EQUITY-LINKED LIFE INSURANCE IN GERMANY: QUANTIFYING THE RISK OF ADDITIONAL POLICY RESERVES

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Abstract

Equity-linked policies under asset value guarantee are discussed quite vividly in Germany. According to recent statements of the German supervisory authority, policy reserves have to be calculated as the maximum of the market value of the policy and the discounted guaranteed sum. This might require additional policy reserves.

This paper discusses two products linked to the DAX30 (German equity index) and providing a guaranteed payment of interest. Because of tax legislation we consider policies with a term of 12 years and 5 annual premiums. Pricing formulas are developed from which the fair rate of index participation can be derived. The risk emerging from additional policy reserves is then investigated by quantifying the corresponding distribution using Monte Carlo simulation methods. The influence of changes in the term structure of interest rate and index volatility is examined and upper analytic bounds for the reserves are derived if meaningful.

Keywords

Equity-linked life insurance, asset value guarantee, rate of index participation, pricing formulas, additional policy reserves, lower partial moments, Monte Carlo simulation.

1 Introduction

The first equity-linked life insurance policy in Germany was sold only recently by Standard Life, a Scottish insurance company. Up to now, there is only one German provider for such equity-linked products, but many will offer similar products soon. All of the products being discussed at the moment have a common feature: The insurance company guarantees to pay back a prefixed (deterministic) amount at maturity. In addition, the policy is linked — via the so called rate of index participation — to a German equity index (here the DAX30, a performance index). Hence, if the index performs well over the term of the policy, a considerably higher amount may be payed back. Of course, there is a great variety of what the payoff at maturity may look like.

Because of tax legislation, policies with a minimum term of 12 years and (at least) 5 annual premiums are preferable. This considerably complicates the whole analysis of the product in contrast to single premium products (which, e.g., dominate the British market).

According to a statement of a member of the German supervisory authority in December 1996, policy reserves for equity-linked products with an asset value guarantee have to be calculated as the maximum of the market value of the policy and the discounted guaranteed sum. For the latter, the discount rate is constant and at most 4% p.a. Hence, whenever the market value drops below the discounted sum, so called additional policy reserves (*APR*) are required. Since these *APR* are stochastic and cause (potentially substantial) additional costs for the insurance company, it is important to quantify the distribution of the *APR* accurately before the policy is sold. Of course these costs are taken from the annual premiums and therefore lower the rate of index participation. Since the *APR* only apply to companies subject to the German supervisory authority, they obviously discriminate those providers compared to others in the European Community.

In [No/Sc 96] a detailed discussion of equity-linked life-insurance products for the German market is given. In particular, for the first time, the expected additional policy reserves as well as a certain risk measure are calculated for a specified product. In this paper we discuss — within a Black/Scholes environment — two further interesting products, a collar-type product and a geometric averaging product. We deduce explicit pricing formulas and show how to evaluate the fair rate of index participation. Then, using Monte Carlo techniques, we quantify the distribution of the additional policy reserves under different market scenarios.

Our paper is organized as follows: In Section 2.1 we explain the product design. In particular, we only deal with the financial aspects of the products. We do not discuss how to calculate the risk premium associated with the — in general random — sum payable at death. This topic will be addressed in a different paper. In Section 2.2 we derive within our model framework explicit pricing formulas for both products under consideration and in Section 2.3 we show how to determine the fair rate of index participation. Here we also give empirical results. In Section 3.1, we properly define the term additional policy reserves and in Section 3.2 we quantify the corresponding distribution. In particular, we calculate the 95%– and 99%–quantiles. The paper closes with a summary and an outlook for further research in Section 4.

2 Product Design and Pricing Formulas

2.1 Product Design

We consider equity-linked life insurance products with a term of 12 years and 5 annual premiums payable at the beginning of the first 5 years ($t = 0, \dots, 4$). All costs (including risk premiums and management fees) are equally distributed over the first four years and subtracted from the five gross premiums. This results in five equal net premiums (NP). These five net premiums are swapped into a single premium which is used to buy a security that pays off A_{12} (as defined below) at expiration $T = 12$ if the insured person is still alive.

We consider two different products with benefit A_T payable at expiration $T = 12$:

$$A_T^1 = NP \sum_{i=1}^{5} \prod_{j=i}^{12} \left(1 + \max\left[i_l, \frac{S_j - S_{j-1}}{S_{j-1}} x_1\right] - \max\left[\frac{S_j - S_{j-1}}{S_{j-1}} x_1 - i_h, 0\right]\right)$$

$$A_T^2 = NP \sum_{i=0}^{4} \max\left[\frac{\sqrt[T-i]{\prod_{j=i+1}^{T} S_j} - S_i}{S_i} x_2, 0\right] + NP \sum_{i=0}^{4} (1 + i_l)^{12-i}.$$

Here S_j denotes the value of the DAX30 (German equity index) at the end of year j after the policy was sold, $x_{1,2}$ are the so-called rates of index participation.

Product one — a collar-type product — pays compound interest on the net premiums. The interest in year j is calculated as x_1 times the DAX30-return in year j but not less than $i_l \geq 0$ and no more than $i_h (> i_l)$.[1]

Product two adds x_2 times the geometric average-return of the DAX30 to a guaranteed sum. Note that this (minimum) sum guaranteed at $T = 12$ is the same for both products, namely

$$G = G(T) = NP \sum_{i=0}^{4} (1 + i_l)^{12-i}$$

[1] Principally, we could also consider $i_l < 0$ in both products, cf., e.g., figures 1 and 2.

