Risk/Arbitrage Strategies: A New Concept for Asset/Liability Management, Optimal Fund Design and Optimal Portfolio Selection in a Dynamic, Continuous-Time Framework
Part I: Securities Markets

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Abstract. Asset/Liability management, optimal fund design and optimal portfolio selection have been key issues of interest to the (re)insurance and investment banking communities, respectively, for some years - especially in the design of advanced risk-transfer solutions for clients in the Fortune 500 group of companies. AFIR 1996 publications dealing with these topics were, e.g., *Optimal Fund Design for Investors with Holding Constraints* (Vol. I, p. 245), *Optimal Portfolio and Optimal Trading in a Dynamic Continuous Time Framework* (Vol. I, p. 275), *Mean-Variance Portfolio Selection under Portfolio Insurance* (Vol. I, p. 347), *Baseline for Exchange Rate - Risks of an International Reinsurer* (Vol. I., p. 395), *Optimizing Investment and Contribution Policies of a Defined Benefit Pension Fund* (Vol. I, p. 593), *Continuous-Time Pension Fund Modelling* (Vol. I, p. 609), *Linear Approach for Solving Large-Scale Portfolio Optimization Problems in a Lognormal Market* (Vol. II, p. 1019), *Options as an Asset Class* (Vol. II, p. 1413), *Optioned Portfolios: The Trade-off between Expected and Guaranteed Returns* (Vol. II, p. 1443), *Optimal Optioned Portfolios with Confidence Limits on Shortfall Constraints* (Vol. II, p. 1497), among others. Taking up some of the new ideas and approaches in this literature we introduce the concept of ***limited risk arbitrage investment management in a general diffusion type securities and derivatives market*** and characterize the corresponding trading and portfolio management strategies (***risk/arbitrage strategies***) as the solutions of a stochastic control problem with constraints on instantaneous investment risk, future portfolio risk dynamics, portfolio time decay dynamics and portfolio value appreciation dynamics. ***Part I: Securities Markets*** presents the necessary mathematical framework (optimal control of Markov diffusion processes in R^n with dynamic programming and continuous-time martingale representation techniques) and lays the foundation for considering financial options as a non-redundant (i.e., securities market completing) asset class in investment management decisions.

Keywords. Limited risk arbitrage investment management, instantaneous investment risk, future risk dynamics, value appreciation dynamics, risk/arbitrage strategies, market parametrization, maximum principle, convex duality, Markovian characterization, market prices of risk, completion premium.

Contents.

1. Introduction

Risk/Arbitrage strategies are quantitative trading or portfolio management strategies in both the traditional bond and stock markets as well as the fast growing and highly competitive derivatives markets that guarantee a limited risk exposure over the entire investment horizon and at the same time achieve a maximum rate of portfolio value appreciation over each individual trading period. In their most general form they are the solutions of the stochastic control problem

$$\max_{(c,\theta)\in A(v)} E[\int_0^H U^c(t,c(t))dt + U^V(V_v^{c\theta}(H))]$$
$$E[\int_0^H \varsigma(t)c(t)dt + \varsigma(H)V_v^{c\theta}(H)] \le v \qquad (1.1)$$

where over the entire investment horizon $[0,H]$ both instantaneous investment risk and future risk dynamics are reduced to values within a given tolerance band, i.e.,

$$|v(t)^T \Delta(t)| \le \delta \text{ and } |v(t)^T \Gamma(t)| \le \gamma\,,\ 0 \le t \le H\,, \qquad (1.2)$$

while at the same time the rate at which the (securities and derivatives) portfolio appreciates in value is greater than a set target rate, i.e.,

$$v(t)^T \Lambda(t) \ge \lambda\,,\ 0 \le t \le H\,. \qquad (1.3)$$

Risk/Arbitrage strategies thus generalize the recently developed securities markets investment management (i.e., optimal fund design) and asset allocation methodologies (based on martingale representation techniques) to include the derivatives markets and therefore highly geared and intrinsically risky assets. The focus is on achieving an efficient allocation of risk in a portfolio context rather than eliminating or reducing derivatives risk exposure on a *single-instrument basis by replication (hedging) with underlying securities.* This new approach to structured portfolio management therefore takes the important role which options play in modern (re)insurance and corporate and investment banking applications as well as in the equilibrium theory of financial markets (note: in a Pareto efficient equilibrium contingent claims are real-world substitutes for Arrow securities) into consideration.

