) to see that if  $x$  is a nonzero element in  $PL_s$  for some  $s > r$  then

$$h_{r,d,K}(x) > h_{s,d,K}(x) = h_{s,s,K}(x) = 0$$

and if  $x$  is in  $PL_s$  for some  $s \leq r$  then

$$h_{r,d,K}(x) \leq h_{s,d,K} = h_{s,s,K} = 0$$

showing that  $PU_r \subseteq U_{r,d}$ , and  $PN_r \subseteq N_{r,d}$ .

We now have established that for any suitable  $d$ , the sets  $U_{r,d}$  and  $N_{r,d}$  are disjoint convex cones containing  $PU_r$  and  $PN_r$ , respectively, thereby meeting conditions 4 in the definitions of  $U_r$  and  $N_r$ -sets.

Now, let  $A$  be any  $U_r$ -set and let  $x$  be any element of  $U_{r,1}$ . Then,  $h_{r,1,K}(x) > 0$  and applying Lemma 3 followed by Lemma 2 we can find  $z$  in  $PL_1$  such that  $x + z$  is in  $PL_s$  for some  $s > r$ , and therefore  $x + z$  is in  $PU_r$  which is contained in  $A$ . We now note that  $-z$  is in  $F$  and since  $x = x + z - z$ ,  $x$  must be in  $A$ , by condition 3 in the definition of a  $U_r$ -set.

The set  $A$  and the convex cone generated by  $PN_r$  are disjoint convex sets and by the Separating Hyperplane Theorem, there is a linear function  $f$  with  $f(A) > 0$  and  $f(PN_r) \leq 0$ . By Lemma 4,  $f$  must be a positive multiple of  $f_r$ . It follows that  $A$  is contained in  $U_{r,r}$ .

Let  $B$  be an  $N_r$ -set. For any  $x$  in  $N_{r,\infty}$  we have from (8) that for  $u = -f_r(x)\delta_k$ ,  $u$  is in  $F$ ,  $x + u$  is in  $PL_r$  and therefore in  $B$ . Hence,  $x = (x + u) - u$  is in  $B$ . Finally, we note that  $B$  is in  $N_{r,r}$ , by an argument analogous to that used above in showing that all  $U_r$ -sets are in  $U_{r,r}$ .  $\square$

Note that the upper bounds in the above theorem immediately imply Axiom 5 since

$$U_{r,r} \cap N_{r,r} = \emptyset$$

Another interesting fact which follows from this, is that we cannot have the absurd situation where both  $x$  and  $-x$  were  $U_r$ -sets, since if so,

$$0 = f_r(x + (-x)) = f_r(x) + f_r(-x) > 0 + 0$$

a contradiction.

## 8 Classification Schemes

A *classification scheme* is a function which assigns to each  $r > 1$  a pair of sets  $\{A_r, B_r\}$  such that

- $A_r$  is a  $U_r$ -set and  $B_r$  is a  $N_r$ -set.
- For  $s \leq r$ ,  $A_r \subseteq A_s$  and  $B_s \subseteq B_r$ .

Finding a classification scheme accomplishes our goal. Given the maximum allowable ratio of  $r$ , a transaction will be usurious if it is in  $A_r$  and nonusurious if it is in  $B_r$ . The second condition in the definition formalizes the requirement of Axiom 6. Therefore, to satisfy all seven of our axioms it is clear that we need to find a classification scheme such that

$$A_r \cup B_r = X \text{ for all } r \tag{9}$$

**Theorem 6 (The Impossibility Theorem).** *There is no classification scheme satisfying (9).*

*Proof.* From Theorem 5 the only way to satisfy (9) is to take  $A_r = U_{r,r}$  and  $B_r = N_{r,r}$  for all  $r$ . But then the second condition is violated. In fact, given any  $r$ , consider the transaction

$$x = (-1, a, b, c)$$

where  $a, b, c$  are such that the polynomial  $-x^3 + ax^2 + bx + c$  has roots  $r_i, i = 1, 2, 3$ , with  $1 \leq r_1 < r_2 < r < r_3$ . Then  $x$  is in  $U_{r,r}$  but not in  $U_{s,s}$  for  $r_1 < s < r_2$ .  $\square$

It is of interest to note that  $\{U_{r,r}, N_{r,r}\}$  is precisely the *scheme* proposed in [1]. It appears however that the author was thinking only of the pure situation in which case the required sensitivity does hold.

