Pricing and Reserving for General Insurance Products

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Abstract Modern insurance products frequently involve contingent benefits that cannot be analysed using traditional actuarial techniques. We formulate and illustrate a general financial model in which pricing and reserving may be done for quite general insurance products. Providing the model is complete in the sense of financial economics, the usual notions of prospective and retrospective reserves are easily understood in the context of financed cash flow streams. The reserve levels and the investment programs which finance them are intimately related and must be recognised as part of the same notion. The model provides a consistent framework in which actuaries may understand and analyse pricing and reserving for modern insurance products. Of practical significance, the reserve formula may also be used for Monte Carlo simulation.

1. Introduction

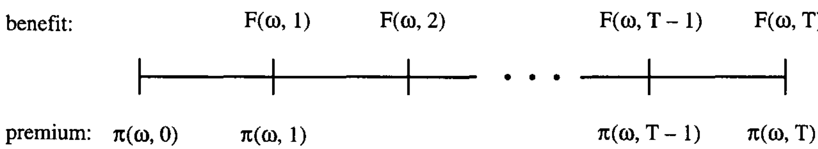
Increasingly, actuaries are faced with pricing and reserving for insurance contracts that involve payments that are contingent on financial variables such as the performance of the stock market and the level of interest rates. Variable annuity and variable life insurance products with embedded minimum death benefit guarantees are two very important examples of such contracts. Although reserving techniques are well understood in the case of the traditional actuarial model of certain cash flows and deterministic interest rates, as analysed in Gerber (1995), actuarial theory is less developed in the more general case of uncertain cash flows. Fortunately, one may draw on modern financial economics to formulate a standardised framework for understanding the pricing and reserving for a general class of insurance products. This framework is based on the techniques for the valuation of uncertain cash flow streams that have been developed in financial economics. The theoretical values for reserves which the model generates must be understood in the context of appropriate investment strategies for the company underwriting the insurance policies. Indeed, the reserve values are meaningful only if the insurance company follows appropriate investment practices.

In section 2 we discuss a benchmark type of insurance policy and the financial economics and associated actuarial theory that can be used to define reserves. Section 3 illustrates the model for the case of a variable annuity with guaranteed minimum death benefit. A more general formulation that can be used to allow for random mortality is briefly mentioned in section 4. Section 5 concludes the paper.

2. Model Formulation

It is assumed we are operating in a discrete-time environment. We shall analyse a general insurance policy with some similarities to the general life insurance policy of Gerber (1995, page 55) but differing in the important aspect that benefit and premium payments are permitted to depend on the state of the financial markets. We shall index the state of the financial market by the symbol ω. The collection of possible states of the financial market is denoted by Ω. The policy we study is assumed to have net cash flows at each of the times 0, 1, 2, . . . , T, with the finite time T being the end of the time horizon[1]. As is traditional in actuarial discussions of reserving, we shall view the policy from the perspective of the policyholder so that policy benefits are recorded as positive cash flows. We shall assume that if the insured dies in year k of the policy and the financial markets are in state ω then a death *benefit* in the amount of $F(\omega, k)$ is paid at time k [the end of the year of death]. Furthermore, if the insured is alive at the beginning of year k and the financial markets are in state ω then a *premium* in the amount of π(ω, k – 1) is paid at time k – 1 [the beginning of the year for which the insured is alive]. Furthermore, the benefit payments {F(ω, k) : k = 1, 2, . . . , T} and the premium payments {π(ω, k) : k = 0, 1, . . . , T} are permitted to be random variables that depend on history of financial market variables such as interest rate levels and stock market performance. The distinction we make between benefits and premiums permits us to accommodate mortality in as convenient a fashion as possible. The techniques we discuss are more general than our definition of this policy might suggest[2]. The benefits and premiums for this insurance policy may be represented as shown in Figure 1.

Figure 1



In order to allow maximum flexibility in the use of this policy, benefit and premium payments are permitted to be negative. For instance, a pure endowment policy for a unit amount payable at time 10 would have $\pi_{10} = -1$. Although we are using the terminology of life insurance, the model may be easily adapted to the case of non-life insurance benefits and premiums.

We shall assume that the insurance company's mortality experience follows the life table. One can motivate this assumption by assuming that the insurance company issues a large number policies and invoke the law of large numbers. As such, we are assuming that mortality is deterministic. Consequently, each new insurance policy may be viewed as an agreement in which the insured receives the cash flow stream

$$c(\omega, k) := {}_{k-1|}q_x \, F(\omega, k), \;\; k = 1, 2, \ldots, T$$

in return for paying to the insurance company[3] the cash flow stream

$$e(\omega, k) := {}_{k}p_x \, \pi(\omega, k), \;\; k = 0, 1, \ldots, T.$$

Mortality can be modelled in a fully random fashion akin to what is done in Gerber (1995) and this is briefly discussed in section 4 below. However, the fundamental issues in reserving for insurance policies with stochastic benefits relate to the behaviour of the underlying financial market variables and not the mortality. This is why we have chosen to treat mortality as deterministic. We now explain the financial economics that will support the notions of pricing and reserving for our class of insurance policies.

It is assumed that a collection of *primitive assets* are available for trading in the financial market. These will consist of assets such as treasury notes and bonds, index funds, and common stocks. These are the assets that the insurance company may purchase in order to finance their liabilities. As is customary in financial modelling, we shall assume that among the primitive assets there exists a non-dividend paying asset, referred to as the *money market account* and denoted by S_0, the price of which evolves according to the equation

$$S_0(\omega, k) := [1 + r(\omega, 0)]\,[1 + r(\omega, 1)] \cdots [1 + r(\omega, k-1)], \qquad (2.1)$$

where the process $\{r(\omega, k)\}$ is referred to as the *short-rate process*. The short-rate process is interpreted as the treasury note rate for the time step size used in the model. The money market account process is interpreted as the accumulated value of one unit continually invested in short paper. The primitive assets, including the money market account, are denoted by the $(N + 1)$-dimensional vector process $\{S(\omega, k)\}$. This notation is a compact way of summarising the vector relation $S(\omega, k) = (S_0(\omega, k), S_1(\omega, k), \ldots, S_N(\omega, k))$. The dividends paid by each of the primitive assets are denoted by the $(N + 1)$-dimensional vector process $\{d(\omega, k)\}$. The i^{th} primitive asset of may be non-dividend paying and in such a case, $d_i(\omega, k) \equiv 0$. In the traditional actuarial model one has a single asset of the form (2.1) with $S_0(\omega, k) = (1 + r)^k$ for some constant value of r.

