Optimal Portfolios of Defaultable Assets

Yukio Muromachi, Daisuke Asahara, and Shuji Tanaka
Financial Research Department
NLI Research Institute
1-1-1 Yurakucho, Chiyoda-ku, Tokyo, 100, Japan

Telephone:03-3597-8431
Facsimile:03-5512-7161
E-mail:muromati@nli-research.co.jp
asahara@nli-research.co.jp
tanaka@nli-research.co.jp

Abstract

Two basic problems are discussed on defaultable assets such as corporate bonds and loans; devising a pricing model of each asset, and optimizing a portfolio consisting of these assets. The pricing model for bonds and loans, which is the naturally expanded version of the ordinary cashflow discount model, is fully based on information observed in the corporate bond market such as market prices of many bonds. Portfolio optimization, one of the most familiar investment tools, is applied to investment in defaultable assets while taking into account future changes in their default probabilities. Some results show that important factors in optimal portfolio selection are not only the default probability of each asset class, but also the transition probabilities between asset classes with different credit ratings and the duration of each asset class.

Keywords
defaultable asset, present value valuation, optimal portfolio

1. Introduction

In recent years, a series of incidents have made us recognize the importance of risk management, especially of credit risk management in the financial industry. Some of the more prominent cases have been mismanagement of Barings Security in the UK and Orange County in the US. In Japan, examples include the bad debt problem, repeated management breakdowns at financial institutions, massive losses of banks and securities companies related to trading in securities and commodities, and so on.

Meanwhile, there has also been progress in Japanese financial deregulation such as removal of restrictions on issuing bonds. Investors are required to adhere to the self-responsibility principle and conduct their own risk management. At present, Japanese financial institutions are bolstering their risk management capabilities. However, their main target is the management of market risk; the management of credit risk is still out of target.

There are two topics in this paper: a valuation method for a defaultable asset and an optimal asset allocation strategy. Present value and profitability valuation is described in section 2. In section 3, the optimization of a portfolio consisting of defaultable assets is discussed. Finally, some implications and remaining problems are summarized briefly in section 4. Methods described in this paper are derived from those proposed by Nishida [1995].

2. Present Value and Profitability of a Defaultable Asset

The first step for risk management is the valuation of present value and profitability of individual assets.

2.1. Basic Concept of Present Value Valuation

Consider the present value of an asset such as a bond or a loan. The most fundamental difference between a bond and a loan is the trading form. Prices of bonds are observable because bonds are usually traded in the market or on the counter. On the other hand, loans are contracted face to face, and much of them are not traded. Therefore, loans have not been valuated from the viewpoint of the present value.

However, loans are not thought to be essentially different from bonds because they are nothing but contracts concerning future cashflows. This concept leads us to a valuation method for the present value of a loan based on bond prices observed in the bond market.

The present value of a bond or a loan PV is expressed as follows:

$$PV = \sum_{i=1}^{n} \frac{C_i}{\left(1 + r_i + \alpha_i\right)^i} \tag{1}$$

or

$$PV = \sum_{i=1}^{n} \frac{E[\widetilde{C}_i]}{\left(1 + r_i + \alpha_i'\right)^i} \tag{2}$$

where C_i ($i = 1,2,\cdots,n$) is the i^{th} contracted cashflow at the time t_i, r_i is the riskless interest rate, $r_i + \alpha_i$ is the theoretical yield, α_i is the theoretical spread, α_i' is the risk premium, $E[]$ denotes the expectation operator, and $\widetilde{x}$ indicates that x is a random variable. Assuming that the risk is limited to the default risk only, the expectation in (2) is given by

$$E[\widetilde{C}_i] = E[\widetilde{S}_i]C_i + E[\widetilde{S}_{i-1}\widetilde{\pi}_i\widetilde{\mu}_i]PV_i \tag{3}$$

where $\widetilde{S}_i$ is the non-default probability until the time t_i, $\widetilde{\pi}_i$ is the default probability from the time t_{i-1} to the time t_i, $\widetilde{\mu}_i$ is the recovery rate at t_i, PV_i is the present value at t_i, and $\widetilde{S}_i$ is written as

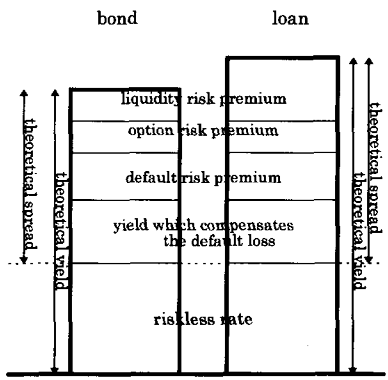
$$\widetilde{S}_i = \prod_{j=1}^{i} (1 - \widetilde{\pi}_j) \tag{4}$$

Equation (2) is more complicated than (1). However, it is useful for clearly seeing how pricing reflects term structures of the default probability and the recovery rate. It is possible to valuate the PVs of various kinds of bonds and loans if α_i is expressed as a function of $\widetilde{\pi}_i$ and $\widetilde{\mu}_i$, and if these in turn are given as functions of the credit rating, the riskless interest rate, the land price, and so on.