Since our definition of classification scheme does not require (9) it allows for certain transactions to be ambiguous with respect to  $r$  and classified as neither usurious or nonusurious. We can show now that classification schemes do indeed exist. For each  $r \geq 1$  we define

$$U_r^* = \{x \in X : f_s(x) > 0 \text{ for } 1 \leq s \leq r\}$$

$$N_r^* = \{x \in X : f_s(x) \leq 0, \text{ for } r \leq s < \infty\}$$

**Theorem 7**  $\{U_r^*, N_r^*\}$  is a classification scheme Moreover for any classification scheme  $\{A_r, B_r\}$ , we must have

$$A_r \subseteq U_r^* \text{ and } B_r \subseteq N_r^*$$

*Proof.* The linearity of  $f$  shows that both sets are convex cones, that  $U_r^*$  is closed under the addition of  $F$  and that  $N_r^*$  is closed under the subtraction of  $F$ .  $N_r^*$  is closed by continuity of  $f_r$ , and  $U_r^*$  is easily shown to be open. For example, if  $x$  and  $x'$  are elements in  $X$  such that  $|x_i - x'_i| \leq \epsilon$  for  $i = 1, 2 \dots K$ , then given  $s$  in the interval  $[1, r]$  we have that  $|f_s(x) - f_s(x')| \leq K\epsilon r^K$ . It is clear that  $U_r^*$  and  $N_r^*$  are disjoint sets containing  $PU_r$  and  $PN_r$  respectively so that the first condition of a classification scheme is met. The second condition is immediate from the definition.

Suppose now that  $\{A_r, B_r\}$  is a classification scheme. If for some  $r > 1$ ,  $x$  is not in  $U_r^*$ , then for some  $q \leq r$ , we have by Theorem 5 and the definition of a classification scheme that

$$x \in \tilde{U}_{q,q} \subseteq \tilde{A}_q \subseteq \tilde{A}_r.$$

We similarly show that  $B_r$  is contained in  $N_r^*$ .

## 9 Weak sensitivity

Theorem 7 implies certain counterintuitive conclusions. For example,  $f_1(x) \leq 0$  implies that  $x$  is not in  $U_r^*$  for any  $r$  and therefore could never be considered as usurious. Moreover any  $x$  with the first nonzero entry as positive cannot be in  $N_r^*$  for any  $r$  since

$$\lim_{r \rightarrow \infty} f_r(x) = \infty.$$

These can therefore never be considered nonusurious. We can imagine situations however, like Example 3 above in the first case, or Example 5 above in the second case where we would want to classify them in such a manner.

A weaker version of Axiom 6 allows us to reduce the ambiguous cases and classify more transactions as usurious or not. Rather than postulating that lowering the maximum rate will preserve usuriousness, we postulate only that a usurious transaction will not become nonusurious, although it may become ambiguous. Precisely we have the following.

A *weak* classification scheme is a function which assigns to each  $r > 1$  a pair of sets  $\{A_r, B_r\}$  such that

- $A_r$  is a  $U_r$ - set and  $B_r$  is a  $N_r$ - set.
- For  $s \leq r, A_r \cap B_s = \emptyset$

Weak sensitive schemes allow us to make quantitative judgements about reasonable rates of interest, and to avoid some of the counterintuitive aspects suggested above. For example, the reason that the transaction of Example 3 is ambiguous, is that for extremely low rates of interest, the benefit to the lender may be minimal. The reason that the transaction of Example 5 is ambiguous is that for extremely high rates of interest, the application fee may present a heavy burden. (This is the significance of the 99.89 % rate mentioned.) Following are examples of weak schemes which allow us to ignore unrealistically extreme values.