At each point in time and state of the world the insurance company may hold a portfolio of the primitive assets. We shall let the vector $\vartheta(\omega, k)$ denote the asset holdings of the insurance company at time k when the financial market is in state ω. The trading dynamics, as are standard in all financial economics models, are that the price of a cash flow stream evolves as

$$\begin{aligned} \vartheta(\omega, k-1) \cdot S(\omega, k) + \vartheta(\omega, k-1) \cdot d(\omega, k) + e(\omega, k) \\ = c(\omega, k) + \vartheta(\omega, k) \cdot S(\omega, k), \end{aligned} \qquad (2.2)$$

for each $k = 0, 1, \ldots, T$ and $\omega \in \Omega$; subject to the boundary conditions $\vartheta(\omega, -1) = 0$ and $\vartheta(\omega, T) = 0$. The left hand side of relation (2.2) is the market value of the investment portfolio at time k plus the dividends and premiums received and relation (2.2) says that this value must equal the benefits paid plus the market value of the investment portfolio at the beginning of the next period. The boundary conditions merely say that there are no portfolio holdings prior to time 0 and that the investment portfolio must be liquidated at the end of the trading horizon [time T]. If this relationship is satisfied at all points in time and all states of the world we say that the trading strategy ϑ *finances* the insurance policy defined by the cash flow streams $c(\omega, k)$ and $e(\omega, k)$.

We shall assume that the investment market is *arbitrage-free* and *complete*. The condition that the market is arbitrage-free can be intuitively described by saying that investors cannot make certain profits by adopting riskless positions. The condition that the market is complete means that every cash flow stream may be obtained by forming an appropriate portfolio of the primitive assets available for trading[4]. In complete markets, each cash flow stream has a unique price. The *price* at time k when the financial market is in state ω of a cash flow stream financed by a trading strategy ϑ is equal to $\vartheta(\omega, k) \cdot S(\omega, k)$,which is the value of the investment portfolio that finances the remaining cash flows. It is possible to characterise this price as a discounted expectation of future remaining cash flows and this is most useful in financial valuation. Indeed, when an investment market is arbitrage-free and complete the theory of financial valuation guarantees the existence of a probability measure, which we shall denote by Q, referred to as the *risk-neutral measure,* such that the price at time k of each uncertain cash flow stream $\{h(\omega, j) : j = k+1, \ldots, T\}$ is given by the following conditional expectation [conditional on all financial market information at time k] under the probability measure Q,

$$E_k^Q\left[\sum_{j=k+1}^{T} \frac{1}{[1+r(\omega,k)]\cdots[1+r(\omega,j-1)]} h(\omega, j)\right] \equiv E_k^Q\left[\sum_{j=k+1}^{T} \frac{S_0(\omega, j)}{S_0(\omega, k)} h(\omega, j)\right]. \tag{2.3}$$

[More specifically, the assumption that the model is arbitrage-free ensures that at least one such valuation measure Q exists and the assumption that the model is complete implies that the valuation measure is unique.] As we have not specified a particular financial model, other than to assume that the model we are working with is arbitrage-free and complete, we cannot say anything more about the nature of the valuation measure Q. In many types of models, Q may be described or constructed from the primitive assets in the model[5]. The model employed in section 3 is an example of this. For the present, we shall continue to work with a general risk-neutral valuation measure.

The *reserve* for an insurance policy is defined as the price of the future net cash flows of the policy with accumulation for survivorship. Consequently, the reserve at time n when the financial market is in state ω is given by the following expression.

$$V(\omega, n) \\ := E_n^Q\Big[\sum_{k=1}^{T-n} \frac{S_0(\omega, n+k)}{S_0(\omega, n)}\, c(\omega, n+k) - \sum_{k=0}^{T-n-1} \frac{S_0(\omega, n+k)}{S_0(\omega, n)}\, e(\omega, n+k)\Big] / {}_n p_x \tag{2.4}$$

This is a *prospective reserve formula*. The reserve will rarely have a closed formula expression which can be computed based on the formula (2.4). However, one can always estimate the reserve through Monte Carlo simulation since the random quantities appearing in equation (2.4) are specified as functions of the underlying random financial market variables. In practice, Monte Carlo simulation can be more complex than might first appear. Indeed, one must often evaluate path dependent insurance benefits such as the periodic premium variable annuity we examine in section 4. Unlike the retrospective formula to be discussed below, the prospective reserve formula (2.4) does not require explicit knowledge of the trading strategy ϑ that finances the insurance policy. The reserve value is the value of the investment portfolio at that point in time and state of the financial market which will ensure that the insurance company will exactly meet its financial obligations under the policy.

The premiums for the policy are called *net* if $V(\omega, 0) = 0$. Since

$$V(\omega, 0) = E^Q\Big[\sum_{k=1}^{T} S_0(\omega, k)\, c(\omega, k) - \sum_{k=0}^{T-1} S_0(\omega, k)\, e(\omega, k)\Big],$$

the condition that the policy premiums are net means that the price of the uncertain benefit cash flow stream is equal to the price of the uncertain premium cash flow stream[6]. Therefore, when premiums are net the insurance company breaks even in the absence of expenses. If ϑ is a trading strategy that finances the insurance policy then equation (2.3) combined with the preceding equation shows that $V(\omega, 0) = 0$ if and only if $\vartheta(\omega, 0) \cdot S(\omega, 0) = e(\omega, 0)$. In other words, policy premiums are net if and only if one has $\vartheta(\omega, 0) \cdot S(\omega, 0) = e(\omega, 0)$.

We now derive the retrospective reserve formula when the premiums for the policy are net[7]. Suppose that the trading strategy ϑ finances the insurance policy defined by the cash flow streams $c(\omega, k)$ and $e(\omega, k)$. The following identity is easily checked.

$$\vartheta(\omega, k-1) \cdot S(\omega, k) - \vartheta(\omega, k) \cdot S(\omega, k) = \Delta[\vartheta(\omega, k-1) \cdot S(\omega, k-1)] - \vartheta(\omega, k-1) \cdot \Delta S(\omega, k-1) \tag{2.5}$$

We may then rewrite (2.5) as follows.

$$0 = \Delta[\vartheta(\omega, k-1) \cdot S(\omega, k-1)] - \vartheta(\omega, k-1) \cdot \Delta S(\omega, k-1) - \vartheta(\omega, k-1) \cdot d(\omega, k) + c(\omega, k) - e(\omega, k) \tag{2.6}$$

Summing from k = 1 to k = n yields

$$0 = \vartheta(\omega, n) \cdot S(\omega, n) - \vartheta(\omega, 0) \cdot S(\omega, 0) - \sum_{k=1}^{n} \vartheta(\omega, k-1) \cdot \Delta S(\omega, k-1) - \sum_{k=1}^{n} \vartheta(\omega, k-1) \cdot d(\omega, k) + \sum_{k=1}^{n} [c(\omega, k) - e(\omega, k)]. \tag{2.7}$$