2. Securities Markets, Trading and Portfolio Management

Complete Securities Markets. Following Karatzas [1] and Cox and Huang [2] we shall first assume a securities market in equilibrium with no arbitrage opportunities, dynamic completeness and a diffusion price process for the basic securities - bonds and stocks - available to investors. For a justification and characterization of these assumptions within the microeconomic theory of securities markets and continuous trading see Huang [3], Duffie [4], Duffie and Huang [5], Krasa and Werner [6], Huang [7], Karatzas, Lehoczky and Shreve [8], Hindy and Huang [9], Foellmer and Schweizer [10], Madan and Milne [11], Karatzas, Lehoczky and Shreve [12], Dybvig and Huang [13], Heath and Jarrow [14], Jarrow and Madan [15], Chatelain and Stricker [16], Stricker [17], Delbaen [18], Lakner [19], Amin and Jarrow [20] and Cheng [21]. Note that in addition to cash we consider two classes of risky assets - the bond prices are as in Heath, Jarrow and Morton [22]. Specifically, an investment in the money market account (cash) appreciates in value over time according to the linear stochastic differential equation

$$dC(t) = C(t)r(t)dt\,,\ C(0) = c\,, \qquad (2.1)$$

while the security price processes (bonds and stocks) are

$$dP_i(t) = P_i(t)[\mu_i(t)dt + \sum_{j=1}^{N} \sigma_{ij}(t)dW_j(t)], \; P_i(0) = p_i. \quad (2.2)$$

The risk-free interest rate $r(t)$ applicable to the money market account and the return rate vector $\mu(t)$ and volatility matrix $\sigma(t)$ of the N-dimensional asset price process are progressively measurable and uniformly bounded (in time and state) stochastic processes on a filtered (augmented) probability space (Ω, Φ, π, F), $F = \{F_t\}_{t\geq 0}$, on which the adapted N-dimensional Wiener process $W(t)$ describing the exogenous sources of market uncertainty is defined. In addition, the covariance matrix $\kappa(t) = \sigma(t)\sigma(t)^T$ of the securities price process is strongly non-degenerate, i.e., there is an $\varepsilon > 0$ such that for all $x \in R^N$,

$$x^T\kappa(t)x \geq \varepsilon\|x\|^2, \quad (2.3)$$

which guarantees existence and boundedness of the inverse matrices $\sigma(t)^{-1}$ and $[\sigma(t)^T]^{-1}$ and therefore allows us to introduce a unique equivalent martingale measure $\tilde{\pi}$ for the price processes of the stocks and bonds in the market, given some investment horizon H [with $F_H = \Phi$]. This Girsanov transformation of probability measure

$$\tilde{\pi}(\varphi) = E[\beta(H)1_\varphi], \; \varphi \in \Phi, \text{ and } \tilde{W}(t) = W(t) + \int_0^t \alpha(s)ds \quad (2.4)$$

[on (Ω, Φ, F)] involves the measurable, bounded and adapted process

$$\alpha(t) = \sigma(t)^{-1}[\mu(t) - r(t)1_N] \quad (2.5)$$

and the exponential martingale

$$\beta(t) = \exp(-\int_0^t \alpha(s)^T dW(s) - \frac{1}{2}\int_0^t \|\alpha(s)\|^2 ds)$$

$$E[\exp(\frac{1}{2}\int_0^t \|\alpha(s)\|^2 ds)] < \infty \qquad d\beta(t) = -\beta(t)\alpha(t)^T dW(t), \; \beta(0) = 1 \quad (2.6)$$

and represents the security price processes in the form

$$dP_i(t) = P_i(t)[r(t)dt + \sum_{j=1}^{N} \sigma_{ij}(t)d\tilde{W}_j(t)] \quad (2.7)$$

or after integration

$$\begin{aligned} P_i(t) &= p_i \exp\left(\int_0^t [\mu_i(s) - \frac{1}{2}\sum_{j=1}^{N}\sigma_{ij}(s)^2]ds + \sum_{j=1}^{N}\int_0^t \sigma_{ij}(s)dW_j(s)\right) \\ &= p_i \exp\left(\int_0^t [r(s) - \frac{1}{2}\sum_{j=1}^{N}\sigma_{ij}(s)^2]ds + \sum_{j=1}^{N}\int_0^t \sigma_{ij}(s)d\tilde{W}_j(s)\right) \end{aligned} \quad (2.8)$$

with the adapted Wiener process $\tilde{W}(t)$ on the risk-neutral probability space $(\Omega, \Phi, \tilde{\pi}, F)$, $F = \{F_t\}_{0\leq t\leq H}$. Furthermore, it makes them martingales after discounting at the money market rate, i.e.,

$$d[D(t)P_i(t)] = D(t)P_i(t)\sigma_{i\cdot}(t)d\tilde{W}(t)$$

$$D(t)P_i(t) = p_i \exp(\int_0^t \sigma_{i\cdot}(s)d\tilde{W}(s) - \frac{1}{2}\int_0^t \|\sigma_{i\cdot}(s)\|^2 ds) \qquad \tilde{E}[\exp(\frac{1}{2}\int_0^t \|\sigma_{i\cdot}(s)\|^2 ds)] < \infty \quad (2.9)$$

where the corresponding term structure of discount factors is

$$D(t) = \exp(-\int_0^t r(s)ds) \quad (2.10)$$

[$dD(t) = -D(t)r(t)dt$, $D(0) = 1$] and discounting follows the general rule

$$dX(t) = X(t)[a(t)dt + b(t)^T dB(t)]$$
$$d[D(t)X(t)] = D(t)X(t)[(a(t) - r(t))dt + b(t)^T dB(t)] \quad (2.11)$$