Choose any  $d$  in  $[1, \infty]$ . The  $(d, *)$  *weak classification scheme* is defined by

$$A_r = \begin{cases} U_{r,d} & \text{if } r \geq d \\ U_{r,r} & \text{otherwise} \end{cases}$$

$$B_r = N_r^*$$

This is indeed weak-sensitive since given any  $r$  and  $x$  in  $A_r$

$$d \leq s \leq r \Rightarrow h_{r,d,K}(x) > 0 \Rightarrow h_{s,s,K}(x) > 0 \Rightarrow x \notin N_s^*$$

$$s < d \leq r \Rightarrow h_{r,d,K}(x) > 0 \Rightarrow h_{d,d,K}(x) > 0 \Rightarrow x \notin N_s^*$$

$$s \leq r < d \Rightarrow h_{r,r,K}(x) > 0 \Rightarrow x \notin N_s^*$$

A  $(d, *)$  classification scheme is not sensitive since if the maximum allowable ratio falls below  $d$  a transaction that was usurious can become ambiguous. But if we choose  $d$  low enough, we can be reasonably assured that this anomalous behaviour won't occur. Such a scheme allows transactions like that of Example 3 to be classified as usurious.

Given any  $e$  between 1 and  $\infty$  we define the  $(*, e)$  *weak classification scheme* by

$$A_r = U_r^*$$

$$B_r = \begin{cases} N_{r,e} & \text{if } r \leq e \\ N_{r,r} & \text{otherwise} \end{cases}$$

These schemes are not sensitive since if the maximum allowable ratio goes above  $e$ , a nonusurious transaction can become ambiguous, but we can reasonably avoid such behaviour by choosing  $e$  sufficiently high. Such a scheme allows transactions such as that in Example 5 to be classified as nonusurious.

One can in fact deal with both low and high rates at once. Given any  $d, e$ , with  $1 \leq d \leq e \leq \infty$ , we define the  $(d, e)$  *weak classification scheme* by

$$A_r = \begin{cases} U_{r,d} & \text{if } r \geq d \\ U_{r,r} & \text{otherwise} \end{cases}$$

$$B_r = \begin{cases} N_{r,e} & \text{if } r \leq e \\ N_{r,r} & \text{otherwise} \end{cases}$$

## 10 Further analysis of the ambiguous case

Let  $M_r$  denote the complement of  $U_r^* \cup N_r^*$ . These are transactions which are necessarily ambiguous and must remain unclassified in any (nonweak) classification scheme. We may further subdivide these as follows. Let

$$UM_r = \{x \in M_r : f_r(x) > 0\}$$

$$NM_r = \{x \in M_r : f_r(x) \leq 0\}$$

Then  $UM_r$  denotes those transactions which are not in themselves usurious but have the capacity to result in a usurious transaction when combined with a nonusurious one. In view of Theorem 5, these are the only elements of  $M_r$  which could possibly be classified as usurious at ratio  $r$ , in a weak classification scheme.

Similarly  $NM_r$  denotes those transactions which are not in themselves nonusurious but have the capacity to result in a nonusurious transaction when combined with a usurious one. In view of Theorem 5 these are the only elements of  $M_r$  which could be possibly considered as nonusurious at ratio  $r$ , in a weak classification scheme

An example of a transaction in  $UM_r$  is that of Example 2. As noted in Section 7, this will be usurious under the  $(d, *)$  weak scheme for any  $d > 1.25$

For another example of a transaction in  $UM_r$  consider  $x = (1, -s)$  where  $1 < s < r$ . This is not in  $U_r^*$  since choosing  $q$  between 1 and  $s$ ,  $f_q(x) < 0$ , and it is not in  $N_r^*$  since choosing  $t$  greater than  $r$ ,  $f_t(x) > 0$ . Moreover, in the  $(d, *)$  weak classification scheme with  $s < d$  this would be in  $U_{r,d}$  and therefore classified as usurious for any  $r \geq d$

This may seem suprising at first glance since this transaction seems not to be a loan at all but merely represents borrowing at a low rate. Recall however that in our approach there is no attempt to identify the lender. To see why this should be in  $UM_r$ , and to illustrate the point made above about combining, we will consider the following scenario. You wish to borrow 1000, and approach a prospective source who charges a very low rate. For each 101 borrowed you pay interest of 6 at the end of the year. However, the minimum amount of a loan from this source is 100,000. To accomodate you, the person agrees to lend you 101,000 and the same time borrow back from you 100,000 at 5 %. You readily agree, and acknowledge that since you are getting such a low rate of interest, it is only fair for the lender to charge the slight spread.