2.2. Theoretical Spreads

Figure 1 shows the structures of theoretical yields of a bond and a loan, assuming that the risks are default risk, option risk and liquidity risk. The default risk premium in Figure 1 is the premium yield which risk-averting investors demand instead of taking the default risk.

Figure 1. Structures of Theoretical Yields



The procedure for deriving the theoretical spread of a loan from that of a similar bond is:

① Classify the risks included in the loan and the bond into two types: risks which bonds and loans have in the same way are called "type A," and other risks are called "type B." In this case only liquidity risk belongs to type B.

② Decompose the theoretical spread of the bond into yields due to individual risks.

③ Yields due to "type A" risks of the loan are equal to those of the bond computed in ②. In this case there are three: the yield which compensates the loss due to the default, the default risk premium, and the *option risk premium*.

④ Estimate yields due to "type B" risks of the loan from some information. In this case the difference of the liquidity risk premium between the loan and the bond is assumed to be equal to the issue cost of the bond.[1]

⑤ Sum up all the yields of the loan. This sum is the theoretical spread of the loan.

Roughly speaking, the theoretical spread of a loan is given by adding the liquidity risk premium difference to the spread of a similar bond observed in the market. This method is correct if all the risks are uncorrelated with each other.

2.3. Profitability Valuation

Profitability is another measure that takes present value valuation a step further by including the investment term. A simple valuation method introduced in this subsection is to compare the effective yield of an each asset with the theoretical yield.

The procedure is as follows:

① Compute the present value PV and the duration Du of the asset.

② Compute the future value FV after Du years on the assumption that PV performs at the theoretical yield during Du years.

③ Compute the effective yield on the assumption that the present amount of the principal of the loan (or the face value of the bond) becomes FV after Du years.

④ Compare the effective yield with the theoretical yield. The former minus the latter is the excess spread.

The present value and duration of cashflow FV after Du years are equal to those of the original asset, respectively.

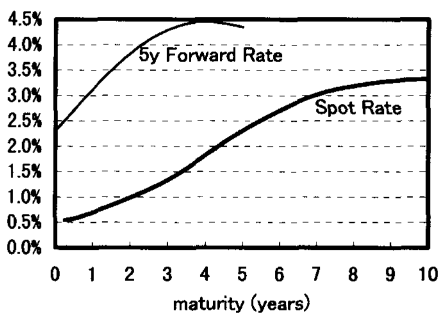
2.4. Examples of Valuation

Four simple loans are valuated in this subsection. Conditions of the loans are summarized in Table 1. All are assumed to be carried out on January 5, 1996. We assume that the spot rate of Japanese Government Bonds (Figure 2) is used as a riskless spot rate, and that the theoretical spreads and the default probabilities are constant, namely, independent of the maturity. They are shown in Table 2.[2]

Table 1. Conditions of Loans

	case 1	case 2	case 3	case 4
credit rating[3]	A	46～55points	46～55points	46～55points
amount of principal	1 billion yen	1 billion yen	1 billion yen	1 billion yen
interest payment	fixed	fixed	floating	fixed
interest rate	3.2%	2.9%	Long Term Prime Rate + 30bp	2.9%
term	10years	5 years	5 years	5 years
redemption	lump sum at maturity	level payment of principal	lump sum at maturity	level payment of principal
recovery rate	0%	0%	0%	80%

Figure 2. Spot Rate Curve and Forward Rate Curve of Japanese Government Bonds



The long-term prime rate r_P, which appears as the base rate in Case 3, is assumed to be expressed as follows:

$$r_P = spot\ rate\ of\ Japanese\ Government\ Bond\ at\ 5years + 29bp \qquad (5)$$

For the JGB spot rate in the future, we use the forward rate calculated from the term structure of the JGB spot rate on January 5, 1996 (see Figure 2). The 29bps in (5) is the difference between the long-term prime rate and the JGB spot rate at 5 years on the day.

Only the recovery rate differs in the conditions of Case 2 and Case 4. In order to valuate loans with different recovery rates systematically, equation (2) is used. The risk premium in equation (2) is calculated so that the present value computed by equation (2) is equal to the value computed by equation (1). Here, assume that the risk premium is expressed as follows:[4]

$$\alpha' = a\left[\pi(1-\mu)\right]^{1/2}\left[(1-\pi)(1-\mu)+\pi\mu\right]^{1/2} \qquad (6)$$

where a is a constant. a is derived from the calculated risk premium, and formula (6) is applied to Case 4.

