

ASSET ALLOCATION

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We start with asset allocation and use this to present an interesting theorem. We get a simple expression for the efficient frontier and we introduce a new way of setting the investment strategy.

KEYWORDS: Efficient frontier, minimum variance portfolio, Poisson distribution, positive definit matrix, expected return and variance of the portfolio, Lagrangian, global optimum, return constraint, redistribution portfolio, and we conclude how to choose the optimum portfolio given a probability of making the return constraint.

Asset allocation

We start with asset allocation and use this to present an interesting theorem. We get a simple expression for the efficient frontier and we introduce a new way to settle the investment strategy.

Introduction

We consider an amount of money, which we are going to invest during a period. This time period could be a day, a week, a month etc. We normalise and define the wealth's value to be equal 1. The investments goal is high return with an acceptable risk. Risk is measured by variance.

The article will presents an expression for the efficient portfolio and the efficient frontier. This is the frontier that gives us the lowest risk of all portfolios with the same rate of expected return. We get this expression through the minimum variance portfolio. For the presentation of the results we use the same technique as Keel and Muller did in Astin Bulletin, vol 25. No 1.1995. The results we present will easily be proved and the readers will also easily understand the application presented.

The efficient portfolio is a function of the minimum variance portfolio and an extra component to make the return constraint. This component is not a real portfolio and we call it the redistribution portfolio. The fascinating result is that this portfolio's expectation is equal to its variance. Until now, the Poisson distribution is the only distribution that has this property. An advantage of this quality is that it is very simple to find the parabola of the efficient frontier.

We take it as given that the investments alternatives are known, and that they have a finite expectation and covariance. The estimation criterias are of no interest for this article.

Definitions

We let for $i, j = 1, 2, \dots, n$:

R_i = "The return or the increase of wealth for investment alternative i , during the considered period."

$$E(R_i) = \mu_i$$

$$\text{Cov}(R_i, R_j) = V_{ij}$$

Let x_i be the amount which we invest in alternative i . We have $x_1 + x_2 + \dots + x_n = 1$. We introduce the unit vector \mathbf{e} ; $\mathbf{e}' = (1, 1, \dots, 1)$ and multiplies with the vector $\mathbf{x}' = (x_1, x_2, \dots, x_n)$;

$$\mathbf{e}'\mathbf{x} = \mathbf{x}'\mathbf{e} = 1.$$

We define further \mathbf{V} as the matrix of covariates between the investments alternatives:

$$\mathbf{V} = \begin{array}{c} \text{Cov}(R_1, R_1), \text{Cov}(R_1, R_2), \dots, \text{Cov}(R_1, R_n) \\ \text{Cov}(R_2, R_1), \text{Cov}(R_2, R_2), \dots, \text{Cov}(R_2, R_n) \\ \dots \\ \text{Cov}(R_n, R_1), \text{Cov}(R_n, R_2), \dots, \text{Cov}(R_n, R_n) \end{array}$$

\mathbf{V} is symmetric, and we suppose that it is positive definit. In a positive definit matrix, the determinant to each principal submatrix is positiv. (Howard Antons 'Elementary Linear algebra' 1991). If not, two lines are linear dependant and we simply remove one of them. This does not make any restrictions for further studies.

Since \mathbf{V} is positive definit, it means that for all \mathbf{x} , with at least one element in the vector not equal to 0: $\mathbf{x}'\mathbf{V}\mathbf{x} > 0$.

We have $ER_i = \mu_i$. We define $\boldsymbol{\mu}' = (\mu_1, \mu_2, \dots, \mu_n)$, so the expected return of the portfolio is given by;

$$r = E(x_1R_1 + x_2R_2 + \dots + x_nR_n) = x_1\mu_1 + x_2\mu_2 + \dots + x_n\mu_n = \boldsymbol{\mu}'\mathbf{x} = \mathbf{x}'\boldsymbol{\mu}.$$

The variance of the portfolio: $\text{var } R(\mathbf{x}) = \mathbf{x}'\mathbf{V}\mathbf{x}$.