Let A_t denote the value of the security at time t and let SW_t denote the value of the above defined swap-contract at time t. Obviously SW_t has always a negative value to the insurance company and hence $A_t + SW_t < 0$ might occur (depending on the index volatility and the term structure of interest rate). Since the value of the policy has to be non negative, we define this value as $V_t = \max[A_t + SW_t, 0]$. This is achieved by buying suitable options. These additional costs of course reduce the rate of index participation[2].

To keep our notations as simple as possible, we assume throughout the paper that the policy is sold immediately after the date of the last balance sheet. In Germany, insurance companies have to provide such statements on a yearly basis, and we therefore only consider integer values for t in our further analysis.

2.2 Pricing Formulas

We will now state pricing formulas for the A_t^k, $t = 0, \ldots, 11$, $k = 1, 2$ under the following assumptions:

- The DAX30 follows a geometric Brownian motion:

$$\frac{dS_t}{S_t} = \mu(t)dt + \sigma dW_t, \tag{1}$$

  where W_t denotes a Wiener process on a probability space. Note, that $\mu(t)$ is time dependent whereas σ is constant (> 0).
  The solution to the stochastic differential equation (1), given an initial value $S_0 > 0$, is, cf. [Ka/Sh 88],

$$S_t = S_0 e^{\int_0^t \mu(s) - \frac{\sigma^2}{2} ds + \sigma W_t}. \tag{2}$$

  In particular, it follows that $\log \frac{S_{t_2}}{S_{t_1}} \sim N\left(\int_{t_1}^{t_2} \mu(s) - \frac{\sigma^2}{2} ds, \sigma^2(t_2 - t_1)\right)$ for $0 \leq t_1 < t_2$, where $N(m, v)$ denotes the normal distribution with mean m and variance v.

- The short rate process $r(t)$ is assumed to be deterministic and to fit the current, riskless term structure of interest rate, i.e.

$$\int_{t_1}^{t_2} r(t)dt = (t_2 - t_1) f_{t_1, t_2},$$

  where f_{t_1,t_2} denotes the continuous, annualized observed forward rate for the period of time $0 \leq t_1 < t_2$.

[2] It will turn out that within our model (deterministic yield curve) these options currently (using the data below) have no value and are therefore not considered in our further analysis.

According to [Ha/Pl 81] the value of A_t^k is given by

$$A_t^k = E_Q\left[e^{-\int_t^T r(s)ds} A_T^k \;|_t\right], \quad 0 \le t \le T, \tag{3}$$

where $E_Q[.\,|_t]$ denotes the conditional expected value under the information available at time t according to a so-called equivalent martingale measure Q. In (1), $\mu(t)$ is substituted by $r(t)$ as a consequence of this transformation of measure, and hence (1) describes the evolution of the DAX30 in a risk-neutral world[3].

Let

$$c_t(\alpha,\beta,\gamma) = \alpha N(d_1) - \beta e^{-\int_t^{t+\gamma} r(s)ds} N(d_2)$$

denote the value (according to the well-known Black-Scholes formula) at time t of a European call option on the DAX30, maturing at time $t+\gamma$, with an index value $S_t = \alpha$ and strike price β. Here

$$d_1 = \frac{\log\frac{\alpha}{\beta} + \int_t^{t+\gamma} r(s) + \frac{\sigma^2}{2} ds}{\sigma\sqrt{\gamma}}, \quad d_2 = d_1 - \sigma\sqrt{\gamma}, \text{ and } N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{s^2}{2}} ds.$$

Let furthermore

$$R_j = \frac{S_j - S_{j-1}}{S_{j-1}} x,$$

$$v_j = 1 + \max[i_l, R_j] - \max[R_j - i_h, 0]$$

and

$$w_j = 1 + i_l + (x+i_l)e^{f_{j-1,j}} c_{j-1}\left(\frac{x}{x+i_l},1,1\right) - (x+i_h)e^{f_{j-1,j}} c_{j-1}\left(\frac{x}{x+i_h},1,1\right).$$

Then $(t = 0,\ldots,11)$

$$A_t^1 = \begin{cases} NPe^{-(T-t)f_{t,T}}\left(\sum_{i=1}^{t}\prod_{j=i}^{t} v_j \prod_{j=t+1}^{12} w_j + \sum_{i=t+1}^{5}\prod_{j=i}^{12} w_j\right) & \text{for } t \le 4 \\ NPe^{-(T-t)f_{t,T}}\sum_{i=1}^{5}\prod_{j=i}^{t} v_j \prod_{j=t+1}^{12} w_j & \text{for } t \ge 5. \end{cases}$$

[3]The (unique) existence of such a measure Q is essentially equivalent to the assumption of a complete, arbitrage-free market.

In particular,

$$A_0^1 = A_0^1(i_l, i_h, x) = NPe^{-Tf_{0,T}} \sum_{i=1}^{5} \prod_{j=i}^{12} w_j.$$

A proof follows from Appendix 1 in [No 96].