Table 2. Fundamental Data for Evaluation of an Asset and Optimization of an Portfolio

① Credit Spreads and Default Probabilities (%)

	Credit Spread		Default Probability	
	Mean	Deviation	Mean	Deviation
AAA	0.1775	0.1054	0	0
AA	0.3162	0.1114	0	0
A	0.4369	0.1133	0	0.0104
BBB	0.5846	0.1361	0.0030	0.0189
56-65 Points	1.0000	0.6500	0.0910	0.1074
46-55 Points	1.2000	1.2000	0.4820	0.3509
36-45 Points	1.4000	2.0000	1.5030	0.8665
-35 points	1.6000	2.5000	2.9830	1.1833

② Correlations of Credit Spreads

	AAA	AA	A	BBB	56-65 Points	46-55 Points	36-45 Points
AAA	1.0000						
AA	0.9547	1.0000					
A	0.9803	0.9770	1.0000				
BBB	0.5958	0.7175	0.7034	1.0000			
56-65 Points	0.5000	0.5000	0.5000	0.5000	1.0000		
46-55 Points	0.5000	0.5000	0.5000	0.5000	0.5000	1.0000	
36-45 Points	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	1.0000

③ Variances and Covariances of Default Probabilities

	AAA	AA	A	BBB	56-65 Points	46-55 Points	36-45 Points
AAA	0.0000						
AA	0.0000	0.0000					
A	0.0000	0.0000	0.0001				
BBB	0.0000	0.0000	0.0000	0.0004			
56-65 Points	0.0000	0.0000	0.0000	0.0019	0.0115		
46-55 Points	0.0000	0.0000	0.0000	0.0058	0.0373	0.1231	
36-45 Points	0.0000	0.0000	0.0002	0.0145	0.0919	0.3030	0.7508

Valuation results of the four examples are summarized in Table 3.

In Case 1, the amount of loss is slightly over 40 million yen on a present-value base. The effective yield Y_5 is about 0.5% less than the theoretical yield Y_4, moreover, it is slightly less than Y_1+Y_2, which indicates that JGBs are more profitable than this loan. In Case 2, the duration is short because of the level payment of principal. As the short duration entails a low yield because of the positive yield curve, the profit on a present-value base is 2.42 million yen, and Y_5 exceeds Y_4 about 0.1%. In Case 3, in spite of its long duration due to the lump sum redemption at maturity, the profit on a present-value base is nearly 40 million yen because of the floating coupon rate. Y_5 is about 0.8% more than Y_4. Case 4 differs from Case 2 only in that the recovery rate changes to 80%. As a result of the decline of Y_4 due to the rise of the recovery rate, the profit on a present-value base is nearly 20 million yen, and Y_5 exceeds Y_4 about 0.8%.

Table 3. Evaluations of Loans

	case 1	case 2	case 3	case 4
present value (million yen)	959.47	1002.42	1036.72	1021.55
profit (million yen)	-40.53	2.42	36.72	21.55
duration (years)	8.549	2.620	4.602	2.620
yield of riskless bond : Y_1	3.2477%	1.1883%	2.1295%	1.1882%
compensation yield[5]: Y_2	0.0000%[6]	0.4820%	0.4820%	0.0968%[7]
compensation yield with 95% confidence level[8]: Y_3	0.0171%	1.0592%	1.0592%	0.2129%
default risk premium	0.2869%	0.4180%	0.4180%	0.0844%
theoretical yield : Y_4	3.6846%	2.3883%	3.3295%	1.6694%
effective yield : Y_5	3.2006%	2.4807%	4.1130%	2.4831%
excess spread (Y_5-Y_4)	-0.4840%	0.0924%	0.7835%	0.8137%
excess spread (Y_5-(Y_1+Y_2))	-0.0471%	0.8104%	1.5015%	1.1981%
excess spread[9] (Y_5-(Y_1+Y_3))	-0.0642%	0.2332%	0.9243%	1.0820%

3. Optimization of a Portfolio Consisting of Defaultable Assets

Optimal asset allocation of a portfolio which consists of defaultable assets such as corporate bonds and loans is discussed using Markowitz's mean-variance approach.

3.1. Application of Markowitz's Mean-Variance Approach

Consider a portfolio consisting of n kinds of defaultable asset classes classified by the credit rating. Mean and variance of the yield to maturity for each class are known. The total return R_p of the portfolio, in which the weight of the i^{th} class is w_i, can be written as follows:

$$R_p = \sum_{i=1}^{n} w_i R_i = \sum_{i=1}^{n} w_i \left[Y_i - P_{id}(1+Y_i)(1-\delta_i) - D_i \sum_{j \neq d} P_{ij}(Y_j - Y_i) \right] \qquad (7)$$
$$\sum_{i=1}^{n} w_i = 1 \qquad (w_i \geq 0)$$

where R_i is the real return in the i^{th} class, Y_i is the yield to maturity, δ_i is the recovery rate, P_{ij} is the transition probability from i^{th} class to j^{th} class, P_{id} is the default probability, and D_i is the duration.[10] The second and the third term in the expression of R_i represent effects of the default loss and the price change due to the credit rating change, respectively.

In the discussion below, influences of the default probability, the transition probability of credit rating, and the duration are shown on the shape of the efficient frontier in the risk-return plots. For simplicity, $\delta_i = 0$ is assumed and all results are calculated by using excess returns, which refer to the yield spreads with respect to the yields of non-defaultable bonds such as Treasury Bonds, instead of using the yields to maturity.

