THE HYPOTHESES UNDERLYING THE PRICING OF OPTIONS: A NOTE ON A PAPER BY BARTELS

Angus S. Macdonald
Department of Actuarial Mathematics and Statistics
Heriot-Watt University
Riccarton
Edinburgh EH14 4AS
United Kingdom

Telephone: +44 131 451 3202
Fax: +44 131 451 3249
e-mail: angus@ma.hw.ac.uk

Abstract

Bartels (1995), following Bergman (1982), stated that derivations of the Black-Scholes PDE which follow the original approach of Black & Scholes (1973) are flawed, in that the time differential of the value of the hedge portfolio omits a term, but that a second error cancels out the first so that the option pricing formula is correct. In this paper we suggest that the error is one of terminology rather than methodology. Changes of value arising from additional investment or consumption should be excluded from rates of return, for example when considering arbitrage, and the original Black-Scholes analysis did employ the correct process. However, the distinction between rate of change of value and rate of return is not always made clear in expositions which follow the PDE approach.

Keywords

Black-Scholes; Self-Financing Portfolios

1 Introduction

Bartels (1995), following Bergman (1982), pointed out that the analysis of Black & Scholes (1973) contained two mathematical errors, which were self-cancelling so that the famous option pricing formulae were nevertheless correct. He also gave a more restrictive condition for the Black-Scholes formulae to hold, namely $r = \mu + \sigma^2/2$. He said:

> "... even many text-books on this subject reproduce the wrong argumentation using erroneously the Black-Scholes hedge portfolio: obviously they missed the fact, that contrary to the claim of Black-Scholes, the change in value of the hedge portfolio in a short time interval is not riskless ..."

and:

> "In studying the existing literature on this subject we find many inconsistencies and I have got the impression that not only "many actuaries find themselves applying a formula which they don't understand properly, and which they have never seen demonstrated" ... but also most of the writers of text-books or of articles on this subject are just in the same unsatisfactory situation."

That the original exposition contained lacunæ is well-known. We revisit the topic for two reasons.

1. Recently Clarkson (1997) cited Bartels (1995) as possible evidence of the inadequacy of the Black-Scholes analysis, in a contribution to the discussion of modern financial mathematics in the U.K. actuarial profession.

2. Bartels (1995) emphasised that the Black-Scholes *formulae* emerged intact once the errors in the original derivation were corrected. We think that this lays emphasis on the wrong point, as do some other recent commentaries on Black & Scholes. In fact, the Black-Scholes *methodology* emerges intact, which is much more important.

2 Deriving the Black-Scholes PDE

Black & Scholes (1973) formed a hedge portfolio containing a fixed number of shares and a varying number of derivatives. We follow what is by now the more usual approach, of holding a fixed number of derivatives and a varying number of shares; the argument is not affected. We use the term 'hedge portfolio' in the original sense just given; many later authors use the term to mean the portfolio containing shares and borrowed cash which replicates the derivative payoff; here we call that the 'replicating portfolio'.

The stock price process S_t is assumed to follow geometric Brownian Motion with constant drift and volatility:

$$dS_t = \mu S_t dt + \sigma S_t dz \tag{1}$$

where z is a standard Brownian motion. (Bartels (1995) used a slightly different parameterisation, but again the argument is not affected.) This is, of course, the original and most basic version of the analysis; these strict assumptions can be relaxed somewhat. Let the price of a European call option with strike price X and exercise time T be denoted f_t; clearly f_t depends on S_t. Assuming that this functional dependence satisfies the conditions for Itô's Lemma to apply, we have:

$$df_t = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu S_t + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S_t^2\right) dt + \frac{\partial f}{\partial S}\sigma S_t dz. \tag{2}$$

Therefore, if we form a hedged portfolio containing one short position in the derivative, and $\partial f/\partial S$ shares, the value of the portfolio, Π_t, follows the process:

$$d\Pi_t = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S_t^2\right) dt \tag{3}$$

which is deterministic. Therefore, the portfolio earns the risk-free rate of return, r, so:

$$\left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S_t^2\right) dt = r\Pi_t dt = r\left(-f + \frac{\partial f}{\partial S}S\right) dt \tag{4}$$

and from these two equations, the Black-Scholes PDE is obtained.