We may now rearrange this expression and use the condition $\vartheta(\omega, 0) \cdot S(\omega, 0) = e(\omega, 0)$, since premiums are net, to obtain the relationship,

$$\vartheta(\omega, n) \cdot S(\omega, n) = \sum_{k=1}^{n} \vartheta(\omega, k-1) \cdot \Delta S(\omega, k-1) + \sum_{k=1}^{n} \vartheta(\omega, k-1) \cdot d(\omega, k) + \sum_{k=0}^{n} e(\omega, k) - \sum_{k=1}^{n} c(\omega, k). \tag{2.8}$$

This formula provides the basis for the retrospective reserve formula for our model. From equation (2.3) we know that

$$\vartheta(\omega, n) \cdot S(\omega, n) = E_n^Q\left[\sum_{k=1}^{T-n} \frac{S_0(\omega, n+k)}{S_0(\omega, n)} c(\omega, n+k) - \sum_{k=1}^{T-n-1} \frac{S_0(\omega, n+k)}{S_0(\omega, n)} e(\omega, n+k)\right]. \tag{2.9}$$

Therefore, we see that the reserve $V(\omega, n)$ as defined by equation (2.4) has an equivalent expression as

$$V(\omega, n) = \frac{\vartheta(\omega, n) \cdot S(\omega, n) - e(\omega, n)}{{}_n p_x}. \tag{2.10}$$

We may use relation (2.8) to express this as

$$V(\omega, n) = [\sum_{k=1}^{n} \vartheta(\omega, k-1) \cdot \Delta S(\omega, k-1) + \sum_{k=1}^{n} \vartheta(\omega, k-1) \cdot d(\omega, k) + \sum_{k=0}^{n-1} e(\omega, k) - \sum_{k=1}^{n} c(\omega, k)] / {}_np_x. \tag{2.11}$$

This is the general *retrospective reserve formula*. The expression (2.11) says that the reserve for the policy is equal to the sum of the capital gains, dividends, premium payments and benefit payments all accumulated for survivorship. Note that there is no accumulation factor applied to the premium and benefit payments. This is "picked up" by the capital gains on the investment holdings. This formula does simplify to the traditional retrospective valuation formula, such as ${}_nV_x = P_x\ddot{s}_{x:\overline{n}|} - {}_nk_x$ for whole life policies, when there is a single asset available as an investment vehicle. However, in general the formula cannot be simplified in a useful fashion since when there are multiple primitive assets the investment holdings [i.e. trading strategy] necessary to finance an insurance policy can be quite complicated. We shall illustrate this in section 4 with a simple example for a variable annuity with guaranteed minimum death benefits. The retrospective formula (2.11) may be used for computing and simulating reserve values if the trading strategy ϑ that finances the insurance policy is known. However, the trading strategy is often unknown and can be complicated to compute. Furthermore, if the valuation involves path dependent securities the computation of the trading strategy is usually too complex to be feasible. We show how the trading strategy may be computed recursively for the simple variable annuity example in section 4. However, variable annuities have path dependent payments and these computations are not feasible in practical applications. In summary, the retrospective formula is generally less useful than the prospective formula because the trading strategy ϑ may be difficult to obtain, especially in cases of practical interest. Nevertheless, just as in the classical case of Gerber (1995) the reserve values may be computed by either of the equivalent prospective or retrospective methods.

If there is a single primitive asset for investing, denoted by $S(\omega, k)$, then the retrospective formula (2.11) may be expressed in a form that is akin to the traditional retrospective formulas presented in Gerber (1995). Let us assume that the single asset pays no dividends. For instance, the asset might be a non-dividend paying stock. Since premiums are net, $e(\omega, 0) = \vartheta(\omega, 0)\, S(\omega, 0)$. Applying (2.2) for $k = 1$ gives $\vartheta(\omega, 0)\, S(\omega, 1) + e(\omega, 1) = c(\omega, 1) + \vartheta(\omega, 1)\, S(\omega, 1)$. This may be rewritten as

$$\vartheta(\omega, 0)\, S(\omega, 0)\, \frac{S(\omega, 1)}{S(\omega, 0)} + e(\omega, 1) = c(\omega, 1) + \vartheta(\omega, 1)\, S(\omega, 1)\,.$$

We may then use the preceding relation, $e(\omega, 0) = \vartheta(\omega, 0)\, S(\omega, 0)$, to express this relation as

$$e(\omega, 0)\, \frac{S(\omega, 1)}{S(\omega, 0)} + e(\omega, 1) = c(\omega, 1) + \vartheta(\omega, 1)\, S(\omega, 1)\,. \tag{2.12}$$

We now repeat the same argument but one period later. Thus, we begin with the relation $\vartheta(\omega, 1)\, S(\omega, 2) + e(\omega, 2) = c(\omega, 2) + \vartheta(\omega, 2)\, S(\omega, 2)$. This may be rewritten as

$$\vartheta(\omega, 1)\, S(\omega, 1)\, \frac{S(\omega, 2)}{S(\omega, 1)} + e(\omega, 2) = c(\omega, 2) + \vartheta(\omega, 2)\, S(\omega, 2)\,. \tag{2.13}$$

We may substitute for $\vartheta(\omega, 1)\, S(\omega, 1)$ in expression (2.13) using the expression in (2.12). On simplifying this yields the relation,

$$\begin{aligned} e(\omega, 0)\, \frac{S(\omega, 2)}{S(\omega, 0)} + e(\omega, 1)\, \frac{S(\omega, 2)}{S(\omega, 1)} + e(\omega, 2) - \left[c(\omega, 1)\, \frac{S(\omega, 2)}{S(\omega, 1)} + c(\omega, 2)\right] \\ = \vartheta(\omega, 2)\, S(\omega, 2). \end{aligned} \tag{2.14}$$

We may proceed by induction to establish that for each n,

$$\begin{aligned} & e(\omega, 0)\, \frac{S(\omega, n)}{S(\omega, 0)} + \cdots + e(\omega, n-1)\, \frac{S(\omega, n)}{S(\omega, n-1)} + e(\omega, n) \\ & - \left[c(\omega, 1)\, \frac{S(\omega, n)}{S(\omega, 1)} + \cdots + c(\omega, n-1)\, \frac{S(\omega, n)}{S(\omega, n-1)} + c(\omega, n)\right] \\ & = \vartheta(\omega, n)\, S(\omega, n). \end{aligned} \tag{2.15}$$

Applying the relationship for the reserve $V(\omega, n)$ noted in equation (2.10), we see that

$$\begin{aligned} V(\omega, n) = & \left[\left[e(\omega, 0)\, \frac{S(\omega, n)}{S(\omega, 0)} + \cdots + e(\omega, n-1)\, \frac{S(\omega, n)}{S(\omega, n-1)}\right]\right. \\ & \left. - \left[c(\omega, 1)\, \frac{S(\omega, n)}{S(\omega, 1)} + \cdots + c(\omega, n-1)\, \frac{S(\omega, n)}{S(\omega, n-1)} + c(\omega, n)\right]\right] /\, {}_n p_x. \end{aligned} \tag{2.16}$$

Relation (2.16) is the usual retrospective formula for reserves except that a more general stochastic factor has replaced the compound interest term. If the single investment vehicle were a fixed interest rate bond at rate i say then $S(\omega, k) = (1 + i)^k$ and (2.16) would reduce to the type of retrospective formula appearing in Gerber (1995). Of course, the relation (2.16) interprets as saying that the reserve is equal to the actuarial accumulated value of the premium contributions less the accumulated cost of the insurance benefits. The instances in which the theory of this paper is most valuable occurs when the insurance policy involves stochastic death benefits which require trading in at least two assets, for example stock and money market account, to finance the death benefits. As such, in all of the interesting cases the retrospective formula will not have a representation of the form (2.16). However, as we have previously noted the retrospective formula (2.11) does have a natural interpretation.