$[B(t) = W(t)$ or $B(t) = \widetilde{W}(t)]$. Note in this context (Brenner and Denny [23]) that if in the above representation

$$P_i(t) = p_i \exp(\int_0^t \mu_i(s)ds) X_i(t) \qquad \begin{aligned} X_i(t) &= \exp(\int_0^t \sigma_{i.}(s)dW(s) - \frac{1}{2}\int_0^t \|\sigma_{i.}(s)\|^2 ds) \\ &= 1 + \int_0^t X_i(s)\sigma_{i.}(s)dW(s) \end{aligned} \quad (2.12)$$

of the asset price processes the factor $X_i(t)$ is (in a Markovian setting) a general positive (continuous local martingale) solution of the stochastic integral equation

$$X_i(t) = 1 + \int_0^t X_i(s)\sigma_{i.}(s, X(s))dW(s) \quad (2.13)$$

$[dX_i(t) = X_i(t)\sigma_{i.}(t, X(t))dW(t),\ X_i(0) = 1]$, then an application of Ito's formula yields

$$dP_i(t) = P_i(t)[\mu_i(t)dt + \sigma_{i.}(t, \begin{bmatrix} \frac{1}{p_1}\exp(-\int_0^t \mu_1(s)ds)P_1(t) \\ \vdots \\ \frac{1}{p_N}\exp(-\int_0^t \mu_N(s)ds)P_N(t) \end{bmatrix})dW(t)] \quad (2.14)$$

and therefore the values of contingent claims will (unlike in the well-known Black & Scholes option pricing framework) in general depend on the return rate process $\mu(t)$ of the underlying securities.

Bond Markets. If $B(t,T)$ is the price of a default-free discount bond with maturity $T \le H$, then the instantaneous forward rate $f(t,T)$ at time t for a corresponding investment beginning at time $T > t$ is

$$f(t,T) = -\frac{\partial \log(B(t,T))}{\partial T} \quad (2.15)$$

and the associated spot rate is $r(t) = f(t,t)$. Forward rates in a financial market evolve according to the differential equation

$$df(t,T) = a(t,T)dt + \sum_{j=1}^N b_j(t,T)dW_j(t) \quad (2.16)$$

which under appropriate integrability conditions satisfied by its coefficients $a(t,T)$ and $b_j(t,T)$ (see Heath, Jarrow and Morton [22]) leads to money market rates

$$r(t) = f(0,t) + \int_0^t a(s,t)ds + \sum_{j=1}^N \int_0^t b_j(s,t)dW_j(s) \quad (2.17)$$

and (not necessarily Markov) discount bond price dynamics

$$dB(t,T) = B(t,T)[(r(t) + c(t,T))dt + \sum_{j=1}^N d_j(t,T)dW_j(t)] \quad (2.18)$$

with determinants

$$c(t,T) = -\int_t^T a(t,s)ds + \frac{1}{2}\sum_{j=1}^N d_j(t,T)^2 \text{ and } d_j(t,T) = -\int_t^T b_j(t,s)ds. \quad (2.19)$$

Given any N traded discount bonds with maturities $0 < T_1 <..< T_i <..< T_N \le H$ the above assumption of no arbitrage opportunities and dynamic market completeness is equivalent to the fact that the linear equation system

$$\begin{bmatrix} d_1(t,T_1) & \cdots & d_N(t,T_1) \\ \vdots & & \vdots \\ d_1(t,T_N) & \cdots & d_N(t,T_N) \end{bmatrix} \begin{bmatrix} m_1(t;T_1,..,T_N) \\ \vdots \\ m_N(t;T_1,..,T_N) \end{bmatrix} + \begin{bmatrix} c(t,T_1) \\ \vdots \\ c(t,T_N) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \quad (2.20)$$

for the corresponding market prices of risk associated with the N exogenous sources of securities market uncertainty has a solution defined on $[0,T_1]$ (existence of an equivalent martingale measure) and that the coefficient matrix

$$\begin{bmatrix} d_1(t,T_1) & \cdots & d_N(t,T_1) \\ \vdots & & \vdots \\ d_1(t,T_N) & \cdots & d_N(t,T_N) \end{bmatrix} \quad (2.21)$$

is non-singular [uniqueness of the equivalent martingale measure $\tilde{\pi}$ for the discount bond price processes $B(t,T_1),..,B(t,T_N)$ spanning the securities market]. Furthermore, we have the representation

$$m(t;T_1,..,T_N) = -\alpha(t), \; 0 \le t \le T_1, \quad (2.22)$$

(standard finance condition) and the forward rate drift restriction

$$a(t,T) = \sum_{j=1}^{N} b_j(t,T)[\alpha_j(t) + \int_t^T b_j(t,s)ds]. \quad (2.23)$$

The risk-neutral spot rate dynamics are then

$$r(t) = f(0,t) + \sum_{j=1}^{N} \int_0^t b_j(s_1,t)[\int_{s_1}^t b_j(s_1,s_2)ds_2]ds_1 + \sum_{j=1}^{N} \int_0^t b_j(s,t)d\tilde{W}_j(s). \quad (2.24)$$

[Because of the above representation

$$m(t;T_1,..,T_N) = -\alpha(t), \; 0 \le t \le T_1,$$

(standard finance condition) we shall in the sequel call the process

$$\alpha(t) = \sigma(t)^{-1}[\mu(t) - r(t)1_N]$$

in the Girsanov transformation of probability measure

$$\tilde{\pi}(\varphi) = E[\beta(H)1_\varphi], \; \varphi \in \Phi \qquad \beta(t) = \exp(-\int_0^t \alpha(s)^T dW(s) - \frac{1}{2}\int_0^t \|\alpha(s)\|^2 ds)$$

the market prices of risk associated with the exogenous sources $W(t)$ of securities market uncertainty.]