For product 2 we get $(t = 0, \ldots, 11)$

$$A_t^2 = \sum_{i=0}^{4} A_{t,i} + G_t,$$

where $G_t = e^{-(T-t)f_{t,T}} G$ and $A_{t,i}$ is given by

$$A_{t,i} = NP\left\{xe^{\eta-(i-t)f_{t,i}} N(d_1) - xe^{-(T-t)f_{t,T}} N(d_2)\right\},$$

for $t \leq i$ with

$$\eta = \frac{s^2}{2} - (T-i)\frac{\sigma^2}{2} - \sum_{j=i+1}^{T} \frac{j-i-1}{T-i}\left(f_{j-1,j} - \frac{\sigma^2}{2}\right),$$

$$s^2 = \sigma^2 \sum_{j=i+1}^{T} \frac{(j-i)^2}{(T-i)^2},$$

$$d_1 = \frac{\eta + \frac{s^2}{2} + (T-i)f_{i,T}}{s}, \qquad d_2 = d_1 - s.$$

For $t > i$:

$$A_{t,i} = e^{-(T-t)f_{t,T}} NP\left\{\frac{x}{S_i} e^{m+\frac{s^2}{2}} N(d_1) - xN(d_2)\right\},$$

with

$$m = \log \sqrt[T-i]{\prod_{j=i+1}^{t} S_j} + \frac{T-t}{T-i} \log S_t + \sum_{j=t+1}^{T} \frac{T-j+1}{T-i}\left(f_{j-1,j} - \frac{\sigma^2}{2}\right),$$

$$s^2 = \sigma^2 \sum_{j=t+1}^{T} \frac{(j-t)^2}{(T-i)^2},$$

$$d_1 = \frac{-\log S_i + m + s^2}{s}, \qquad d_2 = d_1 - s.$$

In particular,

$$A_0^2 = A_0^2(i_l, x) = \sum_{i=0}^{4} NP \left\{ x e^{\eta - i f_{0,i}} N(d_1) - x e^{-T f_{0,T}} N(d_2) \right\} + G_0.$$

A proof for these formulas is given in Appendix A.

2.3 The rate of index participation

Given the yield curve, the index volatility and i_l, i_h, the fair rate of index participation can now be calculated as a solution $x > 0$ of the implicit equation

$$A_0^k = A_0^k(x_k) = \sum_{i=0}^{4} NP e^{-\int_0^i r(s)ds} = \text{present value of net premiums}, \qquad k = 1, 2,$$

if such a solution exists[4].

For all of our empirical results we work with the term structure

t	1	2	3	4	5	6	7	8	9	10	11	12
$f_{0,t}$	3.2	3.49	3.94	4.4	4.81	5.14	5.42	5.63	5.82	5.96	6.04	6.11

and an index volatility of $\sigma = 12.98\%$ as of January 22, 1997.

Varying i_l and i_h, we get the following rates of index participation. In particular, both rates of index participation are quite sensitive with respect to the guaranteed interest rate i_l.

Product 1:

i_l	0%			2%			4%		
i_h	12%	15%	20%	12%	15%	20%	12%	15%	20%
x_1	161.0%	96.2%	78.0%	102.8%	74.4%	66.9%	67.8%	57.9%	54.0%

[4]In case of product 1, there exists no solution if $i_l \geq i_1^*$ for some $i_1^* > 0$. If, however, $i_l < i_1^*$, there is an $i_h^*(i_l)$ such that a unique solution exists if and only if $i_h > i_h^*(i_l)$. Furthermore, $x_1 \to \infty$ for $i_h \searrow i_h^*(i_l)$. For product 2 it is true, that there exists a unique solution if and only if $i_l < i_2^*$ for some $i_2^* > 0$.

Product 2:

i_l	0%	2%	4%
x_2	230.9%	176.2%	110.8%

Figure 1 shows the rate of index participation for product 1 as a function of i_l and i_h, figure 2 shows the rate of index participation for product 2 as a function of i_l.

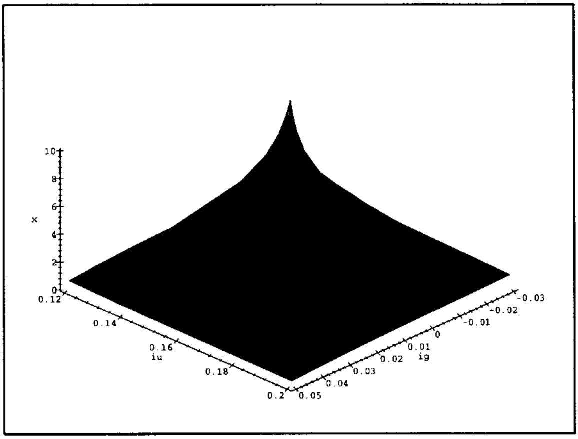

Figure 1: $x_1 = x_1(i_l, i_h)$

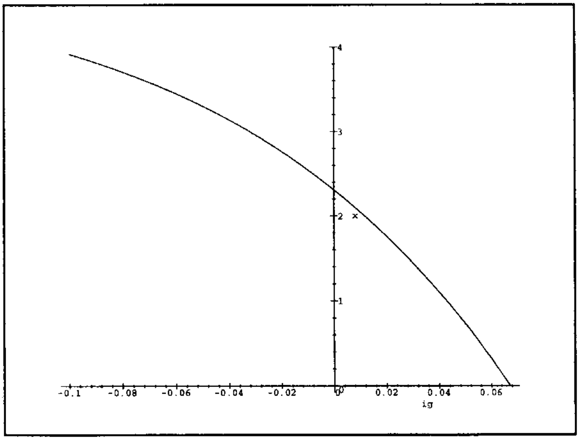

Figure 2: $x_2 = x_2(i_l)$

For $i_l = 2\%$ and $i_h = 12\%$ figures 3 and 4 show the sensitivity of the rate of index participation with respect to changes in the term structure of interest rate and the DAX30-volatility. Here Δr indicates a parallel shift of the given term structure of interest rate by Δr.