It is very important to remember that the notion of reserve that we have developed has a particular meaning in our complete markets setting. The reserve is the amount of money the insurance company needs to have available so that if invested correctly it can meet its future cash flow obligations. Consequently, the reserve is the value of the portfolio the insurance company should be holding to finance its obligations in respect of the insurance policy. The essential issue involves the following of the trading strategy that finances the outflows and inflows associated with the insurance policy in question. If the insurance company adopts an investment strategy that finances the insurance policy then they will have sufficient funds to exactly meet their obligations at all points in time and across all states of the world. If they do not follow such a strategy then they will have insufficient funds to meet their obligations at some points in time and states of the world. Also, as we have noted, if the trading strategy ϑ_1 finances the insurance policy in question then the reserve at time n is given by the quantity $[\vartheta_1(\omega, n) \cdot S(\omega, n) - e(\omega, n)] / {}_n p_x$. If the insurance company were to follow a different investment strategy, say ϑ_2, that did not finance the insurance policy then the retrospective value of the company's investment outcome might give the correct numerical value for the reserve at some point in time but this trading strategy would not correctly hedge the insurance company's risk and thus the numerical value would not be an appropriate measure of the amount of money the insurance company requires to meet its obligations under the policy.

3. Illustrative Example for Variable Annuities

We now illustrate the pricing and reserving model for the practically important case of variable annuities. We consider both the single premium and periodic premium versions of this product. We shall consider a variable annuity with minimum death benefit guarantee equal to a 0% rate of return[8] and for simplicity we will assume that we have a three-period model. We shall assume that mortality may be described as $q_x = 0.05$, ${}_{1|}q_x = 0.05$, and ${}_{2|}q_x = 0.05$. For simplicity, we use the constant interest rate binomial model of Cox, Ross, and Rubinstein (1979). In this model there are two assets, the money market fund and a stock index fund. The investment market outcomes for the model consists of the Cartesian product space $\Omega := \{0, 1\}^T$. A typical market outcome consists of the vector $\omega \equiv (\omega_1, \omega_2, \ldots, \omega_T)$ where each $\omega_i \in \{0, 1\}$. For the case of our three-period model we have $T = 3$ and $\Omega = \{0, 1\}^3$. The stock index is assumed to evolve according to the path independent process

$$S_1(\omega, k) = S_1(0)u^{\omega_1 + \cdots + \omega_k}d^{k - [\omega_1 + \cdots + \omega_k]}. \tag{3.1}$$

Thus, with the passage of each time unit the stock index either increases by the factor u or decreases by the factor d. We shall set $S_1(0) = 800$ as this is the approximate level of the S&P 500 index at present. We shall assume $u = 1.1$ and $d = 0.9$. The constant one-period interest rate will be taken as $r = 0.07$ and therefore, $S_0(\omega, k) = (1 + r)^k$. In practice, the fact that the investment values are path dependent for the periodic premium case will require some type of approximate numerical valuation procedure such as Monte Carlo simulation.

We now briefly describe the details of single premium and periodic premium variable annuities for our example. The *single premium* variable annuity consists of a single investment contribution of $75,000 made at time 0. In return, the insured receives a minimum return guarantee of 0% should he die in any of the years 1 through 3. If the insured survives to the end of the contract period the minimum return guarantee is not effective and the insured gets the market value of his investment portfolio even if his cumulative return is below the assumed guaranteed minimum rate of return of 0% in the event that the insured died during the contract period. The *periodic premium* variable annuity consists of a series of periodic investment contributions of $25,000 each of which is made at the beginning of years 1 through 3. The insured then receives a minimum return guarantee of 0% should he die in any of the years 1 through 3. If the insured survives to the end of the contract period the minimum return guarantee is not effective and the insured gets the value of his investment portfolio even if the cumulative return is below the assumed guaranteed minimum rate of return of 0% in the event that the insured died during the contract period[9]. In practice, the insured pays for the cost of the minimum death benefit guarantee through an annual charge as a percentage of the market value of his portfolio. This charge is referred to as the *mortality expense*. For illustrative purposes, we shall assume that the insured pays for the cost of the minimum death benefit guarantee through level premium payments made at the time of his regular investment contributions. Additional details are provided below.

Example 1 - Single Premium Case The annuitant will make a single premium deposit of \$75,000 at time 0. In addition to this deposit, the annuitant pays at time 0 a premium for the cost of the minimum death benefit guarantee. The amount of this premium is something which the actuary will need to compute. The payments made to the annuitant at times 1, 2, and 3 are as indicated in the following table.

Table 3.1 - Payments to Policyholder for Single Premium Variable Annuity

Time	Payments to Policyholder
1	$75{,}000\, q_x \operatorname{Max}\left(1, \frac{S_1(\omega, 1)}{S_1(0)}\right) = 75{,}000\, q_x \left[\frac{S_1(\omega, 1)}{S_1(0)} + \left(1 - \frac{S_1(\omega, 1)}{S_1(0)}\right)_+\right]$
2	$75{,}000\, {}_{1\vert}q_x \operatorname{Max}\left(1, \frac{S_1(\omega, 2)}{S_1(0)}\right) = 75{,}000\, {}_{1\vert}q_x \left[\frac{S_1(\omega, 2)}{S_1(0)} + \left(1 - \frac{S_1(\omega, 2)}{S_1(0)}\right)_+\right]$
3	$75{,}000\left[{}_{2\vert}q_x \operatorname{Max}\left(1, \frac{S_1(\omega, 3)}{S_1(0)}\right) + {}_3p_x \frac{S_1(\omega, 3)}{S_1(0)}\right]$ $= 75{,}000\left[{}_{2\vert}q_x \left[\frac{S_1(\omega, 3)}{S_1(0)} + \left(1 - \frac{S_1(\omega, 3)}{S_1(0)}\right)_+\right] + {}_3p_x \frac{S_1(\omega, 3)}{S_1(0)}\right]$

The annuity and the minimum death benefit guarantee are funded by the deposit of \$75,000 and the premium for the guaranteed minimum death benefits. In practice, the financing of the minimum death benefit guarantee and the associated reserving of the minimum death benefit guarantee are tracked separately from the financing of the investment return payments and the associated reserving of the investment return payments. We refer to these two reserves as the *guarantee reserve* and the *investment reserve* respectively. The sum of the guarantee and the investment reserve is the reserve for the whole policy. This decomposition is illustrated in Table 3.2. Clearly, there are two ways to make this decomposition, "investment value + put option" or "minimum payment + call option". However, the decomposition is made in terms of the investment value and the put option component because in practice the annuity company often parcels out the insurance on the value of the guarantee and manages the investment value as a standard brokerage arrangement without bearing any investment risk. It is then up to the insurance company accepting the insured values of the minimum death benefit guarantee to price and hedge this risk. We also compute the trading strategies which the insurance company must follow for each of the guarantee reserve and the investment reserve[10]. These are referred to as *guarantee trading strategy* and *investment benefit trading strategy* respectively.