We now have enough information to calculate each portfolio's expected return and variance.

We showed above that since \mathbf{V} is positive definit, it follows that $\text{var } R(\mathbf{x})$ is non-negative. We seek the \mathbf{x} -vector which gives the lowest variance of all the possible \mathbf{x} -vectors with the same expected return.

Optimising with Lagrangian.

We wish to find the \mathbf{x} -vector which solve this problem;

$$\min_{\mathbf{x}} \mathbf{x}'\mathbf{V}\mathbf{x}$$

Given:

$$\begin{aligned} \mathbf{x}'\mathbf{e} &= \mathbf{e}'\mathbf{x} = 1 \\ \mathbf{x}'\boldsymbol{\mu} &= \boldsymbol{\mu}'\mathbf{x} = r \end{aligned}$$

We use the Lagrangian multiplication. Since \mathbf{V} is positive definite this ensures that the Lagrangian gives a global optimum. The equation we are going to solve is as follows:

$$1) \quad L = \mathbf{x}'\mathbf{V}\mathbf{x} - 2\beta_1(\mathbf{x}'\boldsymbol{\mu} - r) - 2\beta_2(\mathbf{x}'\mathbf{e} - 1)$$

Given $\mathbf{x}'\boldsymbol{\mu} = r$ and $\mathbf{x}'\mathbf{e} = 1$.

We differentiate L , and we get a $(n+1)$ matrix, or a vector.

$$1') \quad \frac{dL}{d\mathbf{x}} = 2\mathbf{V}\mathbf{x} - 2\beta_1\boldsymbol{\mu} - 2\beta_2\mathbf{e} = \mathbf{0}$$

So we have a set of three equations:

$$a) \quad \mathbf{V}\mathbf{x} - \beta_1\boldsymbol{\mu} - \beta_2\mathbf{e} = \mathbf{0}$$

$$b) \quad \mathbf{x}'\boldsymbol{\mu} = r$$

$$c) \quad \mathbf{x}'\mathbf{e} = 1$$

In these $n+2$ equations we have $n+2$ unknowns.

The minimal variance portfolio

We shall find the portfolio which gives the lowest risk of all possible portfolios. We are therefore here not interested in the return constraint.

We solve:

$$a1) \quad \mathbf{V}\mathbf{x}_{\min} - \beta_2'\mathbf{e} = \mathbf{0}$$

$$c1) \quad \mathbf{e}'\mathbf{x}_{\min} = 1$$

Since \mathbf{V} is positive definite, exist \mathbf{V}^{-1} so :

$$c2) \quad \mathbf{e}'\mathbf{V}^{-1}\mathbf{V}\mathbf{x}_{\min} = 1$$

We right-multiply $\mathbf{e}'\mathbf{V}^{-1}$ into a1). This gives :

$$a2) \quad \mathbf{e}'\mathbf{V}^{-1}\mathbf{V}\mathbf{x}_{\min} - \beta_2'\mathbf{e}'\mathbf{V}^{-1}\mathbf{e} = 0$$

which gives:

$$a3) \quad 1 = \beta_2'\mathbf{e}'\mathbf{V}^{-1}\mathbf{e}$$

And we get:

$$a3') \quad \beta_2' = (\mathbf{e}'\mathbf{V}^{-1}\mathbf{e})^{-1}$$

It follows from a1) that :

$$a4) \quad \mathbf{x}_{\min} = \frac{\mathbf{V}^{-1}\mathbf{e}}{\mathbf{e}'\mathbf{V}^{-1}\mathbf{e}}$$

$$ER(\mathbf{x}_{\min}) = \boldsymbol{\mu}'\mathbf{x}_{\min}$$

$$\text{Var } R(\mathbf{x}_{\min}) = \mathbf{x}_{\min}'\mathbf{V}\mathbf{x}_{\min} = \frac{1}{(\mathbf{e}'\mathbf{V}^{-1}\mathbf{e})^2} \mathbf{e}'\mathbf{V}^{-1}\mathbf{V}\mathbf{V}^{-1}\mathbf{e} = \frac{1}{(\mathbf{e}'\mathbf{V}^{-1}\mathbf{e})} = \beta^2_2$$

The portfolio with respect to the return constraint.