The only difference between this and the original Black-Scholes analysis, is that the latter constructed a portfolio containing one share and a short position in $(\partial f/\partial S)^{-1}$ derivatives.

2.1 Where is the Mistake?

Equation (3) follows from:

$$d\Pi_t = d\left(-f + \frac{\partial f}{\partial S}S\right) = -df_t + \frac{\partial f}{\partial S}dS_t. \tag{5}$$

However, $\partial f/\partial S$ is also a function of time, so the correct differential ought to be:

$$d\Pi_t = -df_t + d\left(\frac{\partial f}{\partial S}S_t\right) \tag{6}$$

which is not the same. (Bartels gave the corresponding expression under the original development.) The essential point is that, in the Black-Scholes analysis, the hedge ratio, $\partial f/\partial S$, or its inverse, is treated as a constant in obtaining $d\Pi_t$, whereas it is not a constant.

Bartels said "The conclusion, that from the absence of arbitrage opportunities it follows, that the rate of change in value of a riskless portfolio equals the riskless interest rate, is also not true, if one does not assume the extra condition of self-financing for the portfolio under consideration." In other words, equation (5) would be correct for a self-financing portfolio, but (from a proposition of Bergman (1982)) the Black-Scholes hedge portfolio is not self-financing.

2.2 Bartels' Alternative Derivation

Bartels offered an alternative derivation of the Black-Scholes formula "... by complete elementary methods avoiding Itô's calculus." The starting point is the assertion that the stock is consistently priced in the sense that (in our notation):

$$\mathrm{E}[S_{t^*}|S_t] = S_t e^{r(t^*-t)} \qquad \text{for arbitrary } t < t^* \tag{7}$$

while, in the case of geometric Brownian Motion:

$$\mathrm{E}[S_{t^*}|S_t] = S_t e^{\mu(t^*-t)} e^{\sigma^2(t^*-t)/2} \tag{8}$$

and equating these leads to the Black-Scholes formula if $r = \mu + \sigma^2/2$. Note that equation (8) is correct under Bartels' parameterisation; under our parameterisation the second exponential would not appear.

3 Why the Mistake Does Not Invalidate Black-Scholes

In this section we discuss why Black & Scholes were right after all, not only in respect of formulae, but also of methodology. The essential point is that, in deciding whether or not a portfolio generated by some trading strategy is riskless, we should consider only that part of the change of value which can be attributed to changes in asset prices, and exclude changes which can be attributed to external financing or consumption. The resulting quantity is exactly that which appears on the right side of equation (5),

1. We agree with Bartels that the Black-Scholes hedge portfolio is not self-financing, although the portfolio which replicates the payoff is.

2. We agree that the rate of change of value of the hedge portfolio is given by equation (6) and not equation (5), but precisely because the hedge portfolio is not self-financing it is not appropriate to use equation (6) in applying the no-arbitrage condition.

3. The only 'error' in the conventional derivation of the Black-Scholes PDE is in describing the differential on the left side of equations (3) and (5) as the rate of change of portfolio value. However, we note that some authors, including Black & Scholes (1973) and Hull (1997) employ a discrete-time analogy which avoids the difficulty, albeit without saying that that is what they are doing.

4. Bartels' alternative derivation simply gives the condition for hedging to give the same price as expectation. Since it makes use of the equivalent martingale measure, it does not avoid the use of Itô calculus. More

important, it lays undue emphasis on the formula, when the important matter is the methodology.

3.1 Self-financing Portfolios and Itô Calculus

The problem is encapsulated in the following question: what is the rate of return on a portfolio which is not self-financing; that is, which is subject to additional investments or consumption at various times? This is what is needed when applying no-arbitrage arguments, and in fact this is what Black & Scholes (1973) used. Unfortunately they described it as a change of relative *value*, which it is not because their hedge portfolio is not self-financing. In other words, the method was right but the terminology was not.

It is perhaps unfortunate that the Itô calculus is so often presented in differential notation, as in equation (1). This disguises the fact that we are dealing, not with differentials, but with integrals. Sample paths of Itô processes are, almost surely, nowhere differentiable; therefore, symbols like dz and dS_t do not carry their conventional meanings. Equation (1), for example, is not to be interpreted as a differential, it is shorthand for the integral equation:

$$S_t = S_0 + \int_0^t \mu S_u du + \int_0^t \sigma S_u dz. \tag{9}$$

in which the second, stochastic, integral is *not* like its deterministic counterpart, despite appearances. See any introduction to SDEs, such as Øksendal (1995), for details.