3.2. Effects of Loss by Default and Price Change due to Credit Rating Change

Consider the effects of loss by default and price change due to the credit rating change on the efficient frontier. The transition probability matrix is defined as $P = \left(P_{ij}, P_{id}\right)$. As an example, P for Japanese corporations during one year is shown in Table 4[11] (see Takahashi and Moridaira [1996]) .

Figure 3 shows the change in expected excess returns due to the effect of the transition probability matrix in Table 4[12], in which the upward probability is higher than the downward probability except for AAA. In low-rating classes, the change in credit rating contributes to a rise in price and the excess return increases because the effect of upgrades exceeds the effect

Table 4. An Example of Transition Probability Matrix (%)

		To						
		76-	66-75	56-65	46-55	36-45	-35	Bankruptcy
From	76-	81.62	18.19	0.16	0.02	0.01	0.01	0.00
	66-75	22.26	74.56	1.84	0.80	0.39	0.15	0.01
	56-65	7.19	10.90	71.06	9.95	0.38	0.29	0.24
	46-55	0.03	3.14	7.78	82.09	5.96	0.03	0.97
	36-45	0.01	0.28	5.06	5.11	87.31	0.62	1.86
	-35	0.04	0.08	8.02	12.95	12.98	62.95	2.99

Figure 3. Change in Excess Returns (Table 4)

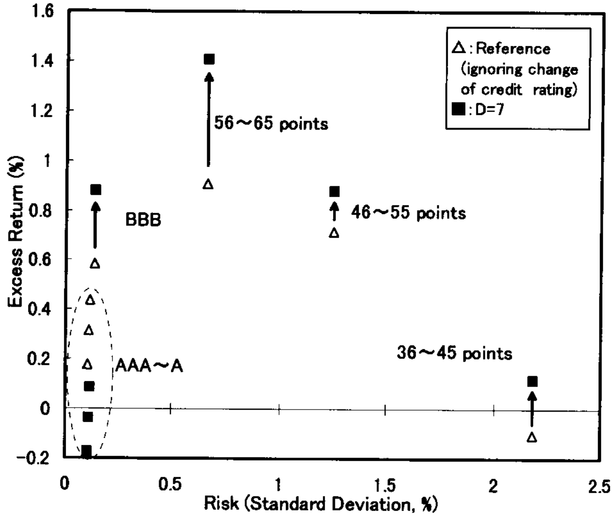
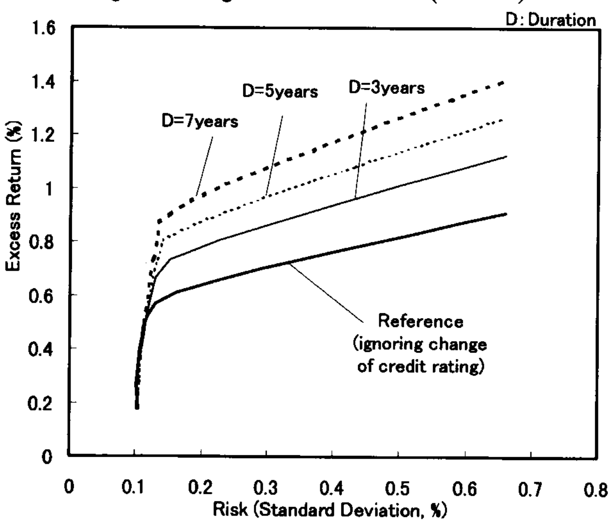


Figure 4. Change in Efficient Frontier (Table 4)



of downgrades. However, the excess return decreases in high-rating classes because the effect of upgrades is small due to the smallness of credit spread difference between high rating classes.

Efficient frontiers computed for various durations are shown in Figure 4. Because of the rise in real returns, efficient frontiers become higher compared with the frontier when the transition probability matrix is not taken into consideration. In addition, the longer the duration, the higher the efficient frontier becomes because the contribution of the price change due to the credit rating change is proportional to the duration.

Optimal asset allocation for $D = 7$ years is shown in Figure 5. Assets of classes with 46-55 or less points[13] are never included in the optimal asset allocation. This implies that the transition probability matrix and the spreads are partly unsuitable for risk-averters. That is, spreads for assets with low credit ratings are too low to invest in them if based on the transition probability matrix. The relation between spreads and the matrix is discussed in the next subsection.

Figure 5. Optimal Asset Distribution (D=7years, Table 4)

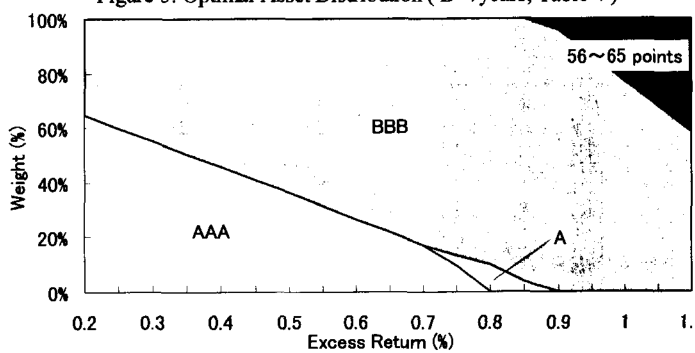


Table 5. Another Example of Transition Probability Matrix (%)