We shall find: $\mathbf{x}_{\text{opt}} = \mathbf{x}_{\min} + \beta_1 \mathbf{z}^*$. We put this into equation a), b) and c) and find:

$$a) \quad \mathbf{V}(\mathbf{x}_{\min} + \beta_1 \mathbf{z}^*) - \beta_1 \boldsymbol{\mu} - \beta_2 \mathbf{e} = \mathbf{0}$$

$$b) \quad \boldsymbol{\mu}'(\mathbf{x}_{\min} + \beta_1 \mathbf{z}^*) = r$$

$$c) \quad \mathbf{e}'(\mathbf{x}_{\min} + \beta_1 \mathbf{z}^*) = 1$$

Equation b) gives:

$$\beta_1 = \frac{r - \boldsymbol{\mu}'\mathbf{x}_{\min}}{\boldsymbol{\mu}'\mathbf{z}^*} = r \frac{ER(\mathbf{x}_{\min})}{ER(\mathbf{z}^*)}$$

Equation c) gives $\mathbf{e}'\mathbf{x}_{\min} + \beta_1 \mathbf{e}'\mathbf{z}^* = 1$. Since \mathbf{x}_{\min} is a portfolio: $\mathbf{e}'\mathbf{x}_{\min} = 1$, so $\mathbf{e}'\mathbf{z}^* = 0$. Out of this, we see that \mathbf{z}^* is not a real portfolio but a redistribution of the minimal-variance portfolio.

From a):

$$d) \quad \mathbf{V}(\mathbf{x}_{\min} + \beta_1 \mathbf{z}^*) = \beta_1 \boldsymbol{\mu} + \beta_2 \mathbf{e}$$

$$e) \quad \mathbf{z}^* = \frac{\mathbf{V}^{-1}(\beta_1 \boldsymbol{\mu} + \beta_2 \mathbf{e} - \mathbf{V}\mathbf{x}_{\min})}{\beta_1}$$

We right-multiplies e) with $\beta_1 \mathbf{e}'$:

$$f) \quad \beta_1 \mathbf{e}'\mathbf{z}^* = 0 = \beta_1 \mathbf{e}'\mathbf{V}^{-1}\boldsymbol{\mu} + \beta_2 \mathbf{e}'\mathbf{V}^{-1}\mathbf{e} - \mathbf{e}'\mathbf{V}^{-1}\mathbf{V}\mathbf{x}_{\min}$$

Since $\mathbf{e}'\mathbf{x}_{\min} = 1$:

$$g) \quad -\beta_1 \mathbf{e}'\mathbf{V}^{-1}\boldsymbol{\mu} = \beta_2 \mathbf{e}'\mathbf{V}^{-1}\mathbf{e} - 1$$

From e) and solve for \mathbf{x}_{\min} we find that \mathbf{z}^* :

$$\begin{aligned} \text{h) } \mathbf{z}^* &= \frac{\mathbf{V}^{-1}}{\beta_1} (\beta_1 \boldsymbol{\mu} + \beta_2 \mathbf{e} - \frac{\mathbf{V} \mathbf{V}^{-1} \mathbf{e}}{\mathbf{e}' \mathbf{V}^{-1} \mathbf{e}}) \\ \text{i) } &= \mathbf{V}^{-1} (\boldsymbol{\mu} + \frac{1}{\beta_1} (-\beta_1 \mathbf{e}' \mathbf{V}^{-1} \boldsymbol{\mu}) \mathbf{e}) \end{aligned}$$

We see that \mathbf{z}^* is independent of r and given by:

$$\mathbf{z}^* = \mathbf{V}^{-1} (\boldsymbol{\mu} - \frac{\mathbf{e}' \mathbf{V}^{-1} \boldsymbol{\mu}}{\mathbf{e}' \mathbf{V}^{-1} \mathbf{e}} \mathbf{e})$$