As one would intuitively expect, the initial investment of $75,000 is exactly sufficient to finance the investment cash flows without the minimum death benefit guarantee. This is true for an arbitrary single premium variable annuity as the following argument shows. Suppose we have a T-period single premium variable annuity with initial investment of Π. Applying equation (2.3) to the non-dividend paying asset S_1, yields $E^Q[(1+r)^{-n} S_1(\omega, n)] = S_1(0)$, for $n = 1, 2, \ldots, T$. Therefore, the price of the investment benefits without the minimum death benefit guarantee are $\Pi\,[\sum_{k=1}^{T} {}_{k-1|}q_x + {}_Tp_x] = \Pi$. Consequently, the single premium exactly finances the investment benefits without the minimum death benefit guarantee. Therefore, the primary emphasis of risk management is on the reserve associated with the minimum death benefit guarantee.

Table 3.2 - Decomposition of Payments to Policyholder for Single Premium Variable Annuity

Time	Minimum Death Benefit Guarantee Payment	Investment Return Portion of Payment to Policyholder
1	$75{,}000\, q_x \left(1 - \frac{S_1(\omega, 1)}{S_1(0)}\right)_+$	$75{,}000\, q_x \frac{S_1(\omega, 1)}{S_1(0)}$
2	$75{,}000\, {}_{1\|}q_x \left(1 - \frac{S_1(\omega, 2)}{S_1(0)}\right)_+$	$75{,}000\, {}_{1\|}q_x \frac{S_1(\omega, 2)}{S_1(0)}$
3	$75{,}000\, {}_{2\|}q_x \left(1 - \frac{S_1(\omega, 3)}{S_1(0)}\right)_+$	$75{,}000 \left[{}_{2\|}q_x \frac{S_1(\omega, 3)}{S_1(0)} + {}_3p_x \frac{S_1(\omega, 3)}{S_1(0)}\right]$

For convenience, we report the numerical values for the reserves and trading strategies using the common device of tree diagrams. The state of the financial market is solely determined by the price history of the stock index fund and this is indicated at each node in the tree diagram by the number of up and down movements long each path through the tree.

The premium for the cost of the minimum death benefit guarantee for this example is $96.87. Therefore, the total premium deposit the insured makes at time 0 is equal to $75,096.87.

Figure 2 - Guarantee Reserve

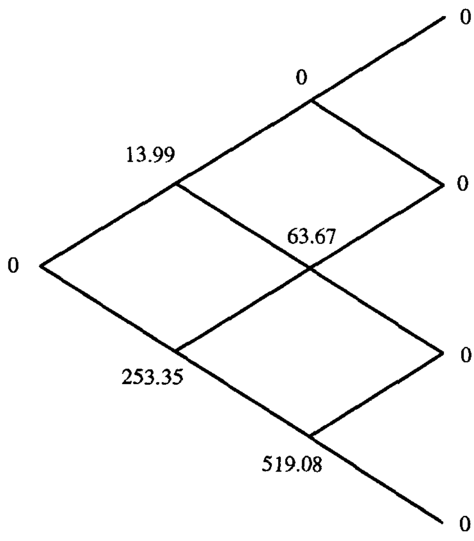


Figure 3 - Investment Reserve

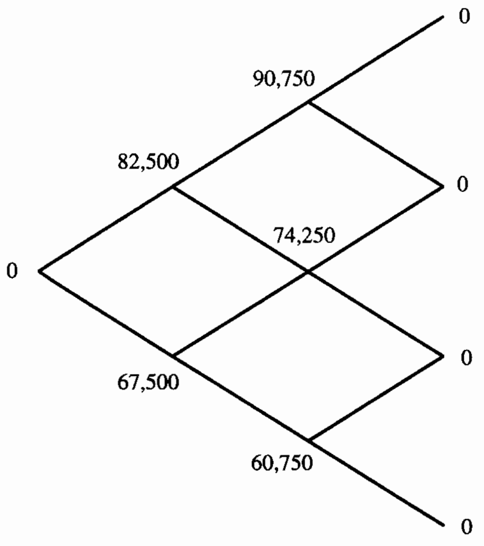


Figure 4 - Guarantee Trading Strategy

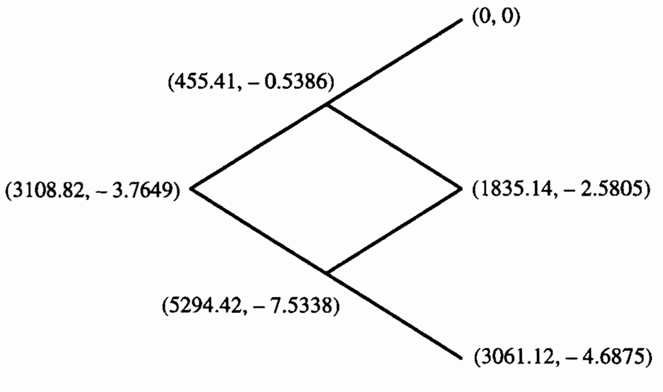
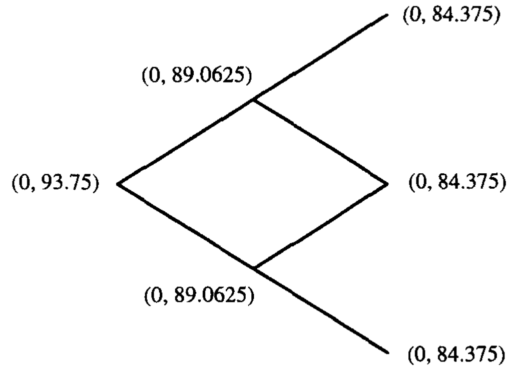


Figure 5 - Investment Benefit Trading Strategy



Example 2 - Periodic Premium Case We assume that there are premium deposits[11] of \$25,000 made at each of times 0, 1, and 2. These deposits are used to fund the annuity but additional level premiums must be paid to finance the minimum death benefit guarantee. As was the case for the single premium variable annuity, from the point of view of risk management the reserve corresponding to the guaranteed minimum death benefit is of the greatest significance in reserving for the periodic premium variable annuity.