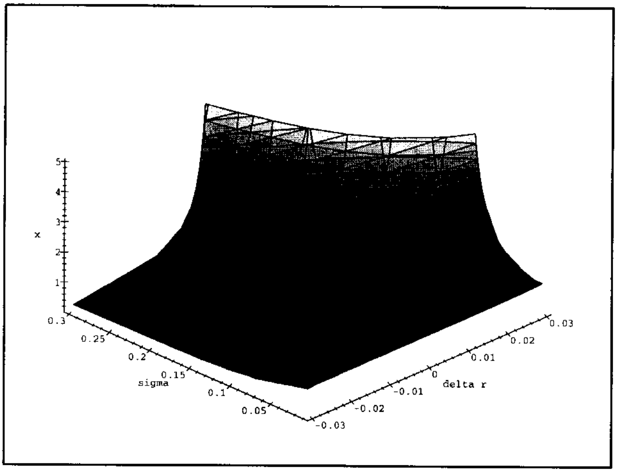

Figure 3: $x_1 = x_1(\sigma, \Delta r)$

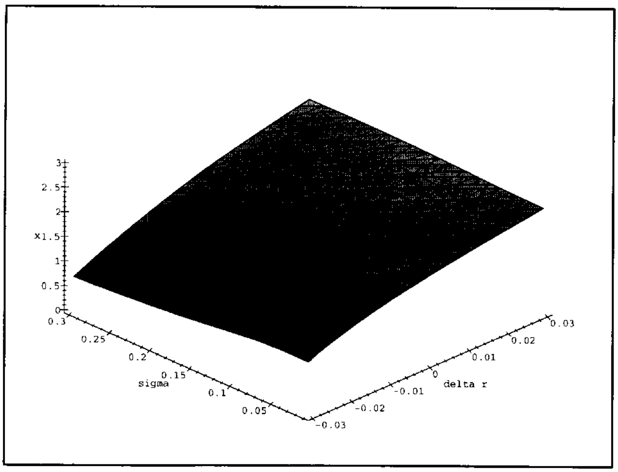

Figure 4: $x_2 = x_2(\sigma, \Delta r)$

3 Additional Policy Reserves

3.1 Definition of additional policy reserves

According to recent statements of a member of the German supervisory authority, policy reserves for equity-linked policies with an asset value guarantee have to be calculated as the maximum of the market value of the policy (V_t) and the discounted guaranteed sum (reduced by the discounted future premiums) [5]. There are still controversial discussions which discount rate R has to be used for discounting the guaranteed sum, since the German accounting rules are not definit on this point. Some experts argue, that $R = 4\%$ (the maximum discount rate for calculating policy reserves) is always allowed, others claim that R may not be higher than the guaranteed rate of interest (here i_l). We will discuss both alternatives and show, that the additional policy reserves are highly sensitive with respect to R.

First, we define the term additional policy reserves. Let G denote the payment guaranteed at $t = T = 12$ as above. Then, its present value at time $t = 1, \ldots, 11$ (with respect to the discount rate R) is given by

$$G(t) = G(1 + R)^{-(T-t)},$$

while the present value of the future payment of premiums is given by

$$P(t) = \sum_{i=t}^{4} NP(1 + R)^{-(i-t)} \quad (= 0 \text{ for } t > 4).$$

Hence, policy reserves at time $t = 1, \ldots, 11$ have to be calculated as

$$PR(t) = \max[G(t) - P(t), V_t]$$

and the term

$$APR(t) = PR(t) - V_t$$

is called additional policy reserves[6].

[5] We will not go into the legal issues here. A detailed overview is given in [He 97] and [No/Sc 96].

[6] Recall that $V_t = \max[A_t + SW_t, 0]$ and $SW_t = \sum_{i=t}^{4} NPe^{-\int_t^i r(s)ds}$.

3.2 Quantifying the distribution of additional policy reserves

For further empirical results we fix $i_l = 2\%$ for both products (which approximately equals the current rate of inflation in Germany), $i_h = 12\%$ in product 1 and $NP = 20000 DEM$ (Deutsche Mark). We use the interest rate and volatility data given above.

For the first product, a meaningful upper bound for $APR(t)$ can easily be derived, defining A_t^* as the value of A_t given that $\max\left[i_l, \frac{S_j - S_{j-1}}{S_{j-1}} x_1\right] = i_l \ \forall j \leq t$. Writing $V_t^* = \max[A_t^* + SW_t, 0]$ we set $PR^*(t) = \max[G(t) - P(t), V_t^*]$ and hence

$$APR^*(t) = PR^*(t) - V_t^* \geq APR(t), \quad t = 1, \ldots, 11.^7$$

Figure 5 shows $APR^*(t)$ for $i_l = 2\%$, $i_h = 12\%$, $R = 2\%$ and $\Delta r = 0\%, \ldots, 5\%$[8]. The higher Δr, the higher $APR^*(t)$. Figure 6 shows the same for $R = 4\%$. In this case $APR(t) = 0 \ \forall t$ for $\Delta r = 0\%,\ 1\%,\ 2\%$[9]. This shows that the risk of additional policy reserves depends highly on the discount rate R.