		To							
		AAA	AA	A	BBB	56-65	46-55	36-45	Bankruptcy[14]
From	AAA	92.10	7.20	0.70	0.00	0.00	0.00	0.00	0.00
	AA	1.10	91.60	6.90	0.30	0.20	0.00	0.00	0.00
	A	0.10	2.50	91.50	5.20	0.60	0.20	0.00	0.00
	BBB	0.00	0.20	5.20	88.30	5.30	0.80	0.10	0.10
	56-65	0.00	0.10	0.40	4.70	85.90	6.90	0.40	1.50
	46-55	0.00	0.10	0.10	0.50	5.50	83.50	2.00	8.20
	36-45	0.00	0.40	0.40	0.80	2.30	5.40	70.50	20.30

Figure 6. Change in Excess Return (Table 5)

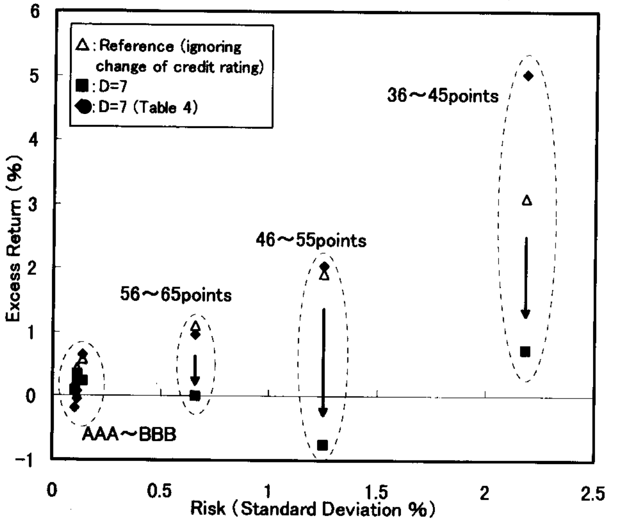
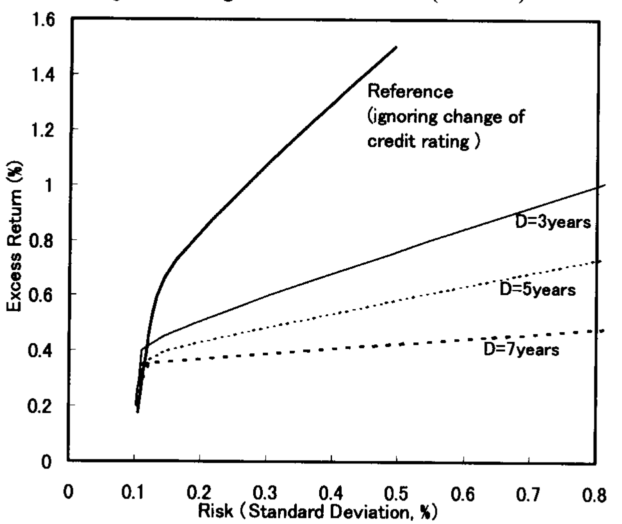


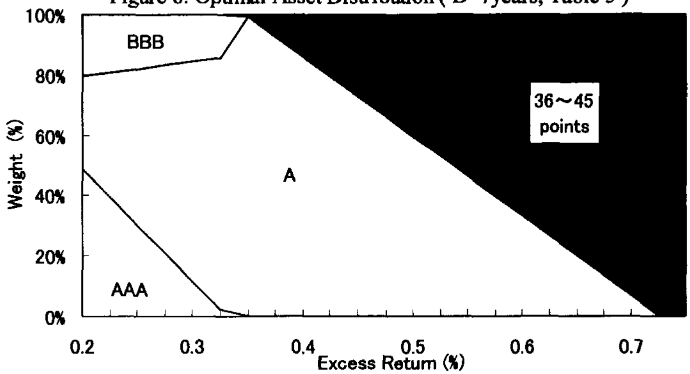
Figure 7. Change in Efficient Frontier (Table 5)



Next, we use a transition probability matrix where the downward probability is relatively higher than the upward probability as a whole. Such a matrix is shown in Table 5.[15]

Figure 6 shows the change in expected excess returns due to the effect of the transition probability matrix in Table 5.[16] Contrary to Figure 3, expected excess returns are almost reduced. Therefore, efficient frontiers decline compared with the frontier when the transition probability matrix is not taken into consideration. In addition, the longer the duration, the lower the efficient frontier becomes (see Figure 7). Figure 8 shows that the ratio of assets with 36-45 points in the optimal asset allocation becomes high in the high return region. The reason is clearly shown in Figure 6.

Figure 8. Optimal Asset Distribution (D=7years, Table 5)



To sum up, the shape of an efficient frontier changes according to the transition probability matrix. Compared to when the matrix is not taken into consideration, an efficient frontier increases when the upward probability is relatively high, and falls down in the opposite case. It is the credit spreads that determine whether the upward probability is higher than the downward probability or not. Optimal asset allocation is also determined synthetically from the transition probability matrix and the credit spreads. However, the amount of change of the efficient frontier is simply correlated to the duration.