Stock Markets. In the general (complete securities market) case there are $0 < M < N$ discount bonds $B(t,T_1),..,B(t,T_M)$ and the stocks $S_1(t),..,S_{N-M}(t)$ where the ex-dividend stock price dynamics are given by the system of linear stochastic differential equations

$$dS_i(t) = S_i(t)[(\gamma_i(t) - \delta_i(t))dt + \sum_{j=1}^{N} \sigma_{ij}(t)dW_j(t)] \quad (2.25)$$

and the corresponding cum-dividend stock price processes are

$$d\hat{S}_i(t) = \hat{S}_i(t)[\gamma_i(t)dt + \sum_{j=1}^{N} \sigma_{ij}(t)dW_j(t)] \qquad \left[\hat{S}_i(t) = S_i(t)\exp(\int_0^t \delta(s)ds)\right]. \quad (2.26)$$

[In the incomplete securities market case considered later where the number of exogenous sources of market uncertainty is greater than the number of available securities there will be additional market state variables $Q(t)$ that are not prices of traded assets.] The linear equation system for the market prices of risk

$$\begin{bmatrix} d_1(t,T_1) & \cdots & d_N(t,T_1) \\ \vdots & & \vdots \\ d_1(t,T_M) & \cdots & d_N(t,T_M) \\ \sigma_{11}(t) & \cdots & \sigma_{1N}(t) \\ \vdots & & \vdots \\ \sigma_{N-M1}(t) & \cdots & \sigma_{N-MN}(t) \end{bmatrix} \begin{bmatrix} m_1(t;T_1,..,T_M) \\ \vdots \\ m_M(t;T_1,..,T_M) \\ m_1(t;\hat{S}_1,..,\hat{S}_{N-M}) \\ \vdots \\ m_{N-M}(t;\hat{S}_1,..,\hat{S}_{N-M}) \end{bmatrix} + \begin{bmatrix} c(t,T_1) \\ \vdots \\ c(t,T_M) \\ \gamma_1(t)-r(t) \\ \vdots \\ \gamma_{N-M}(t)-r(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (2.27)$$

again has a (up to modifications) unique solution which is related to the equivalent martingale measure $\tilde{\pi}$ for the discount bond price processes $B(t,T_1),..,B(t,T_M)$ and the stock price processes $\hat{S}_1(t),..,\hat{S}_{N-M}(t)$ via the standard finance condition

$$m(t;T_1,..,T_M,\hat{S}_1,..,\hat{S}_{N-M}) = -\alpha(t), \; 0 \le t \le T_1. \quad (2.28)$$

Trading Strategies. A trading or portfolio management strategy involving the basic assets - bonds and stocks - available to investors in the securities market is an N-dimensional progressively measurable stochastic process

$$\theta(t) = [\theta_1(t) \; \cdots \; \theta_i(t) \; \cdots \; \theta_N(t)]^T \text{ with } \int_0^H \|\theta(t)\|^2 dt < \infty. \quad (2.29)$$

$\theta_i(t)$, $\theta_i(t) = \nu_i(t)P_i(t)$, is the amount invested in security i at time t and $\nu_i(t)$ is the corresponding number of securities i in the investor's portfolio. [Note that the number $\nu(t)$ of securities in the investor's portfolio is a predictable stochastic process, i.e., has left-continuous sample paths with right limits.] If $V(t)$ is the value of the investor's portfolio at time t, then

$$V(t) - \sum_{i=1}^{N} \theta_i(t) \quad (2.30)$$

is the amount held in cash at time t. Liquid funds which the investor withdraws from the portfolio over time (e.g., future liabilities, costs associated with projects, etc.) are modelled by a non-negative, progressively measurable stochastic process $c(t)$ that satisfies

$$\int_0^H c(t)dt < \infty. \quad (2.31)$$

Given the above asset price dynamics, the value process of the investor's portfolio follows the linear stochastic differential equation

$$\begin{aligned} dV(t) &= \theta(t)^T[\mu(t)dt + \sigma(t)dW(t)] + [V(t) - \sum_{i=1}^{N}\theta_i(t)]r(t)dt - c(t)dt \\ &= [\theta(t)^T[\mu(t) - r(t)1_N] + V(t)r(t) - c(t)]dt + \theta(t)^T\sigma(t)dW(t) \qquad (2.32) \\ &= [V(t)r(t) - c(t)]dt + \theta(t)^T\sigma(t)d\tilde{W}(t), \; V(0) = v \end{aligned}$$

which has the unique strong solution

$$\begin{aligned} V(t) &= \frac{1}{D(t)}\left[v + \int_0^t D(s)[\theta(s)^T[\mu(s) - r(s)1_N] - c(s)]ds + \int_0^t D(s)\theta(s)^T\sigma(s)dW(s)\right] \\ &= \frac{1}{D(t)}[v - \int_0^t D(s)c(s)ds + \int_0^t D(s)\theta(s)^T\sigma(s)d\tilde{W}(s)]. \end{aligned} \quad (2.33)$$