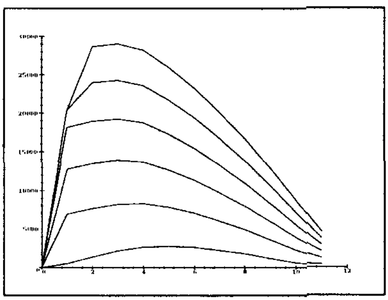

Figure 5: $APR^*(t),\ R = 2\%$

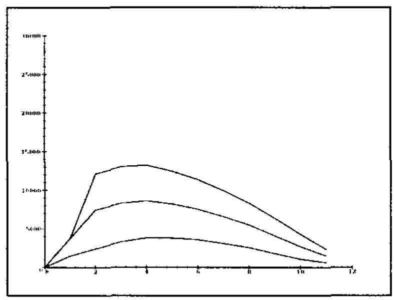

Figure 6: $APR^*(t),\ R = 4\%$

To quantify the risk of additional policy reserves more precisely, we calculate the so called lower partial moments, cf. [Ha 91], and define (if existent)

$$LPM_n = LPM_n(X, a) = \int_{-\infty}^{a^-} (a - x)^n dF_X(x)$$

for any real number a and any random variable X with distribution function F_X. With $X = V_t$ and $a = G(t) - P(t)$ we obviously obtain

$$LPM_0 = P(V_t < G(t) - P(t)) = P(APR(t) > 0)$$

[7] Note that according to our pricing formulas A_t^* and hence $APR^*(t)$ are deterministic.

[8] Here the shift Δr is assumed to occur after the product has been sold, i.e. the rate of participation has been calculated with $\Delta r = 0$ at $t = 0$.

[9] We note that for $\Delta r = -1\%,\ -2\%$ we have $APR^*(t) = 0 \ \forall t$ in figures 5 and 6.

$LPM_1 = E\left(APR(t)\right)$ = expected value of additional policy reserves at time t

$LPM_2 = E\left(APR(t)^2\right)$,

which is the so called semi target variance. It is a more adequate risk measure than the usual variance as it quantifies only the downside risk of missing the target. We furthermore define $q_\alpha(t)$ to be the $\alpha-$quantile of the distribution of $APR(t)$.

The following results are derived using our explicit pricing formulas as well as Monte Carlo techniques to simulate the DAX30 up to time t. For this we take, cf. (2), $\mu(t) = r(t) + 6.87\%$, where 6.87% is the long-term, annualized, continuous spread between the annual DAX30-return and the one-year spot-rate. We use 10000 simulations for each of the following tables which here is sufficient talking about accuracy. Each of the shifts in $r(t)$ and σ is assumed to occur immediately after the product is sold.

First we show our results for product 1:

Product 1, $i_l = 2\%$, $i_h = 12\%$, $\Delta r = 0\%$, $\Delta\sigma = 0\%$, $R = 2\%$						
t	$APR^*(t)$	$LPM_0(t)$	$LPM_1(t)$	$\sqrt{LPM_2(t)}$	$q_{95}(t)$	$q_{99}(t)$
1	515.54	0.3563	166.27	287.31	515.54	515.54
2	1357.74	0.1616	158.71	434.06	1357.74	1357.74
3	2153.48	0.0978	95.28	403.21	694.52	2153.48
4	2651.93	0.0295	33.08	241.08	0	1363.87
5	2756.18	0.0065	7.87	123.67	0	0
6	2605.69	0.0010	1.52	56.57	0	0
7	2206.88	0.0002	0.09	8.74	0	0
8	1789.16	0	0	0	0	0
9	1155.89	0	0	0	0	0
10	593.13	0	0	0	0	0
11	367.97	0	0	0	0	0

Product 1, $i_l = 2\%$, $i_h = 12\%$, $\Delta r = 1\%$, $\Delta\sigma = 0\%$, $R = 2\%$						
t	$APR^*(t)$	$LPM_0(t)$	$LPM_1(t)$	$\sqrt{LPM_2(t)}$	$q_{95}(t)$	$q_{99}(t)$
1	6932.17	1.0000	5801.74	5859.70	6932.17	6932.17
2	7672.67	1.0000	4121.11	4534.11	7672.67	7672.67
3	8224.99	0.5691	1766.72	2868.89	6407.07	8224.99
4	8319.44	0.1830	491.98	1461.52	3895.82	6458.08
5	7830.27	0.0360	90.85	601.87	0	3486.99
6	7066.66	0.0062	13.08	216.6	0	0
7	6028.19	0.0007	1.11	47.84	0	0
8	4931.61	0.0001	0.01	1.30	0	0
9	3582.43	0	0	0	0	0
10	2257.19	0	0	0	0	0
11	1221.45	0	0	0	0	0

Product 1, $i_l = 2\%$, $i_h = 12\%$, $\Delta r = 0\%$, $\Delta\sigma = 2\%$, $R = 2\%$						
t	$APR^*(t)$	$LPM_0(t)$	$LPM_1(t)$	$\sqrt{LPM_2(t)}$	$q_{95}(t)$	$q_{99}(t)$
1	884.72	0.4297	332.74	529.63	884.72	884.72
2	1760.61	0.2091	263.95	639.02	1760.61	1760.61
3	2580.65	0.1319	170.72	574.35	1587.81	2580.65
4	3080.88	0.0458	67.75	376.28	0	2527.77
5	3158.35	0.0120	17.39	192.88	0	504.12
6	2967.73	0.0024	4.04	95.61	0	0
7	2515.36	0.0007	0.79	38.47	0	0
8	2041.23	0.0001	0.06	5.59	0	0
9	1339.03	0	0	0	0	0
10	708.50	0	0	0	0	0
11	430.24	0	0	0	0	0