3.3. Credit Spreads and Transition Probabilities

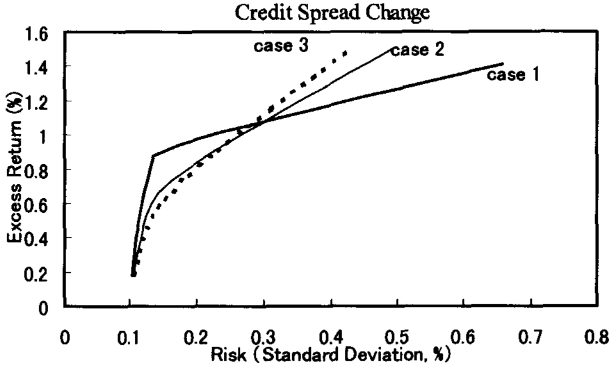
Next, we consider the shape change of the efficient frontier due to the change of credit spreads.[17] In this subsection, Table 4 is used as a transition probability matrix. Table 6 shows three kinds of credit spreads for asset classes with relatively low ratings. Case 1 is used in the previous subsection.

Table 6. Credit Spread for Lower Rating Asset Classes (%)

	case 1	case 2	case 3
56-65 Points	1.0	1.2	1.4
46-55 Points	1.2	2.4	2.8
36-45 Points	1.4	4.6	5.5
-35 Points	1.6	7.8	10.0

Figure 9 shows the shape change of the efficient frontier in $D = 7$ years. As the credit spreads for low rating classes become high, the real returns decline in higher rating classes, while they rise in lower rating classes; this is a general tendency. In the highest rating class, the real return decreases since any change of class is necessarily a downgrading. On the other hand, whether the real return of another class increases or not is determined by the net effect of upgrades and downgrades. Contribution of these effects is dependent on the transition probability matrix and the difference of the credit spreads. Figure 9 shows the case where the upward probabilities are relatively large.

Figure 9. Shape Change of Efficient Frontier due to Credit Spread Change



These results indicate that the transition probability matrix and the credit spreads should be taken into consideration simultaneously. When the transition probability matrix and the credit spreads of lower rating classes are unsuitable for risk-averters, changing the credit spreads of lower rating classes may create a new problem in the higher rating classes; for example, the real returns become too low to invest in the higher rating classes.

3.4. Duration and Excess Return

Finally, consider the relation between the duration and excess return. As described in subsection 3.2, the duration is not only an important factor which determines the shape change of the efficient frontier, but also a useful measure of the investment term.

Figures 10 and 11 show relations between the duration and excess return when the transition probability matrix is Table 4 and Table 5, respectively. Generally speaking, the excess return depends weakly on duration in the high rating class because of the high probability of remaining at the same rating class, and because of the small influence of the change of the rating on the price. On the other hand, the excess return depends strongly on the duration in the low rating class. It suggests that the duration of low-rating assets should remain short for avoiding risk.

However, we must not decide the investment terms only on the basis of the above analysis because risk analysis is lacking in it. In addition, it is a serious problem that this analysis takes no account of the variation of the transition probability matrix.

Figure 10. Duration and Excess Return (Table 4)

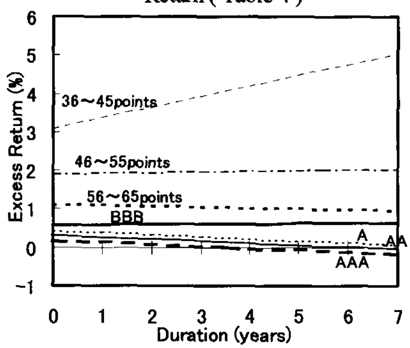
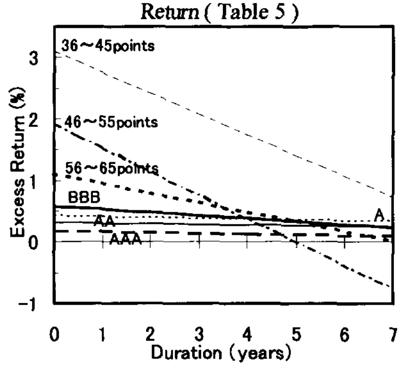


Figure 11. Duration and Excess Return (Table 5)



4. Discussion

4.1. Present Value Valuation

Although the present value and profitability valuation described in section 2 is simple and useful, they must be used with some cautions.

First is the increasing importance of an option premium valuation. The excess return obtained by an investor instead of taking a credit risk is very thin in the high-rating assets. Therefore, if an option risk is taken into consideration, there may be a case where the real excess return is negative although the excess return ignoring the option risk is positive. Strict valuation is needed for not only an option risk but also other risks especially in investment in the high-rating class assets.