[In the incomplete securities market case the uniform boundedness of the coefficients $r(t)$, $\mu(t)$ and $\sigma(t)$ of the market model will be replaced by appropriate integrability conditions which ensure the existence of a unique strong solution of the above evolution equation for the portfolio value process.] The corresponding gains from trade are

$$dG(t) = \theta(t)^T[\mu(t)dt + \sigma(t)dW(t)] + [G(t) - \sum_{i=1}^{N}\theta_i(t)]r(t)dt$$
$$= [\theta(t)^T[\mu(t) - r(t)1_N] + G(t)r(t)]dt + \theta(t)^T\sigma(t)dW(t) \quad (2.34)$$
$$= G(t)r(t)dt + \theta(t)^T\sigma(t)d\widetilde{W}(t),\ G(0) = 0$$

or (after integration)

$$G(t) = \frac{1}{D(t)}\left[\int_0^t D(s)\theta(s)^T[\mu(s) - r(s)1_N]ds + \int_0^t D(s)\theta(s)^T\sigma(s)dW(s)\right]$$
$$= \frac{1}{D(t)}\int_0^t D(s)\theta(s)^T\sigma(s)d\widetilde{W}(s). \quad (2.35)$$

[Note that in the dynamically complete securities market setting assumed here for an option replication or hedging strategy

$$D(t)V(t) = v + \int_0^t d[D(s)G(s)] = v + \int_0^t v(s)^T d[D(s)P(s)]$$

holds, i.e., the trading strategy is self-financing.]

Arrow-Debreu Prices. The exponential martingale $\beta(t)$ defining the equivalent risk-neutral probability measure $\widetilde{\pi}$,

$$\beta(t) = E[\frac{d\widetilde{\pi}}{d\pi}\Big|F_t], \quad (2.36)$$

represents portfolio value, gains from trade and funds withdrawn in the form

$$\frac{\widetilde{E}[D(t)V(t)|F_s]}{D(s)} = \frac{E[\beta(t)D(t)V(t)|F_s]}{\beta(s)D(s)},\ 0 \le s \le t \le H$$
$$\frac{\widetilde{E}[D(t)G(t)|F_s]}{D(s)} = \frac{E[\beta(t)D(t)G(t)|F_s]}{\beta(s)D(s)},\ 0 \le s \le t \le H \quad (2.37)$$
$$\frac{\widetilde{E}[\int_t^u D(s)c(s)ds|F_t]}{D(t)} = \frac{E[\int_t^u \beta(s)D(s)c(s)ds|F_t]}{\beta(t)D(t)},\ 0 \le t \le u \le H$$

which means that the stochastic process

$$\xi(t) = \beta(t)D(t) = \exp(-\int_0^t \alpha(s)^T dW(s) - \int_0^t [r(s) + \frac{1}{2}\|\alpha(s)\|^2]ds)$$
$$d\xi(t) = -\xi(t)[r(t)dt + \alpha(t)^T dW(t)],\ \xi(0) = 1 \quad (2.38)$$

is the unique implicit Arrow-Debreu price system [consistent with the risk-neutral term structure of discount factors $D(t)$] associated with the initially given probability measure π (reference probability), i.e., multiplication with $\xi(t)$ relates the time t value of a securities market variable [e.g., $V(t)$, $G(t)$, $c(t)$, etc.] to its equivalent value at time 0.

Admissibility. Given an initial wealth $v \ge 0$ a pair (c,θ) consisting of a fund withdrawal rate $c(t)$ and a trading strategy $\theta(t)$ is admissible, $(c,\theta) \in A(v)$, if the associated discounted portfolio value process

$$D(t)V_v^{c\theta}(t) = v - \int_0^t D(s)c(s)ds + \int_0^t D(s)\theta(s)^T\sigma(s)d\widetilde{W}(s) \quad (2.39)$$

satisfies

$$D(t)V_v^{c\theta}(t) \ge 0,\ 0 \le t \le H. \quad (2.40)$$

This admissibility condition implies the natural budget constraint

$$\widetilde{E}[D(\tau)V_v^{c\theta}(\tau)+\int_0^\tau D(s)c(s)ds]\le v \qquad (2.41)$$

which holds for any stopping time $\tau\in[0,H]$. It also rules out all arbitrage opportunities in the securities market, i.e., trading strategies in bonds and stocks that with positive probability achieve intertemporal funding of a positive liability stream or positive final wealth with zero initial investment. Furthermore, if we define

$$\tau=\inf\{t: V_v^{c\theta}(t)=0\}, \qquad (2.42)$$

then because the discounted portfolio value process $D(t)V_v^{c\theta}(t)$ is a continuous, non-negative supermartingale under the equivalent risk-neutral probability measure $\widetilde{\pi}$ we have