Since $\mathbf{x}_{\min} = \frac{\mathbf{V}^{-1} \mathbf{e}}{\mathbf{e}' \mathbf{V}^{-1} \mathbf{e}}$ gives that $\mathbf{x}'_{\min} = \frac{\mathbf{e}' \mathbf{V}^{-1}}{\mathbf{e}' \mathbf{V}^{-1} \mathbf{e}}$ and $(\mathbf{V}^{-1})' = \mathbf{V}^{-1}$

The result is $\mathbf{z}^* = \mathbf{V}^{-1} (\boldsymbol{\mu} - \text{ER}(\mathbf{x}_{\min}) \mathbf{e})$.

Theorem:

Let \mathbf{z}^* be the redistribution portfolio, then

$$\text{E R}(\mathbf{z}^*) = \text{var R}(\mathbf{z}^*)$$

Prove:

We right-multiplies with $(\mathbf{z}^*)' \mathbf{V}$:

$$\begin{aligned} (\mathbf{z}^*)' \mathbf{V} \mathbf{z}^* &= (\mathbf{z}^*)' \mathbf{V} \mathbf{V}^{-1} (\boldsymbol{\mu} - \text{ER}(\mathbf{x}_{\min}) \mathbf{e}) = (\mathbf{z}^*)' (\boldsymbol{\mu} - \text{ER}(\mathbf{x}_{\min}) \mathbf{e}) \\ &= \text{ER}(\mathbf{z}^*) - \text{ER}(\mathbf{x}_{\min}) (\mathbf{z}^*)' \mathbf{e} = \text{ER}(\mathbf{z}^*) - 0 \end{aligned}$$

Therefore: $\text{var R}(\mathbf{z}^*) = (\mathbf{z}^*)' \mathbf{V} \mathbf{z}^* = \text{ER}(\mathbf{z}^*)$, and the prove is complete.

QED.

A directly result is that the expected return of the redistribution-portfolio is positive.

Properties of the optimum portfolio.

We have found the optimum investment strategy which generates the efficient frontier:

$$\text{j) } \mathbf{x}_{\text{opt}}(r) = \mathbf{x}_{\min} + \beta_1 \mathbf{z}^*$$

$$j') \quad = \mathbf{x}_{\min} + \frac{(r - ER(\mathbf{x}_{\min}))}{ER(\mathbf{z}^*)} \mathbf{z}^*$$

We confirm that:

$$k) \quad ER(\mathbf{x}_{\text{opt}}(r)) = ER(\mathbf{x}_{\min}) + \frac{(r - ER(\mathbf{x}_{\min}))}{ER(\mathbf{z}^*)} ER(\mathbf{z}^*)$$

$$k') \quad = ER(\mathbf{x}_{\min}) + (r - ER(\mathbf{x}_{\min})) = r.$$

We shall now find the variance to $R(\mathbf{x}_{\text{opt}})$, so we need $\text{Cov}(R(\mathbf{x}_{\min}), R(\mathbf{z}^*))$, which is given by:

$$l) \quad \text{Cov}(R(\mathbf{x}_{\min}), R(\mathbf{z}^*)) = \mathbf{z}^{*'} \mathbf{V} \mathbf{x}_{\min}$$

$$l') \quad = \mathbf{z}^{*'} \mathbf{V} \mathbf{V}^{-1} \mathbf{e} = \mathbf{z}^{*'} \mathbf{e} =$$

$$0 \quad \frac{\mathbf{e}^* \mathbf{V}^{-1} \mathbf{e}}{\mathbf{e}^* \mathbf{V}^{-1} \mathbf{e}}$$

The covariance between the minimal variance portfolio and the redistribution portfolio equals 0, independent of the return constraint. This makes sense, since they are both independent of r .