Next, we must not make investment decisions only on the basis of present value and profitability valuation. The aim is not only to increase the return of one's portfolio but also to reduce risk. In some cases, it may be possible to incorporate an asset into one's portfolio to reduce risk even if its profitability is low. Therefore, investment decisions should be made based on not only on present value and profitability valuation, but also the mean-variance approach shown in section 3 and other analyses.

In addition, in order to introduce and spread such an valuation method we will need to develop appropriate modelings of factors such as a floating rate in the future, risk premium, and liquidity risk premium.

4.2. Optimizing a Portfolio

The simplified analyses of portfolio optimization described in section 3 have some defects.

First, almost all the correlations between random variables such as Y_i, δ_i, P_{ij} and P_{id} are ignored in optimization except for correlations of spreads between different classes. Moreover, the change of the standard deviations due to the change of the credit rating is also ignored. Therefore, the calculated results in section 3 are not strictly correct. However, we think that even if we took their effects into account, the main results would not change qualitatively, although the shape change of the efficient portfolio would become more complicated.

Second, since excess returns are used to calculate the optimal portfolio instead of total returns, we mainly discuss the diversification effect between assets with different credit rating. If total returns are used, the correlations between the credit spread and the riskless interest rate becomes an important issue, and a more general diversification effect can be obtained. This point warrants further study.

Third, although transition probabilities and default probabilities are treated as constants, they are random variables dependent on the economic environment and other factors. Appropriate modelings of these variables also warrant further study.

Moreover, the effect of diversification into various industries is important for optimizing a portfolio because some of the correlation coefficients between random variables in different industries may be negative. Analyses of credit spreads, default probabilities, and transition probabilities between industries are also important.

The most serious problem in this optimization calculation might be smallness of liquidity of the individual assets, which implies that the reallocation of assets is quite difficult. From this reason, some people say that the discussion about optimal asset allocation is meaningless.

However, we do not think that the claim is essential, since the fact that the reallocation of assets is difficult might be translated as a kind of constraint in the optimization; for example, a restriction on the change of the weight of each asset in the portfolio or a reallocation cost proportional to the change of the weight. The analyses in this paper are the most fundamental portion of the framework of optimization. Discussion about the effects of such constraints is left to future study.

4.3. Concluding Remarks

Some people claim that the present value of a loan is subjective because an objective price such as a market price is not available due to the lack of a loan market. This is an essential problem about the present value valuation and optimization described in this paper.

Nontheless, it is desirable to ascertain present values of assets such as loans. At present, management of assets belonging to the trading account is based on the present value. Therefore it is quite natural that the present value valuation is introduced to assets belonging to the banking account if the integrated management of all assets is the ultimate goal. Furthermore, it may suddenly happen that an asset in the banking account must be converted into cash. Moreover, a suitable present value valuation is needed when the securitization of an asset is planned. Thus it is doubtful that the present value valuation of assets is meaningless.

Future developments in the securitization of assets and credit derivatives are important for the present value valuation and optimization of a portfolio consisting of defaultable assets. It may be possible that these developments will bring us a realistic market of defaultable assets. Existence of a market where defaultable assets are traded is one of the most desirable conditions for the integrated risk management of all assets.

References

Carty, L. V, and J. S. Fons, Measuring Changes in Corporate Credit Quality, The Journal of Fixed Income, June, pp.27-41, 1994

Nishida S., New Developments in ALM (in Japanese) , Nihon Keizai Shimbun Inc., 1995

Takahashi H. and S. Moridaira, Prospects for Credit Risk Management (in Japanese) , Institute for Monetary and Economic Studies BOJ, 1996

Appendix. Interest-Rate-Sensitivity of the Price of a Defaultable Asset

Consider the price of a defaultable bond. Assume that its coupon rate is expressed as $aF_{i-1,i}+b$ where $F_{i-1,i}$ is the forward rate of the riskless bond from the time t_{i-1} to the time t_i, and a and b are constants. In the case where $a=0$, the bond has a fixed rate coupon, while in the case where $a\neq 0$, it has a floating-rate coupon. In this section we consider only the price of a bond, however, all the results described below can be also used for a loan.

Using the cashflow discount model, the price of a defaultable bond is written in the following way:

$$P=\sum_{i=1}^{n}\frac{aF_{i-1,i}+b}{(1+r_i+\alpha_i)^i}+\frac{1}{(1+r_n+\alpha_n)^n} \tag{A-1}$$

where P is the price, r_i is the spot rate of the riskless bond at the time t_i, α_i is the credit spread of the defaultable bond, and $F_{i-1,i}$ is given as

$$F_{i-1,i}=\frac{(1+r_i)^i}{(1+r_{i-1})^{i-1}}-1 \tag{A-2}$$

From the above two equations, the first-order variation of the bond price is expressed in terms of the change of r_i and α_i:

$$\begin{aligned}dP=&\sum_{i=1}^{n}\frac{i}{(1+r_i+\alpha_i)^{i+1}}\left[a\left\{1+\alpha_i\left(\frac{1+r_i}{1+r_{i-1}}\right)^{i-1}\right\}-b\right]dr_i-\frac{n}{(1+r_n+\alpha_n)^{n+1}}dr_n\\&-a\sum_{i=1}^{n-1}\frac{i}{(1+r_i+\alpha_i)^{i+1}}\left(\frac{1+r_i+\alpha_i}{1+r_{i+1}+\alpha_{i+1}}\right)^{i+1}\left(\frac{1+r_{i+1}}{1+r_i}\right)^{i+1}dr_i\\&-\sum_{i=1}^{n}\frac{i\left[a\left\{\dfrac{(1+r_i)^i}{(1+r_{i-1})^{i-1}}-1\right\}+b\right]}{(1+r_i+\alpha_i)^{i+1}}d\alpha_i-\frac{n}{(1+r_n+\alpha_n)^{n+1}}d\alpha_n\end{aligned} \tag{A-3}$$