$$V_v^{c\theta}(t)=0,\ \tau\le t\le H, \qquad (2.43)$$

[in the case where $\tau<H$ holds]. Moreover, given any initial wealth $v\ge 0$ and fund withdrawal rate $c(t)$ which satisfies $\widetilde{E}[\int_0^H D(t)c(t)dt]\le v$ there exists a trading strategy $\theta(t)$,

$$\theta(t)=\frac{[\sigma(t)^T]^{-1}\psi(t)}{D(t)} \text{ and } \widetilde{E}[\int_0^H D(s)c(s)ds\Big|F_t]-\widetilde{E}[\int_0^H D(s)c(s)ds]=\int_0^t \psi(s)^T d\widetilde{W}(s), \qquad (2.44)$$

such that $(c,\theta)\in A(v)$. If equality holds, then this trading strategy is unique (up to stochastic equivalence) and the associated discounted portfolio value process satisfies

$$D(t)V_v^{c\theta}(t)=\widetilde{E}[\int_t^H D(s)c(s)ds\Big|F_t] \qquad (2.45)$$

[and therefore $V_v^{c\theta}(H)=0$]. Equally, given any stopping time $\tau\in[0,H]$ and any non-negative, F_τ-measurable random variable V_τ with $\widetilde{E}[D(\tau)V_\tau]\le v$ there is a fund withdrawal rate $c(t)$,

$$c(t)=\frac{v-\widetilde{E}[D(\tau)V_\tau]}{D(t)\tau}, \qquad (2.46)$$

and a trading strategy $\theta(t)$,

$$\theta(t)=\frac{[\sigma(t)^T]^{-1}\psi(t)}{D(t)} \text{ and } \widetilde{E}[D(\tau)V_\tau|F_t]-\widetilde{E}[D(\tau)V_\tau]=\int_0^t \psi(s)^T d\widetilde{W}(s), \qquad (2.47)$$

such that $(c,\theta)\in A(v)$ and $V_v^{c\theta}(\tau)=V_\tau$. If equality holds, then $c=0$, the trading strategy is unique and the discounted portfolio value process has the representation

$$D(t)V_v^{c\theta}(t)=\widetilde{E}[D(\tau)V_\tau|F_t]. \qquad (2.48)$$

Utility Functions. Investors' preferences for intertemporal fund consumption $c(t)$ and final wealth $V_v^{c\theta}(H)$ can be made precise in mathematical terms by using utility functions $U^c(t,c)$ and $U^V(V)$ that are strictly increasing and strictly concave in the two corresponding (real-valued) arguments c and V, furthermore twice continuously differentiable in these arguments with

$$\frac{\partial^2 U^c(t,c)}{\partial c^2} \text{ and } \frac{d^2U^V(V)}{dV^2}$$

non-decreasing in c and V and moreover

$$\lim_{c\downarrow 0}\frac{\partial U^c(t,c)}{\partial c}=\lim_{V\downarrow 0}\frac{dU^V(V)}{dV}=+\infty \text{ and } \lim_{c\uparrow\infty}\frac{\partial U^c(t,c)}{\partial c}=\lim_{V\uparrow\infty}\frac{dU^V(V)}{dV}=0. \qquad (2.49)$$

(For recent alternatives to the time-additive von Neumann-Morgenstern preferences considered here see for instance Hindy and Huang [9], Duffie and Epstein [24] and Detemple and Zapatero [25]. Under additional assumptions both equilibrium security prices - along the lines of Huang [3], Duffie [4], Duffie and Huang [5], Huang [7] and Karatzas, Lehoczky and Shreve [8] - and solutions of the consumption/investment problem can be derived for these alternative preference structures in a way analogous to the one described here.) It is also important to note in this context (Wang [26], He and Leland [27] and He and Huang [28]) that the utility functions $U^c(t,c)$ and $U^V(V)$ (in the representative agent framework that applies in a complete securities market) and the coefficients $\mu(t,P(t))$ and $\sigma(t,P(t))$ of the diffusion type equilibrium asset price process $P(t)$ (i.e., the market portfolio that is held by the representative agent in equilibrium) are related to each other via the linear boundary value problem

$$\begin{gathered}\frac{\partial V}{\partial t}+A^T\nabla V+\frac{1}{2}\operatorname{tr}(BB^T\nabla^2V)-P^T(\mu-r1_N)-Vr+I^c(t,X)=0\\ V(0,P,X)=\sum_{i=1}^N p_i \qquad V(H,P,X)=I^V(X)\end{gathered}\tag{2.50}$$

where

$$I^c(t,c)=[\frac{\partial U^c(t,c)}{\partial c}]^{-1} \qquad I^V(V)=[\frac{dU^V(V)}{dV}]^{-1} \tag{2.51}$$

and

$$A=\begin{bmatrix}P_1\mu_1\\ \vdots\\ P_N\mu_N\\ -Xr\end{bmatrix} \qquad B=\begin{bmatrix}P_1\sigma_{11} & \cdots & P_1\sigma_{1N}\\ \vdots & & \vdots\\ P_N\sigma_{N1} & \cdots & P_N\sigma_{NN}\\ -X\alpha_1 & \cdots & -X\alpha_N\end{bmatrix} \qquad \nabla=\begin{bmatrix}\frac{\partial}{\partial P_1}\\ \vdots\\ \frac{\partial}{\partial P_N}\\ \frac{\partial}{\partial X}\end{bmatrix}=\begin{bmatrix}\nabla_P\\ \nabla_X\end{bmatrix} \tag{2.52}$$