Product 1, $i_l = 2\%$, $i_h = 12\%$, $\Delta r = 1\%$, $\Delta\sigma = 2\%$, $R = 2\%$						
t	$APR^*(t)$	$LPM_0(t)$	$LPM_1(t)$	$\sqrt{LPM_2(t)}$	$q_{95}(t)$	$q_{99}(t)$
1	7582.17	1.0000	6500.72	6554.36	7582.17	7582.17
2	8352.71	1.0000	4958.03	5318.67	8352.71	8352.71
3	8919.58	0.6466	2437.58	3570.98	7616.99	8919.58
4	8999.71	0.2497	794.18	1966.16	5159.81	7939.63
5	8459.97	0.0593	168.76	871.51	519.11	5086.18
6	7631.59	0.0117	30.73	353.20	0	837.44
7	6513.49	0.0023	4.75	128.56	0	0
8	5331.57	0.0005	0.95	47.68	0	0
9	3882.10	0	0	0	0	0
10	2454.04	0	0	0	0	0
11	1326.04	0	0	0	0	0

If we let $R = 4\%$ all of the above values become 0.
We note that for $\Delta\sigma = -1\%$ all of the above values are decreasing compared to $\Delta\sigma = 0\%$, cf. also footnote 9.

For product 2 we get the following results for $R = 2\%$ and $R = 4\%$:

Product 2, $i_l = 2\%$, $\Delta r = 0\%$, $\Delta\sigma = 0\%$, $R = 2\%$					
t	$LPM_0(t)$	$LPM_1(t)$	$\sqrt{LPM_2(t)}$	$q_{95}(t)$	$q_{99}(t)$
1	0.3078	499.99	1122.38	2888.21	4214.60
2	0.2314	774.15	2008.55	5332.12	8237.47
3	0.1621	844.48	2601.58	6819.02	11989.06
4	0.1147	758.29	2782.82	6641.70	14134.80
5	0.0698	579.42	2760.02	3523.90	15869.69
6	0.0392	347.71	2198.12	0	13179.48
7	0.0232	209.62	1687.71	0	9571.46
8	0.0131	109.97	1177.05	0	3482.47
9	0.0069	47.84	704.34	0	0
10	0.0036	16.89	349.91	0	0
11	0.0012	4.15	137.88	0	0

Product 2, $i_l = 2\%$, $\Delta r = 1\%$, $\Delta\sigma = 0\%$, $R = 2\%$					
t	$LPM_0(t)$	$LPM_1(t)$	$\sqrt{LPM_2(t)}$	$q_{95}(t)$	$q_{99}(t)$
1	0.6839	2052.22	2923.91	5982.45	7370.63
2	0.3665	1598.47	3245.01	8136.30	11214.72
3	0.2113	1268.69	3425.30	9077.36	14559.55
4	0.1324	949.19	3271.02	8167.70	16150.65
5	0.0713	626.32	2976.83	3831.87	17129.17
6	0.0369	341.42	2265.05	0	13193.52
7	0.0200	190.79	1677.66	0	8359.76
8	0.0106	92.97	1146.81	0	834.76
9	0.0046	38.03	667.86	0	0
10	0.0019	11.54	313.33	0	0
11	0.0008	2.67	116.36	0	0

Product 2, $i_l = 2\%$, $\Delta r = 0\%$, $\Delta\sigma = 2\%$, $R = 2\%$					
t	$LPM_0(t)$	$LPM_1(t)$	$\sqrt{LPM_2(t)}$	$q_{95}(t)$	$q_{99}(t)$
1	0.3134	545.99	1204.23	3103.21	4420.02
2	0.2609	979.62	2357.79	6286.98	9097.18
3	0.1998	1175.26	3225.50	8710.57	13677.21
4	0.1526	1180.09	3681.85	9956.14	16915.78
5	0.1017	977.34	3782.74	8489.19	20180.24
6	0.0682	663.47	3159.32	3841.47	18434.31
7	0.0436	437.59	2536.77	0	16089.91
8	0.0281	259.06	1845.38	0	11523.63
9	0.0177	132.98	1194.74	0	5585.16
10	0.0090	57.57	678.03	0	0
11	0.0050	17.70	282.90	0	0

Product 2, $i_l = 2\%$, $\Delta r = 1\%$, $\Delta\sigma = 2\%$, $R = 2\%$					
t	$LPM_0(t)$	$LPM_1(t)$	$\sqrt{LPM_2(t)}$	$q_{95}(t)$	$q_{99}(t)$
1	0.6714	2200.24	3147.03	6437.80	7834.30
2	0.4008	1966.86	3794.95	9353.64	12378.49
3	0.2553	1753.06	4267.25	11406.95	16706.80
4	0.1727	1485.65	4382.75	11851.21	19443.25
5	0.1054	1088.49	4176.16	9341.39	21983.33
6	0.0649	680.82	3354.94	3397.89	19699.13
7	0.0390	424.01	2622.25	0	16532.38
8	0.0246	234.37	1858.32	0	10567.41
9	0.0134	111.66	1165.55	0	3047.16
10	0.0070	45.40	631.90	0	0
11	0.0030	10.96	238.70	0	0

Product 2, $i_l = 2\%$, $\Delta r = 0\%$, $\Delta\sigma = 0\%$, $R = 4\%$					
t	$LPM_0(t)$	$LPM_1(t)$	$\sqrt{LPM_2(t)}$	$q_{95}(t)$	$q_{99}(t)$
1	0	0	0	0	0
2	0	0	0	0	0
3	0.0013	1.12	40.74	0	0
4	0.0081	20.35	287.84	0	0
5	0.0157	70.44	711.83	0	2377.16
6	0.0113	65.70	744.21	0	1270.49
7	0.0092	50.31	652.45	0	0
8	0.0056	32.58	521.82	0	0
9	0.0034	15.79	325.68	0	0
10	0.0017	5.79	165.82	0	0
11	0.0008	1.92	72.60	0	0