Here we define self-financing portfolios and ask the question: what is the rate of return on a portfolio which is *not* self-financing?

Suppose, for the sake of argument, we have n assets with prices S_t^i ($i = 1, \ldots, n; t = 0, 1, \ldots, T$). We form a portfolio which at time t contains θ_t^i units of the i^{th} asset, so the the portfolio has value $V_{t^+} = \sum_i \theta_t^i S_t^i$. No trading takes place during each time period, so at time $t+1$ the portfolio has value $V_{t+1^-} = \sum_i \theta_t^i S_{t+1}^i$. Trading then takes place, which may include the investment of new capital or consumption, after which the portfolio contains θ_{t+1}^i units of the i^{th} asset, and has value $V_{t+1^+} = \sum_i \theta_{t+1}^i S_{t+1}^i$.

Changes in the value of the portfolio arise from two sources:

1. changes in asset prices during each time period; and
2. investment or consumption at the start of each time period.

and the value of the portfolio at time t, say, just after investment or consumption, is:

$$V_{t^+} = \sum_i \theta_0^i S_0^i + \sum_{s=0}^{t-1} \sum_i \theta_s^i (S_{s+1}^i - S_s^i) + \sum_{s=0}^{t-1} \sum_i (\theta_{s+1}^i - \theta_s^i) S_{s+1}^i \tag{10}$$

which, after some rearrangement, we can write as:

$$V_{t^+} = \sum_i \theta_0^i S_0^i + \sum_{s=0}^{t-1} \sum_i \theta_s^i \Delta S_s^i + \sum_{s=0}^{t-1} \sum_i S_s^i \Delta\theta_s^i + \sum_{s=0}^{t-1} \sum_i \Delta\theta_s^i \Delta S_s^i. \tag{11}$$

A portfolio is self-financing if the last term in equation (10) is absent, or the last two terms in equation (11). Then we have:

$$V_{t^-} = V_{t^+} = \sum_i \theta_0^i S_t^i + \sum_{s=0}^{t-1} \sum_i \theta_s^i (S_{s+1}^i - S_s^i) = \sum_i \theta_0^i S_t^i + \sum_{s=0}^{t-1} \sum_i \theta_s^i \Delta S_s^i. \tag{12}$$

The answer to the question — what is the rate of return over one time period, of a portfolio which is not necessarily self-financing? — is given by:

$$\text{rate of return} = \frac{\sum_i \theta_s^i (S_{t+1}^i - S_t^i)}{\sum_i \theta_t^i S_t^i} \tag{13}$$

which only depends on the increments of the process on the right side of equation (12), *whether or not the portfolio is self-financing.* If the portfolio is not self-financing, the process does not represent the portfolio value, but it does represent, through its increments, the rate of return with external investment and consumption stripped out. Over more than one time period, questions of (for example) time-weighted returns *versus* money-weighted returns might arise, but over one time period there are no such problems.

Now turn to continuous time, and suppose we have n assets with prices S_t^i $(i = 1, \ldots, n)$. We form a portfolio which at time t contains θ_t^i units if the

i^{th} asset. Let the portfolio have value $V_t = \sum_i \theta_t^i S_t^i$. Applying Itô's Lemma to this, we have:

$$V_t = \sum_i \theta_t^i S_t^i = \sum_i \theta_0^i S_0^i + \sum_i \int_0^t \theta_s^i dS_s^i + \sum_i \int_0^t S_s^i d\theta_s^i + \sum_i \int_0^t d\theta_s^i dS_s^i \tag{14}$$

in exact analogy with equation (11). The portfolio is self-financing if the last two terms are absent, equivalently if there is no additional investment or consumption:

$$V_t = \sum_i \theta_0^i S_0^i + \sum_i \int_0^t \theta_s^i dS_s^i. \tag{15}$$

Now ask the question: what is the instantaneous rate of return at time t, meaning that part of the rate of change of value attributable to changing asset prices? Proceeding from the discrete case to the continuous limit, as in Merton (1990), we see that this depends only on the increments of the process:

$$\sum_i \int_0^t \theta_s^i dS_s^i. \tag{16}$$

whether or not the portfolio is self-financing.