As one would intuitively expect, the three periodic premium deposits of \$25,000 are exactly sufficient to finance the benefit payments without the minimum death benefit guarantee. This is true for an arbitrary periodic premium variable annuity as the following argument shows. Suppose we have a T-period periodic premium variable annuity with periodic premium deposit of π. Applying equation (2.3) to the non-dividend paying asset S_1 yields,

$$E^Q\left[(1+r)^{-n}\left(\frac{S_1(\omega,n)}{S_1(0)}+\frac{S_1(\omega,n)}{S_1(\omega,1)}+\cdots+\frac{S_1(\omega,n)}{S_1(\omega,n-1)}\right)\right]=\sum_{j=0}^{n-1}(1+r)^{-j}=\ddot{a}_{\overline{n}|}\ ,$$

for $n = 1, 2, \ldots, T$. Therefore, the price of the benefit payments without the minimum death benefit guarantee are $\pi\,[\sum_{k=1}^{T} {}_{k-1|}q_x\, \ddot{a}_{\overline{k}|} + {}_{T}p_x\, \ddot{a}_{\overline{T}|}]$. The price of the premium deposits are $\pi \sum_{k=0}^{T-1} {}_{k}p_x\,(1+r)^{-k}$. Using summation by parts[12] yields the relation,

$$\sum_{k=0}^{T-1} {}_kp_x (1+r)^{-k} = \sum_{k=0}^{T-1} {}_kp_x \, \Delta\ddot{a}_{\overline{k}|}$$

$$= {}_kp_x \, \ddot{a}_{\overline{k}|}\Big|_0^T - \sum_{k=0}^{T-1} \ddot{a}_{\overline{k+1}|} \, \Delta_k p_x = {}_Tp_x \, \ddot{a}_{\overline{T}|} + \sum_{k=1}^{T} \ddot{a}_{\overline{k}|} \, {}_{k-1|}q_x$$

Consequently, the periodic premiums exactly finance the benefit payments without the minimum death benefit guarantee. Therefore, the primary emphasis of risk management is on the reserve associated with the minimum death benefit guarantee.

Table 3.3 - Payments to Policyholder for Periodic Premium Variable Annuity

Time	Payments to Policyholder
1	$25{,}000\, q_x \,\mathrm{Max}\left(1, \frac{S_1(\omega,1)}{S_1(0)}\right) = 25{,}000\, q_x \left[\frac{S_1(\omega,1)}{S_1(0)} + \left(1 - \frac{S_1(\omega,1)}{S_1(0)}\right)_+\right]$
2	$25{,}000\, {}_{1\vert}q_x \,\mathrm{Max}\left(2, \frac{S_1(\omega,2)}{S_1(0)} + \frac{S_1(\omega,2)}{S_1(\omega,1)}\right)$ $= 25{,}000\, {}_{1\vert}q_x \left[\frac{S_1(\omega,2)}{S_1(0)} + \frac{S_1(\omega,2)}{S_1(\omega,1)} + \left(2 - \frac{S_1(\omega,2)}{S_1(0)} - \frac{S_1(\omega,2)}{S_1(\omega,1)}\right)_+\right]$
3	$25{,}000\, {}_{2\vert}q_x \,\mathrm{Max}\left(3, \frac{S_1(\omega,3)}{S_1(0)} + \frac{S_1(\omega,3)}{S_1(\omega,1)} + \frac{S_1(\omega,3)}{S_1(\omega,2)}\right)$ $+ 25{,}000\, {}_3p_x \left(\frac{S_1(\omega,3)}{S_1(0)} + \frac{S_1(\omega,3)}{S_1(\omega,1)} + \frac{S_1(\omega,3)}{S_1(\omega,2)}\right)$ $= 25{,}000\, {}_{2\vert}q_x \left[\frac{S_1(\omega,3)}{S_1(0)} + \frac{S_1(\omega,3)}{S_1(\omega,1)} + \frac{S_1(\omega,3)}{S_1(\omega,2)}\right]$ $+ 25{,}000\, {}_{2\vert}q_x \left(3 - \frac{S_1(\omega,3)}{S_1(0)} - \frac{S_1(\omega,3)}{S_1(\omega,1)} - \frac{S_1(\omega,3)}{S_1(\omega,2)}\right)_+$ $+ 25{,}000\, {}_3p_x \left(\frac{S_1(\omega,3)}{S_1(0)} + \frac{S_1(\omega,3)}{S_1(\omega,1)} + \frac{S_1(\omega,3)}{S_1(\omega,2)}\right)$

The level premium for the cost of the minimum death benefit guarantee for this example is \$21.18. Therefore, the total premium deposit the insured makes at each of times 0, 1, and 2 is equal to \$25,021.18.