$[\alpha(t)=\sigma(t)^{-1}[\mu(t)-r(t)1_N]$ and $X(t)$ is the representative marginal utility of wealth]. By using this equilibrium relationship it is for example possible to recover the determinants of (unobservable) investor preferences from observable securities market data. The above properties of (continuous) von Neumann-Morgenstern utility functions guarantee the existence of the inverse functions

$$I^c(t,c)=[\frac{\partial U^c(t,c)}{\partial c}]^{-1} \text{ and } I^V(V)=[\frac{dU^V(V)}{dV}]^{-1}$$

which play a central role in the investment management and asset allocation problems considered here, are continuously differentiable and convex in the variables c and V and moreover satisfy

$$U^c(t,I^c(t,c_2))\geq U^c(t,c_1)+c_2(I^c(t,c_2)-c_1),\ c_1\geq 0,\ c_2>0 \tag{2.53a}$$

$$U^V(I^V(V_2))\geq U^V(V_1)+V_2(I^V(V_2)-V_1),\ V_1\geq 0,\ V_2>0 \tag{2.53b}$$

as well as

$$\frac{\partial}{\partial c}U^c(t,I^c(t,c))=c\frac{\partial I^c(t,c)}{\partial c},\ c>0,\text{ and } \frac{d}{dV}U^V(I^V(V))=V\frac{dI^V(V)}{dV},\ V>0. \tag{2.54}$$

Liability Funding. The important investment problem of optimally funding a future liability stream $c(t)$, $E[\int_0^H U^c(t,c(t))^- dt] < \infty$, with a given initial capital $v > 0$ is then to find a trading strategy $\theta(t)$, $(c,\theta) \in A(v)$, which solves the stochastic control problem

$$\begin{aligned} &\max_{(c,\theta)\in A(v)} E[\int_0^H U^c(t,c(t))dt] \\ &E[\int_0^H \xi(t)c(t)dt] \le v \end{aligned} \qquad (2.55)$$

(***optimal fund design***). Such a strategy can be found by using standard Lagrangian methods. The function $H^c(y) = E[\int_0^H \xi(t) I^c(t, y\xi(t))dt]$, $0 < y < \infty$, is continuous and strictly decreasing with $\lim_{y\downarrow 0} H^c(y) = +\infty$ and $\lim_{y\uparrow\infty} H^c(y) = 0$ and has therefore an inverse $[H^c(y)]^{-1}$ that maps $v \xrightarrow[1\text{-}1]{} y_v$. The corresponding fund withdrawal rate $c_v(t) = I^c(t, y_v \xi(t))$ satisfies $E[\int_0^H \xi(t)c_v(t)dt] = v$ (y_v is the associated Lagrange multiplier) and therefore there is a unique trading strategy $\theta_v(t)$, $(c_v, \theta_v) \in A(v)$, that solves the above liability funding problem. The value function of the optimal investment portfolio can be represented in the form

$$V_v^{c_v,\theta_v}(t) = \frac{1}{\xi(t)} E[\int_t^H \xi(s)c_v(s)ds \,|\, F_t] \qquad (2.56)$$

and therefore has the property $V_v^{c_v,\theta_v}(H) = 0$.

Asset Allocation. If on the other hand our objective is to optimally manage a securities portfolio with a given final wealth target in mind (i.e., to determine an optimal asset allocation), then the associated stochastic control problem is

$$\begin{aligned} &\max_{(c,\theta)\in A(v)} E[U^V(V_v^{c\theta}(H))] \\ &E[\xi(H)V_v^{c\theta}(H)] \le v \end{aligned} \qquad (2.57)$$

(***optimal portfolio selection***). Here we consider the (continuous and strictly decreasing) function $H^V(y) = E[\xi(H) I^V(y\xi(H))]$, $0 < y < \infty$, the inverse $[H^V(y)]^{-1}$ of which maps $v \xrightarrow[1\text{-}1]{} y_v$. The F_H-measurable random variable $V_H^v = I^V(y_v \xi(H))$ satisfies $E[\xi(H)V_H^v] = v$ and therefore there is a unique trading strategy $\theta_v(t)$, $(0,\theta_v) \in A(v)$, that achieves the specified portfolio management objectives. The value function of the corresponding optimal asset allocation portfolio is

$$V_v^{\theta_v}(t) = V_v^{0\theta_v}(t) = \frac{1}{\xi(t)} E[\xi(H)V_H^v \,|\, F_t]. \qquad (2.58)$$