Product 2, $i_l = 2\%$, $\Delta r = 1\%$, $\Delta\sigma = 0\%$, $R = 4\%$					
t	$LPM_0(t)$	$LPM_1(t)$	$\sqrt{LPM_2(t)}$	$q_{95}(t)$	$q_{99}(t)$
1	0	0	0	0	0
2	0	0	0	0	0
3	0.0054	9.82	173.80	0	0
4	0.0130	44.04	488.43	0	1175.51
5	0.0173	93.58	898.64	0	3636.64
6	0.0112	75.18	871.06	0	1284.54
7	0.0080	51.54	732.51	0	0
8	0.0047	32.20	575.07	0	0
9	0.0024	14.42	348.86	0	0
10	0.0011	4.78	167.87	0	0
11	0.0005	1.23	68.81	0	0

Product 2, $i_l = 2\%$, $\Delta r = 0\%$, $\Delta\sigma = 2\%$, $R = 4\%$					
t	$LPM_0(t)$	$LPM_1(t)$	$\sqrt{LPM_2(t)}$	$q_{95}(t)$	$q_{99}(t)$
1	0	0	0	0	0
2	0	0	0	0	0
3	0.0024	3.15	76.14	0	0
4	0.0169	50.11	482.33	0	1940.64
5	0.0283	153.66	1121.61	0	6687.71
6	0.0235	147.64	1165.14	0	6525.33
7	0.0188	124.06	1060.36	0	5870.28
8	0.0140	83.23	837.24	0	3104.24
9	0.0094	47.54	561.45	0	0
10	0.0059	22.83	333.90	0	0
11	0.0035	7.87	150.71	0	0

Product 2, $i_l = 2\%$, $\Delta r = 1\%$, $\Delta\sigma = 2\%$, $R = 4\%$					
t	$LPM_0(t)$	$LPM_1(t)$	$\sqrt{LPM_2(t)}$	$q_{95}(t)$	$q_{99}(t)$
1	0	0	0	0	0
2	0.0002	0.02	1.53	0	0
3	0.0136	27.00	300.58	0	722.39
4	0.0295	111.31	828.04	0	4468.11
5	0.0324	210.23	1448.39	0	8490.79
6	0.0241	177.96	1406.48	0	7790.15
7	0.0182	137.87	1229.61	0	6312.75
8	0.0120	86.52	947.51	0	2148.02
9	0.0079	45.02	612.65	0	0
10	0.0045	19.49	343.20	0	0
11	0.0019	5.43	142.30	0	0

We note that for $\Delta r = -1\%$ and/or $\Delta\sigma = -1\%$ all of the above values ($R = 2\%$, 4%) are decreasing.

The results in particular show that the expected additional policy reserves $LPM_1(t)$ are quite sensitive with respect to an increase in the yield curve. While writing this paper, interest rates in Germany are at an all-time low, and an increase becomes more and more likely. Furthermore, we see that in case of $R = 4\%$ there only remains a risk in product 2.

4 Summary

This paper looked at two equity-linked life insurance products: a collar-type one, where the annual interest earned lies in a prefixed interval $[i_l, i_h]$ and a geometric averaging product, where the geometric average index-return is added to a guaranteed sum (calculated with i_l).

In the first part of the paper, we derived explicit pricing formulas for both products and showed, how to calculate the fair rate of index participation x, if the exogenous variables (yield curve, index volatility, i_l, i_h) are given. In both cases x depends heavily on i_l, the minimum guaranteed interest rate.

Of course, the reader might ask, why we look at the geometric average instead of the arithmetic average. For the latter, no explicit pricing formulas are available, hence all calculations have to be done using extensive, time consuming Monte Carlo simulations. This is, in particular, unfavourable when calculating additional policy reserves at time t, as we do in Section 3. On the other hand it turns out, that using the data given in the paper, the rate of index participation in case of using an arithmetic average is almost identical to the one calculated for the geometric average product. Hence, it appears that the geometric average product can profitably be used in a control variate algorithm in order to speed up the Monte Carlo simulations considerably. So far, our discussion only focused on the product from an investment point of view.

In the second part of the paper, we calculated the additional policy reserves which are required, whenever the market value of the policy drops below the discounted sum guaranteed at expiration on a date of the balance sheet. It turns out that the distribution of the additional policy reserves is quite sensitive with respect to changes in the interest rate level. Hence, it might be worthwhile using a stochastic interest-rate model — like the one-factor no-arbitrage model by Hull and White, cf. e.g. [Hu/Wh 94] — instead of a deterministic one.

A Proof of the Pricing Formula for the Geometric Average Product

We consider a security with the following payoff at maturity $T = 12$.

$$A_T = \sum_{i=0}^{4} NP \max\left[\frac{\sqrt[T-i]{\prod_{j=i+1}^{T} S_j} - S_i}{S_i} x, \rho_i\right] =: \sum_{i=0}^{4} A_{T,i}, \quad \rho_i \geq 0,\ x > 0.$$

Let $A_{t,i}$ denote the value of $A_{T,i}$ at time $t = 0, \ldots, 11$, i.e. $A_{t,i} = E_Q\left[e^{-\int_t^T r(s)ds} A_{T,i} \mid_t\right]$. In calculating this expected value, we follow an idea in [Tu/Wa 91].