3.2 Application to the Black-Scholes Hedge Portfolio

The Black-Scholes hedge portfolio is not self-financing. This is obvious, since the portfolio contains:

1. an unchanging short position in one derivative; and

2. a changing quantity of shares, which involves sales and/or purchases.

(In the original derivation, it would be the other way round.) That is, the asset price processes are $S_t^1 = f_t$ and $S_t^2 = S_t$, and the portfolio is given by $\theta_t^1 = -1$ and $\theta_t^2 = \partial f / \partial S$. From the discussion above, the rate of return depends on the increments of the process:

$$-\int_0^t df + \int_0^t \frac{\partial f}{\partial S} dS \tag{17}$$

or, in the differential notation, on:

$$-df + \frac{\partial f}{\partial S} dS. \tag{18}$$

and this is correct whether the hedge portfolio is self-financing or not. Bartels is clearly correct in saying that the change in *value* of the Black-Scholes hedge portfolio is given by

$$d\Pi_t = -df_t + d\left(\frac{\partial f}{\partial S} S_t\right)$$

but that is not the relevant quantity when considering rates of return and when applying the no-arbitrage assumption.

We suggest that the most that can be said is that it would be an error to describe the differential (18) as the change of value of the hedge portfolio. We agree with Bartels that this might not always be made clear by authors, although we hesitate to say that this is because authors are unaware of the point. The important point is that the Black-Scholes methodology emerges intact, whether or not the quantity in (18) is always described with sufficient care.

If we examine closely the argument, due to Merton (1990) and cited by Bartels (1995), that led us to conclude that the last two terms of equations (11) or (14) were irrelevant for measuring rates of return, we run into subtleties whose resolution depends on the construction of the Itô integral and its *non-anticipating nature*. If, therefore, the Black-Scholes analysis is presented without a discussion of Itô integrals, which is usual when the author chooses the PDE route, it is understandable if the issue is ducked. We would agree with Bartels (1995) to the extent that this has much potential to mislead.

It is worth noting that Black & Scholes (1973) referred to the "change in the value of the equity [in the position] in a short interval Δt". Hull (1997) refers to the "change in the value of the portfolio in time Δt". It is perhaps unfair to call these errors when they are couched in terms of discrete time intervals Δt and not infinitesimal intervals dt, since that is tantamount to reliance on the discrete-time analogy, where the distinction is immaterial over any one time interval. Willmott *et al.* (1995), who also approach derivatives *via* PDEs, are quite explicit that the hedge ratio is held constant during each small time interval or other terms would arise.

3.3 The Replicating Portfolio and the Martingale Approach

Although the hedge portfolio is not self-financing, the replicating portfolio is. That is, we can construct a portfolio containing $\partial f/\partial S$ shares, and an amount $-\Pi_t$ of cash (strictly, the risk-free asset) which replicates the derivative payoff. This can be shown to be self-financing. Its value satisfies the Black-Scholes PDE, which agrees with Bergman's equivalence theorem cited by Bartels (1995).

A more illuminating approach to the whole problem is to note the existence of a self-financing strategy replicating the payoff using a martingale approach (see for example, Baxter & Rennie (1996, Chapter 3)) and then find by direct computation that it gives the Black-Scholes hedge. This approach is founded explicitly on the representation of gain processes as Itô integrals, and therefore avoids the subtle pitfalls of the PDE approach.

3.4 Bergman's Portfolio

Bartels (1995) gave an example from Bergman (1982), as follows: form a portfolio, not self-financing, with $e^{\Lambda t}/2S_t$ shares and $e^{\Lambda t}/2f_t$ derivatives. Then the portfolio has value $e^{\Lambda t}$, and is riskless, but since Λ can be made arbitrarily large, the return on the portfolio is not necessarily equal to r.

This trading strategy is problematical. With positive probability, the call option expires worthless. In such a case, the investor must purchase unbounded quantities of options as expiry approaches. However, that is beside the point here: in considering the rate of return on the portfolio, it is only the following process that should be considered:

$$\frac{e^{\Lambda t}}{2S_t}dS_t + \frac{e^{\Lambda t}}{2f_t}df_t. \tag{19}$$

We see that, although the portfolio's value is deterministic, the rate of return on it is not, so this is not a riskless portfolio at all. Calling it such suggests confusion between rate of return and relative rate of change of value.