The lender is therefore lending money at about 5.9%, hardly usurious under present standards, and borrowing at 5%. The combined effect however is that you receive a net loan of 1000, and must repay a net amount of 2000 at the end of one year. The interest rate is 100%, and the net loan is clearly usurious under the present Canadian maximum of 60%. Were we to classify the borrowing transaction as nonusurious, we would violate Axiom 3.

It is of interest to note that the main activity of most financial institutions is to go through this very same type of parlay, borrowing at one rate and lending at a higher one. These do not generally involve the same party of course, but sometimes they do. Is it not untypical in Canada for people to receive a loan from a Trust Company and put up the Company's very own guaranteed investment certificates as collateral.

## 11 Concluding remarks

We conclude with a few remarks dealing with the practical implementations of the theory.

The choice of a classification scheme could depend on the particular objective. If one wishes a classification scheme which includes as few unclassified transactions as possible then the choice is  $\{U_r^*, N_r^*\}$ . For the purposes of the Canadian scene however one might wish to give the lender the benefit of the doubt and include as *criminal*, only those transactions which must be so classified. In other words it could be argued that the appropriate classification scheme is  $\{U_{r,1}, N_r^*\}$ . or even the (1,1) weak classification scheme  $\{U_{r,1}, N_r\}$ . The latter leads to a very simple numerical procedure. A transaction  $x$  is usurious iff  $h_{1,r,N}(x) > 0$ , nonusurious iff  $h_{r,r,N}(x) \leq 0$ , and ambiguous iff neither of these hold. ( Here ,  $N$  is the time of the last nonzero payment which can obviously replace  $K$  in the calculation ).

Recall, as remarked above, that looking at the various zeros of the polynomial  $f$  can lead to inconsistencies if one is not careful. Using the theory developed however, we now can give some criteria for analyzing a transaction in terms of these zeroes.

Given a transaction  $x$ , let  $Z = \{t > 0 : f_t(x) = 0\}$ . Suppose that  $Z = \emptyset$ . If  $f$  is positive, then  $x$  is in  $U_{r,1}$  and therefore usurious for all  $r > 1$ . If  $f$  is negative then  $x$  is in  $U_{r,\infty}$  and therefore nonusurious for all  $r > 1$ . If  $Z$

is nonempty, we have the following five possibilities, which all follow easily from the definitions.

- If  $Z \cap [1, r] = \emptyset$ , and  $f_1(x) > 0$  then  $x$  is in  $U_r^*$
- If  $Z \cap [1, r] = \emptyset$ , and  $f_1(x) < 0$  then  $x$  is in  $NM_r$ .
- If  $Z \cap (r, \infty) = \emptyset$ , and  $f < 0$  on  $(r, \infty)$ , then  $Z$  is in  $N_r^*$
- If  $Z \cap (r, \infty) = \emptyset$ , and  $f > 0$  on  $(r, \infty)$ , then  $x$  is in  $UM_r$ .
- If  $Z$  intersects both  $[1, r]$  and  $(r, \infty)$  then  $x$  is in  $UM_r$  if  $f_r(x) > 0$ , or  $NM_r$  if  $f_r(x) < 0$ .

## References

- [1] B. Burnell, "The Criminal Rate of Interest", *Proceedings of the Canadian Institute of Actuaries*, XVII,(1985),304-311.
- [2] S.G. Kellison, *The Theory of Interest*, 2nd ed., Richard D. Irwin Inc, (1991).
- [3] S.D. Promislow "A new approach to the theory of Interest", *Transactions, Society of Actuaries*, XXXII,(1980), 53-92
- [4] S.D. Promislow and D.Spring "Axioms for the Internal Rate of Return of an Investment Project", *Proceedings 26-th Annual Actuarial Research Conference, ARCH*, 1992.1, 327-335.
- [5] D. Teichroew, A.A. Robicheck and M. Montalbano, "Mathematical Analysis of Rates of Return Under Certainty", *Management Science(Ser.A)* XI,(1965) 395-403