(A-3) gives a general form of the sensitivity of the bond price on the riskless interest rate and the credit spread.

Hereafter, to simplify the expressions, assume that the two term structures are flat.

In the case where $a=0$, (A-3) leads us to the following equation:

$$\frac{dP}{P}=-\frac{\displaystyle\sum_{i=1}^{n}\frac{ib}{(1+r')^i}+\frac{n}{(1+r')^n}}{\displaystyle\sum_{i=1}^{n}\frac{b}{(1+r')^i}+\frac{1}{(1+r')^n}}\cdot\frac{dr'}{1+r'}=-D\cdot\frac{dr'}{1+r'} \tag{A-4}$$

where $r'=r+\alpha$ and D is the duration of the bond. (A-4) is a well-known relationship between the bond price and the yield to maturity.

In the case where $a=1$, (A-3) gives the following equation:

$$dP=-\left[\sum_{i=1}^{n}\frac{i\cdot(b-\alpha)}{(1+r+\alpha)^i}\right]\cdot\frac{dr}{1+r+\alpha}-\left[\sum_{i=1}^{n}\frac{i\cdot(r+b)}{(1+r+\alpha)^i}+\frac{n}{(1+r+\alpha)^n}\right]\cdot\frac{d\alpha}{1+r+\alpha} \tag{A-5}$$

From (A-5), it is clear that the price variation according to the change of the riskless interest rate is zero when $b=\alpha$. However, the second term (the price variation according to the change of the credit spread) still remains. Since the form of the second term is similar to (A-4), the duration according to the credit spread may be defined from the expression of dP / P given from (A-5).

---

[1] This assumption is based on the idea that the bond issue competes with loans so that there is no arbitrage opportunity between them ideally. This assumption is also proposed by Nishida [1995].

[2] Default probabilities and their variances and covariances are obtained by the analyses of bankruptcy data offered by Teikoku Data Bank in Japan (See endnote 11 in detail). Credit spreads from AAA to BBB and their correlations are based on the analyses of the corporate bond market and the assumption that the liquidity risk premium of a loan is 15bp (30bp) larger than that of a bond with a credit rating from AAA to BBB (under BBB). Other values are assumed on the basis of some information.

[3] Points are based on the data offered by Teikoku Data Bank in Japan. See endnote 10 in more detail.

[4] The form of equation (6) is derived from the following assumptions; ① the loss ratio is $\pi(1-\mu)$, ② the correlation between π and μ is zero, ③ the credit risk premium is proportional to the standard deviation of the loss ratio, and ④ π and μ have binomial distributions. Although simplified, these assumptions should be suitable for our purposes.

[5] This is the yield which compensates the expected loss due to the default probability.

[6] This probability is not zero, however, so small.

[7] This value is about one fifth of that in case 2 because of the increase of recovery rate.

[8] This is the yield which compensates the expected loss due to the default probability with 95% one-sided confidence level.

[9] Contribution of the duration change due to the change of the default probability is ignored.

[10] In this paper, it is assumed that the duration is constant for simplicity. Interest-rate-sensitivity of the price of a defaultable asset is generally expressed in Appendix.

[11] This transition probability matrix is derived by Takahashi and Moridaira [1995] from the data offered by Teikoku Data Bank whose main service is the credit analysis of corporations. Teikoku Data Bank gives a corporation a credit point ranging from 0 to 100, and follows the change of the credit point and the bankruptcy of corporations. Hereafter, "points" in this paper denotes the credit point given by Teikoku Data Bank.

[12] In the following calculations, transition probabilities of the class with more than 76 points in Table 4 are used as those of AAA, AA, and A classes, and those of the class with 66-75 points are used as those of BBB. In addition, transitions between AAA, AA and A classes are ignored.

[13] See footnote 11. "Points" in this paper denotes the credit point given by Teikoku Data Bank.

[14] Falling down under 35 Points is included in this class.

[15] This transition probability matrix is made on the basis of the analysis by Carty and Fons [1994] . They investigate the default probabilities and the transition probabilities using Moody's data.

[16] In order to clarify the effect of downgrading, Case 2 in Table 6 is used as credit spreads for low-rating classes.

[17] Because of the lack of a junk bond market in Japan, the credit spreads of bonds with low credit ratings cannot

be observed. Therefore, each investor must search for the appropriate values on the basis of the transition probability matrix and his or her risk preference.