Figure 6 - Guarantee Reserve

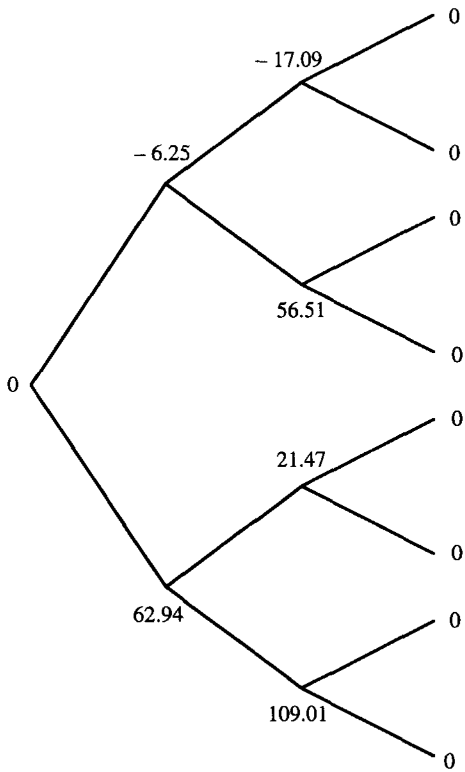


Figure 7 - Investment Reserve

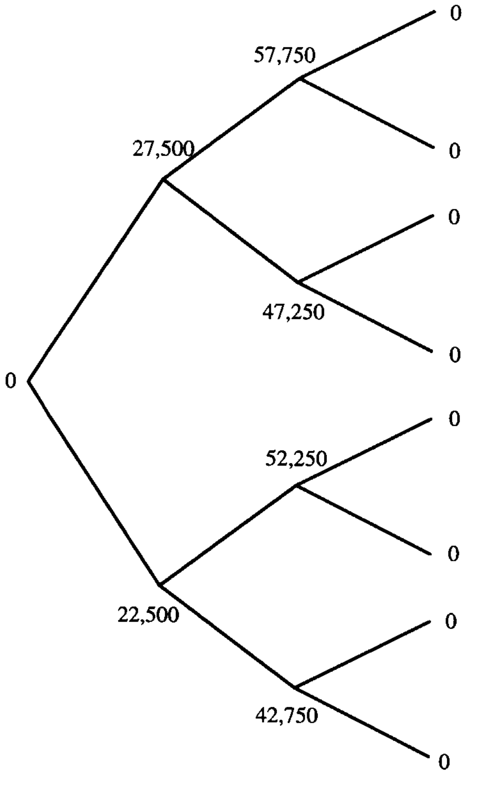


Figure 8 - Guarantee Trading Strategy

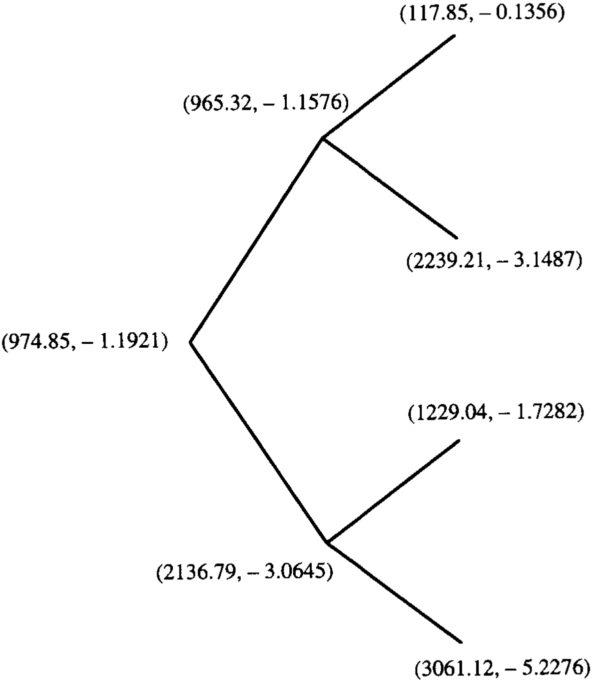
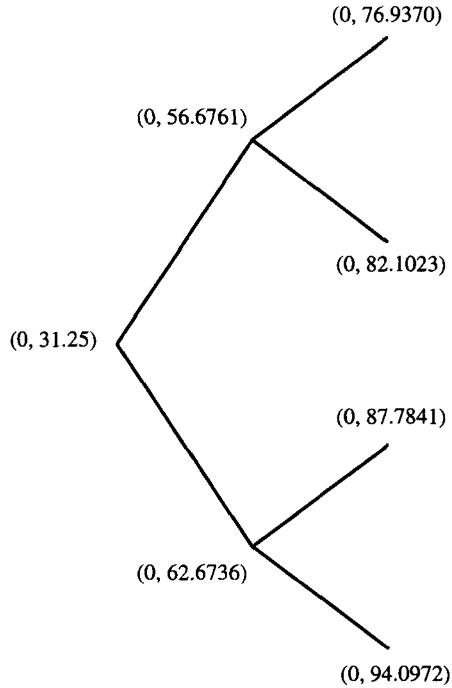


Figure 9 - Investment Benefit Trading Strategy



The most important information in the reserving calculations for both examples is the behaviour of the reserve corresponding to the minimum death benefit guarantee component of both types of annuity. Notice the wide relative variation in reserve levels as one moves through each tree diagram for the guarantee reserves. Although these amounts are small relative to the overall policy reserve, they can be significant if enough policies are written. Furthermore, the guarantee reserve levels can be large relative to the premium the insurance company charges to cover the cost of the minimum death benefit guarantee. This serves to emphasise the importance of following appropriate investment strategies. An important practical consideration is an insurance company may effectively act as a reinsurer by bearing only the liabilities from the minimum investment return guarantee risk. In this case the insurer's reserve levels for this line of business will fluctuate wildly and are directly proportional to the amount of reinsurance accepted. Of course, there is no risk to the insurance company if it follows an appropriate hedging strategy. Lastly, we remind the reader that it is the cost of this hedging strategy which the reserve level reflects.

4. Model Formulation with Random Mortality

It is possible to formulate the above model with mortality as a random component. One may still consider the model to be complete because of the near deterministic aspect to large pools of insureds but we must now index the states of the world by financial market outcomes and mortality outcomes. As the financial market outcomes and the mortality outcomes are unrelated, the space of outcomes for the model may be taken as the Cartesian product $\Omega = \Omega^{(1)} \times \Omega^{(2)}$ where $\Omega^{(1)}$ represents the space of financial market outcomes and $\Omega^{(2)}$ represents the space of mortality outcomes.
A valuation measure for this model will be of the form $Q = Q^{(1)} \times P^{(2)}$, where $Q^{(1)}$ is a risk-neutral valuation measure for the embedded financial market model and P is the measure governing mortality exposure.

Uncertain payments in this model will depend upon both the state of the financial market and the mortality outcome and a typical uncertain payment will be indexed by both outcome variables. For example, the uncertain payments for a periodic premium variable annuity with level periodic payments of π and a 0% minimum guaranteed return might be expressed as

$$c(\omega, k) = \pi \operatorname{Max}\left(\frac{S_1(\omega^{(1)}, k)}{S_1(\omega^{(1)}, 0)} + \cdots + \frac{S_1(\omega^{(1)}, k)}{S_1(\omega^{(1)}, k-1)}, k \right) 1_{\{K(\omega^{(2)}) = k-1\}} . \tag{4.1}$$

The addition of random mortality does not add much to the model and has no practical significance. The deterministic model of section 2 may be obtained from this more general framework by conditioning on the mortality outcome.

5. Concluding Remarks

We have presented a method for pricing and reserving for insurance products when the financial market is complete. The method ignores insurance company expenses and transaction costs but is a theoretically correct procedure for reserving under the idealised conditions of a frictionless market. In the United States, there are legislated actuarial standards for reserving for insurance liabilities that bear in little relation to the financial economics which underlies the funding of most insurance company liabilities. Some of these rules for reserving, including the case of variable annuities, are described in Tullis and Polkinghorn (1996). A discussion of reserving for the guaranteed minimum death benefits in variable annuities may be found in the *Record of the Society of Actuaries*, volume 21 number 2, pages 65 through 81. It is hoped that this paper will assist the actuary in both recognising the importance of financial theory in reserving problems and in applying it to practical problems to obtain the same type of yardstick reserve values that one obtains from the classical actuarial theory as presented in Gerber (1995).

It is understandable that there is some confusion in the actuarial profession as to how reserving should be approached when insurance liabilities are uncertain. The discussion in Corby (1977) illustrates some differing points of view on the issue of reserving for variable annuity products. We have shown that reserve values may be assigned that are unambiguous and have a definite financial rationale. The fundamental concept is that the reserve represents the amount of money the insurance company must have properly invested in order to fund the remaining liabilities. The utility of this notion of reserve depends on the company following appropriate investment strategies and this point is absolutely fundamental. Despite the idealised assumptions underlying the model we have developed, the reserve values we have described represent the same type of yardstick assessment of reserve requirements that one obtains from the traditional actuarial model.