This gives us ($\beta_1 = \beta_1(r)$):

$$m) \quad \text{var } R(\mathbf{x}_{\text{opt}}(r)) = \text{var } R(\mathbf{x}_{\min}) + \text{var } R(\beta_1(r) \mathbf{z}^*)$$

$$m') \quad = \text{var } R(\mathbf{x}_{\min}) + (r - ER(\mathbf{x}_{\min}))^2 \text{var}$$

$$R(\mathbf{z}^*) \quad \frac{(ER(\mathbf{z}^*))^2}{ER(\mathbf{z}^*)}$$

$$n) \quad = \text{var } R(\mathbf{x}_{\min}) + \frac{(r - ER(\mathbf{x}_{\min}))^2}{ER(\mathbf{z}^*)}$$

The last transit follows from the theorem, and of course $\text{var } R(\mathbf{x}_{\text{opt}}(r))$ is non-negative. To describe the efficient portfolio, we prefer the standard deviation (σ), instead of variance:

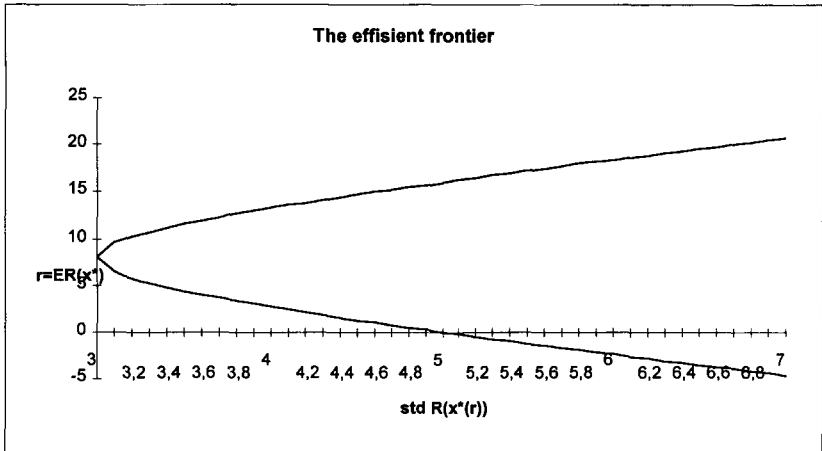
$$l) \quad \sigma(R(\mathbf{x}_{\text{opt}}(r))) = \sqrt{\text{var } R(\mathbf{x}_{\min}) + \frac{(r - ER(\mathbf{x}_{\min}))^2}{ER(\mathbf{z}^*)}}$$

If we solve l) with respect to r we find that:

$$m) \quad r = ER(\mathbf{x}_{\min}) \pm \sqrt{ER(\mathbf{z}^*) [\sigma(R(\mathbf{x}_{\text{opt}}))^2 - \text{var } R(\mathbf{x}_{\min})]}$$

We have now showed the equation for the parabola, and it easily shows the symmetry around

$r = ER(x_{\min})$. If we choose values: i.e. with $\text{var } R(x_{\min}) = 9$, $ER(x_{\min}) = 8$ and $ER(z^*) = 4$, then we have a graph, which is the efficient frontier.



It can easily be shown that the set which is bounded by the graph is a convex set. This is obvious because we have a parabola.

The theory doesn't say which risk- or return- level we should choose. It is normal to use utility theory for this purpose. We introduce a new way to find this point after the example.

Example

$$\mu = (3, 5)' \quad V = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \quad V^{-1} = \frac{2/5 \ 1/5}{1/5 \ 3/5} = 1/5 \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

V is positive definite because the determinant to the first principal = 3, and the second principal = 6 - 5 = 1, both are positive.

$$x_{\min} = \frac{V^{-1}e}{e'V^{-1}e}$$

$$V^{-1}e = 1/5 \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1/5 \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \text{This gives } e'V^{-1}e = 7/5.$$

And $x_{\min} = (3/7, 4/7)'$. We confirm that $x_1 + x_2 = 3/7 + 4/7 = 1$.