Case 1: $t = i$

Given the information available at time t, we get

$$\log \sqrt[T-i]{\prod_{j=i+1}^{T} S_j} = \log S_i + \sum_{j=i+1}^{T} \frac{T-j+1}{T-i} \log \frac{S_j}{S_{j-1}} \sim N(m, s^2),$$

with

$$m = \log S_i + \sum_{j=i+1}^{T} \frac{T-j+1}{T-i} \left(f_{j-1,j} - \frac{\sigma^2}{2} \right)$$

$$s^2 = \sigma^2 \sum_{j=i+1}^{T} \frac{(T-j+1)^2}{(T-i)^2} = \sigma^2 \sum_{j=i+1}^{T} \frac{(j-i)^2}{(T-i)^2},$$

according to (2) (with $\mu(t) = r(t)$, cf. our remark following (3)).

Hence (with $\epsilon \sim N(0,1)$),

$$\log \sqrt[T-i]{\prod_{j=i+1}^{T} S_j} \stackrel{d}{=} m + s\epsilon$$

$$= \log(S_i e^{\eta}) + \sum_{j=i+1}^{T} \left(f_{j-1,j} - \frac{(\frac{s}{\sqrt{T-i}})^2}{2} \right) + \frac{s}{\sqrt{T-i}} \sqrt{T-i}\,\epsilon,$$

with

$$\eta = \frac{s^2}{2} - (T-i)\frac{\sigma^2}{2} - \sum_{j=i+1}^{T} \frac{j-i-1}{T-i} \left(f_{j-1,j} - \frac{\sigma^2}{2} \right).$$

This is the distribution at time T of the value of a stock with current stock price (at time t) $S_i e^{\eta}$ and volatility $\frac{s}{\sqrt{T-i}}$ in a risk-neutral world. Applying the formula of Black and Scholes, we get

$$\begin{aligned} A_{t,i} &= e^{-(T-i)f_{i,T}} NP E_Q \left[\rho_i + \frac{x}{S_i} \max \left[\sqrt[T-i]{\prod_{j=i+1}^{T} S_j} - S_i(1 + \frac{\rho_i}{x}), 0 \right] |_t \right] \\ &= e^{-(T-i)f_{i,T}} NP \left\{ \rho_i + e^{\eta} e^{(T-i)f_{i,T}} x N(d_1) - (x + \rho_i) N(d_2) \right\}, \qquad (4) \end{aligned}$$

with

$$d_1 = \frac{\eta - \log(1 + \frac{\rho_i}{x}) + \frac{s^2}{2} + (T-i)f_{i,T}}{s}, \quad d_2 = d_1 - s.$$

Case 2: $t < i$

In this case we obviously get from (3) and (4)

$$A_{t,i} = E_Q\left[e^{-\int_t^i r(s)ds} E_Q\left[e^{-\int_i^T r(s)ds} A_{T,i} \mid_i\right] \mid_t\right] = e^{-(i-t)f_{t,i}} A_{i,i}.$$

Case 3: $t > i$

Given the information available at time t, we get

$$\begin{aligned} \log \sqrt[T-i]{\prod_{j=i+1}^{T} S_j} &= \log \sqrt[T-i]{\prod_{j=i+1}^{t} S_j} + \frac{T-t}{T-i} \log S_t + \sum_{j=t+1}^{T} \frac{T-j+1}{T-i} \log \frac{S_j}{S_{j-1}} \\ &\sim N(m, s^2), \end{aligned}$$

with

$$m = \log \sqrt[T-i]{\prod_{j=i+1}^{t} S_j} + \frac{T-t}{T-i} \log S_t + \sum_{j=t+1}^{T} \frac{T-j+1}{T-i} \left(f_{j-1,j} - \frac{\sigma^2}{2}\right)$$

$$s^2 = \sigma^2 \sum_{j=t+1}^{T} \frac{(T-j+1)^2}{(T-i)^2} = \sigma^2 \sum_{j=t+1}^{T} \frac{(j-t)^2}{(T-i)^2}.$$

Hence (with $\epsilon \sim N(0,1)$),

$$\log \sqrt[T-i]{\prod_{j=i+1}^{T} S_j} \stackrel{d}{=} m + s\epsilon$$

$$= m + \frac{s^2}{2} - (T-t)f_{t,T} + \sum_{j=t+1}^{T} \left(f_{j-1,j} - \frac{(\frac{s}{\sqrt{T-t}})^2}{2}\right) + \frac{s}{\sqrt{T-t}} \sqrt{T-t}\epsilon.$$

This is the distribution at time T of the value of a stock with current stock price (at time t) $e^{m+\frac{s^2}{2}-(T-t)f_{t,T}}$ and volatility $\frac{s}{\sqrt{T-t}}$ in a risk-neutral world. Applying the formula of Black and Scholes, we get

$$\begin{aligned} A_{t,i} &= e^{-(T-t)f_{t,T}} NP E_Q \left[\rho_i + \frac{x}{S_i} \max \left[\sqrt[T-i]{\prod_{j=i+1}^{T} S_j} - S_i(1+\frac{\rho_i}{x}), 0 \right] |_t \right] \\ &= e^{-(T-t)f_{t,T}} NP \left\{ \rho_i + \frac{x}{S_i} e^{m+\frac{s^2}{2}} N(d_1) - (x+\rho_i) N(d_2) \right\}, \end{aligned}$$

with

$$d_1 = \frac{-\log(S_i(1+\frac{\rho_i}{x})) + m + s^2}{s}, \quad d_2 = d_1 - s.$$

Letting $\rho_i = 0 \ \forall i$ we get the results of Section 2. Furthermore, if $r(t) \equiv const$, our formulas for $t \geq i$ coincide with a result in [Tu/Wa 91].

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