In practice, the pricing and reserving for a periodic premium variable annuity, such as was illustrated in section 4, will present computational difficulties since the benefit payments are path dependent. One approach that may be taken is for the actuary to utilise Monte Carlo simulation to obtain estimates of the reserve levels. In practical cases, the complexity of the pricing and reserving problems faced by an actuary for general insurance policies will require some knowledge of numerical techniques such as Monte Carlo simulation.

The model that we developed in section 2 involves mortality. Although we presented the theory from the perspective of a life insurance policy, the theory is in no way restricted to this environment. Suitable modifications would permit similar formulas to be applied to property/casualty problems. The only restriction would be that the financial model is complete. This rules out the pricing and reserving for products such as catastrophe risk bonds in the framework of our model. Further theory must be drawn on to handle such products.

References

Corby, F. (1977). Reserves for Maturity Guarantees Under Unit-Linked Policies. *Journal Institute of Actuaries* 104, 259-273.

Cox, J., Ross, S. and M. Rubinstein (1979). Option Pricing: A Simplified Approach. *Journal of Financial Economics* 7, 229-263.

Gerber, H. (1995). *Life Insurance Mathematics, second edition.* Springer-Verlag, New York.

Tullis, M. and P. Polkinghorn (1996). Valuation of Life Insurance Liabilities. ACTEX Publications, Winsted, Connecticut.

Appendix - Binomial Option Pricing for Variable Annuity Example

As we described in section 3, there are two assets traded in this model, the money market account which is denoted by S_0 and the stock fund which is denoted by S_1. The constant interest rate is denoted by r. Consider a generic uncertain cash flow stream denoted by c. There exists a trading strategy, denoted $(\vartheta_0, \vartheta_1)$, which describes the investment strategy that finances this cash flow stream. ϑ_0 denotes the number of units of money market account held and ϑ_1 denotes the number of units of the equity fund that are held. We may solve for the trading strategy at each node in the associated information tree. The basic equation is as follows.

$$\begin{bmatrix} (1+r)^{k+1} & uS(\omega, k) \\ (1+r)^{k+1} & dS(\omega, k) \end{bmatrix} \begin{bmatrix} \vartheta_0(\omega, k) \\ \vartheta_1(\omega, k) \end{bmatrix} = \begin{bmatrix} c((\omega_1, \ldots, \omega_k, 1, \ldots), k+1) \\ c((\omega_1, \ldots, \omega_k, 0, \ldots), k+1) \end{bmatrix} \qquad \text{(A-1)}$$

It is customary to write this equation as

$$\begin{bmatrix} (1+r)^{k+1} & uS \\ (1+r)^{k+1} & dS \end{bmatrix} \begin{bmatrix} \vartheta_0 \\ \vartheta_1 \end{bmatrix} = \begin{bmatrix} c^u \\ c^d \end{bmatrix}, \qquad \text{(A-2)}$$

where c^u is the contingent payment made if the economy evolves to the upstate and c^d is the contingent payment made if the economy evolves to the downstate and S is the current price/level of the equity fund. The formula for the inverse of a 2×2 matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{A-3}$$

We may apply (A-3) to solve the system (A-2) for the investment holdings. This gives the expression for the local investment holdings

$$\begin{bmatrix} \phi \\ \vartheta \end{bmatrix} = \frac{1}{[d - u]\, S\, (1 + r)^{k+1}} \begin{bmatrix} dS & -uS \\ -(1 + r)^{k+1} & (1 + r)^{k+1} \end{bmatrix} \begin{bmatrix} c^u \\ c^d \end{bmatrix}. \tag{A-4}$$

Simplifying this expression yields the local formula

$$\begin{bmatrix} \phi \\ \vartheta \end{bmatrix} = \begin{bmatrix} \dfrac{uc^d - dc^u}{[u - d]\,(1 + r)^{k+1}} \\ \\ \dfrac{c^u - c^d}{[u - d]\, S} \end{bmatrix}. \tag{A-5}$$

If we wish to determine the local price of the cash flows we need only weight the trading strategy with the value of the assets as $\phi\,(1 + r)^k + \vartheta\, S$. Simplifying the algebra yields the formula

$$\text{price} = \frac{[(1 + r) - d]\, c^u + [u - (1 + r)]\, c^d}{(1 + r)\,[u - d]}. \tag{A-6}$$

Now let $q := \dfrac{(1 + r) - d}{u - d}$. Then (A-6) may be expressed as

$$\text{price} = \frac{q\, c^u + (1 - q)\, c^d}{(1 + r)}. \tag{A-7}$$

This formula is the one-step or local risk neutral expectation commonly employed in option pricing calculations.

The local risk-neutral probability q may be used to endow Ω with a probability measure by $Q(\omega) := q^{\omega_1 + \cdots + \omega_T}[1 - q]^{T - [\omega_1 + \cdots + \omega_T]}$. This is important for carrying out Monte Carlo simulation for path dependent cash flow streams.

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[1] The assumption that the time horizon is finite may be relaxed.

[2] Depending on the nature of the alternative policy arrangements, it may be necessary to alter the definition of the reserve formula. All such modifications are a matter of convention rather than theory.

[3] If $e(\omega, k)$ is negative then the insured receives the payment from the insurance company.

[4]The condition of completeness is really one of spanning in the sense of linear algebra. This is illustrated in equation (A-2) of the model discussed in the appendix.

[5]If a term structure model is used it is more typical to assume that Q, or the local probabilities under Q, is given and the primitive assets for the model, the bonds, are derived in terms of this assumed data.

[6]The reference to uncertainty is with respect to uncertainty in the underlying financial variables and not mortality since the mortality risk has been assumed away by adopting mortality factors in this model.

[7]The reader may rework our analysis in the case where the premiums are not net to obtain a "retrospective formula" but this formula would not agree with the prospective reserve formula.

[8]This is the minimum death benefit guarantee offered by Fidelity Investments. The insuring of the minimum death benefit guarantee is not born by Fidelity though, it is parceled out to a life insurance company.

[9]In some cases, the pure endowment component of the contract is guaranteed as well. This is not common in the United States although there was a brief period in which an investor could purchase "mutual fund insurance". As one would expect, the cost of the guarantee is dramatically increased if the pure endowment component is also guaranteed. Since many investors only purchase variable annuities for the tax shelter, they want the mortality expense to be as minimal as possible.

[10]The reader may use these trading strategies to check that the retrospective reserve formula gives the same values as the prospective reserve formula.

[11]These might also be referred to as investment contributions.

[12]The formula for summation by parts is based on the simple identity $\Delta(a_k b_k) = a_{k+1}\Delta b_k + b_k \Delta a_k$.