$$\text{Var } R(\mathbf{x}_{\min}) = (\mathbf{e}' \mathbf{V}^{-1} \mathbf{e})^{-1} = 5/7, \text{ with expectation } ER(\mathbf{x}_{\min}) = 1/7 \quad (3,5) \quad \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 29/7.$$

$$\mathbf{z}^* = \mathbf{V}^{-1}(\boldsymbol{\mu} - 29/7 \mathbf{e}) = \mathbf{V}^{-1} (1/7) \quad \begin{bmatrix} -8 \\ 6 \end{bmatrix} = 10/35 \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

We see that $z_1 + z_2 = 0$.

$$\text{The theorem says: } \text{var } R(\mathbf{z}^*) = ER(\mathbf{z}^*) = \boldsymbol{\mu}' \mathbf{z}^* = (10/35) (3,5) \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 4/7.$$

And we have the results which we need to calculate the efficient portfolio.

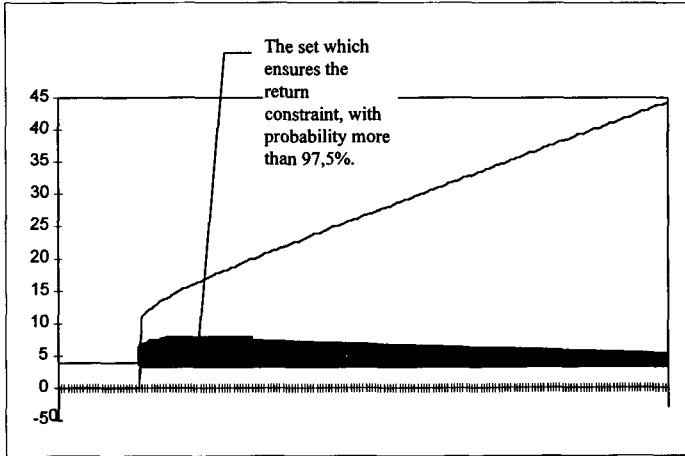
Investments with respect to the return constraint.

If we assume that the stochastic return could be described by a normal distribution, then the probability of making the return constraint can easily be found.

It is well-known that if Y_i is normally distributed, then approximately 95% of the observed Y_1, Y_2, \dots, Y_n will be found within $EY_i \pm 2 \text{std } Y_i$. If we use this technique, a company will make the return constraint A, with probability 97,5% with all the portfolios which solve the inequality:

$$ER(\mathbf{x}_{\min}) + \sqrt{ER(\mathbf{z}^*) [P(R(\mathbf{x}_{\text{opt}}(t)))^2 - \text{var}R(\mathbf{x}_{\min})]} - 2P(R(\mathbf{x}_{\text{opt}}(t))) > A$$

If an insurance company has a 4% return constraint, we could then produce a new graph, which solves and shows this inequality. It gives us a new and interesting way to settle the main strategy.



Let us find the optimum which gives us the highest possible return, with respect to a given probability of making the return constraint. This means mathematically that we have to solve the inequality:

$$n) ER(\mathbf{x}_{\min}) + \sqrt{ER(\mathbf{z}^*)[P(R(\mathbf{x}_{\text{opt}}(r)))^2 - \text{var}R(\mathbf{x}_{\min})]} - k \cdot P(R(\mathbf{x}_{\text{opt}}(r))) > A$$

With respect to $P(R(\mathbf{x}_{\text{opt}}(r)))$.

This gives us the optimum $P_{\text{opt}}(R(\mathbf{x}_{\text{opt}}(r)))$:

$$o) P_{\text{opt}}(R(\mathbf{x}_{\text{opt}}(r))) = k \sqrt{\frac{\text{var} R(\mathbf{x}_{\min})}{k^2 - ER(\mathbf{z}^*)}}$$

The choice of k , gives the selected probability to reach the return constraint. We see that the expression demands that $k^2 - ER(\mathbf{z}^*) > 0$.

The same theory, is valid for other assumed distributions of the return.