3.5 Bartels' Alternative Derivation

Bartels' alternative derivation arises from the two expectations (7) and (8) The first is the expectation under the equivalent martingale measure, while the latter is the expectation under the 'real world' probabilities; they are not equal unless $r = \mu + \sigma^2/2$. Perhaps matters are clearer if we write explicitly:

$$\mathrm{E}_{\mathbf{Q}}[S_{t^*}|S_t] = S_t e^{r(t^*-t)} \tag{20}$$

and:

$$\mathrm{E}_{\mathbf{P}}[S_{t^*}|S_t] = S_t e^{\mu(t^*-t)} e^{\sigma^2(t^*-t)/2} \tag{21}$$

where $\mathbf{P}$ is the 'real-world' measure and $\mathbf{Q}$ is the equivalent martingale measure. In general these are different, and cannot be equated. An analagous error would be to note that any two Poisson distributions (on the same sample space) are equivalent probability measures, in the same sense as that in which $\mathbf{P}$ and $\mathbf{Q}$ are equivalent probability measures, and to suppose that their expected values are therefore equal.

Of course, if by chance Bartels' relation $r = \mu + \sigma^2/2$ did hold (under his parameterisation) then the Black-Scholes formula would be obtained under either of two methods; pricing under $\mathbf{Q}$ or pricing under $\mathbf{P}$. In the first case it is assumed that hedging takes place, in the second that hedging does not take place. However, if hedging takes place the price given by $\mathbf{Q}$ is obtained regardless of the drift of $\mathbf{P}$, so the relation is no more than the necessary condition for an actuarial risk-taker to calculate the same net price as a Black-Scholes hedger.

Such 'proofs' of the Black-Scholes formula, purporting to avoid the use of Itô calculus, miss the point — it is the methodology which matters, not the formula. (In this instance, note that equation (20) depends on the construction of the equivalent martingale measure, in which Itô's calculus is needed.) That is why it is important to restate that it is not simply the *formula* but the *methodology* which emerges intact when the lacunæ of the original exposition are addressed.

4 Comments

This paper would not have been written but for the fact that, after much argument, the actuarial profession in the U.K. is on the brink of accepting the need to catch up with modern financial mathematics. It is curious that, in a supposedly mathematical profession, much of the argument should have involved the mathematics itself, especially the use of stochastic calculus (Clarkson, 1997) which one would have thought could have been examined fairly objectively. In fairness, it is hard for the actuary who has not already begun to study the modern mathematics of finance to assess the worth of doing so, but it does seem to us that recourse to the judicial approach — weighing up opposing arguments rather than examining them forensically — has proved more tempting than recourse to the textbooks among substantial numbers of actuaries. Unfortunately, there is no saying whether correct or incorrect mathematics will weigh more heavily in scales like these. Opposing views there certainly are, and Bartels (1995) has been cited by opponents of modern financial mathematics. That is our reason for revisiting the question here.

Finally, if Black & Scholes were ever in error, we can only wish that our own errors should be as good as theirs!

References

BARTELS, H.-J. (1995). The hypotheses underlying the pricing of options. *Proceedings of the 5th International AFIR Colloquium, Brussels*, **1**, 3–15.

BAXTER, M. & RENNIE, A. (1996). *Financial calculus.* Cambridge University Press.

BERGMAN, Y.Z. (1982). *Pricing of contingent claims in perfect and imperfect markets.* Ph.D. thesis, University of California, Berkeley.

BLACK, F. & SCHOLES, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, **81**, 637–659.

CLARKSON, R.S. (1997). An actuarial theory of option pricing. To appear in *British Actuarial Journal*, **3**.

HULL, J.C. (1997). *Options, futures and other derivatives (3rd edition).* Prentice-Hall.

MERTON, R.C. (1990). *Continuous-time finance.* Blackwell, Cambridge MA & Oxford U.K..

ØKSENDAL, B. (1995). *Stochastic differential equations (4th edition).* Springer-Verlag, Berlin & Heidelberg.

WILLMOTT, P., HOWISON, S. & DEWYNNE, J. (1995). *The mathematics of financial derivatives.* Cambridge University Press.