General Investment Management. The general investment management and asset allocation problem in the given securities market is a combination of optimal liability funding and optimal intertemporal portfolio management under the natural budget constraint (a situation where the admissible trading strategies have to satisfy additional constraints will be considered later, see ***Part II: Securities and Derivatives Markets***) and can therefore be expressed in terms of the stochastic control problem

$$\begin{aligned} &\max_{(c,\theta)\in A(v)} E[\int_0^H U^c(t,c(t))dt + U^V(V_v^{c\theta}(H))] \\ &E[\int_0^H \xi(t)c(t)dt + \xi(H)V_v^{c\theta}(H)] \le v \end{aligned} \qquad (2.59)$$

(***asset/liability management*** - with the focus on the asset side in this paper, see also *Baseline for Exchange Rate - Risks of an International Reinsurer*, AFIR 96, Vol. I, p. 395). It follows immediately that the continuous and strictly decreasing function

$$H(y) = H^c(y) + H^V(y) = E[\int_0^H \xi(t)I^c(t, y\xi(t))dt + \xi(H)I^V(y\xi(H))] \quad (2.60)$$

has an inverse $[H(y)]^{-1}$ that maps $v \xrightarrow[1-1]{} y_v$. If we set $v_c = H^c(y_v)$ and $v_V = H^V(y_v)$, $v = v_c + v_V$, then the corresponding optimal fund withdrawal rate $c_v(t) = I^c(t, y_v\xi(t))$ and final wealth $V_H^v = I^V(y_v\xi(H))$ have the properties $E[\int_0^H \xi(t)c_v(t)dt] = v_c$ and $E[\xi(H)V_H^v] = v_V$. Therefore there is a unique trading strategy $\theta_v(t) = \theta_{v_c}(t) + \theta_{v_V}(t)$, $(c_v, \theta_v) \in A(v)$, that achieves the investor's set [in terms of the given utilities $U^c(t,c)$ for intertemporal fund consumption and $U^V(V)$ for final wealth] investment management and asset allocation objectives. The value function of the optimal portfolio has the representation

$$V_v^{c_v,\theta_v}(t) = \frac{1}{\xi(t)} E[\int_t^H \xi(s)c_v(s)ds + \xi(H)V_H^v | F_t]. \quad (2.61)$$

With the investor's marginal utility of wealth $X(t) = y_v\xi(t)$ we can then write

$$V_v^{c_v,\theta_v}(t) = V(t, P(t), X(t)) \quad (2.62)$$

where the function $V(t, P, X)$ is twice continuously differentiable in the state variables P and X and once continuously differentiable with respect to time t if the coefficients of the securities market model are of the form

$$r(t) = r(t, P(t)),\ \mu(t) = \mu(t, P(t)) \text{ and } \sigma(t) = \sigma(t, P(t)) \quad (2.63)$$

and satisfy appropriate differentiability, growth and Lipschitz conditions in the state variable P (Markovian characterization, see Cox and Huang [2]). If we now apply Ito's formula, we have

$$dV = [\frac{\partial V}{\partial t} + A^T\nabla V + \frac{1}{2}tr(BB^T\nabla^2 V) - \nabla V^T B\alpha]dt + \nabla V^T B d\tilde{W} \quad (2.64)$$

where

$$A = \begin{bmatrix} P_1\mu_1 \\ \vdots \\ P_N\mu_N \\ -Xr \end{bmatrix},\ B = \begin{bmatrix} P_1\sigma_{11} & \cdots & P_1\sigma_{1N} \\ \vdots & & \vdots \\ P_N\sigma_{N1} & \cdots & P_N\sigma_{NN} \\ -X\alpha_1 & \cdots & -X\alpha_N \end{bmatrix} \text{ and } \nabla = \begin{bmatrix} \frac{\partial}{\partial P_1} \\ \vdots \\ \frac{\partial}{\partial P_N} \\ \frac{\partial}{\partial X} \end{bmatrix} = \begin{bmatrix} \nabla_P \\ \nabla_X \end{bmatrix} \quad (2.65)$$

[note that the above two parameters $A(t)$ and $B(t)$ express the relevant state dynamics - which include the marginal utility of wealth - in terms of the initially given Wiener process $W(t)$ whereas the following considerations are all based on its Girsanov transform $\tilde{W}(t)$ with $-\nabla V^T B\alpha$ being the appropriate correction term], thus applying Ito's formula again we obtain

$$d[DV] = D[\frac{\partial V}{\partial t} + A^T\nabla V + \frac{1}{2}tr(BB^T\nabla^2 V) - \nabla V^T B\alpha - Vr]dt + D\nabla V^T B d\tilde{W} \quad (2.66)$$

and therefore

$$\int_0^t D(s)\theta_v(s)^T\sigma(s)d\widetilde{W}(s) = \int_0^t D(s)c_v(s)ds + \int_0^t d[D(s)V(s,P(s),X(s))] =$$

$$\int_0^t D(s)[\frac{\partial V}{\partial s} + A^T\nabla V + \frac{1}{2}tr(BB^T\nabla^2 V) - \nabla V^T B\alpha - Vr + c_v](s)ds + \qquad (2.67)$$

$$\int_0^t D(s)[\nabla V^T B](s)d\widetilde{W}(s).$$

Since the two stochastic integrals with respect to the Wiener process in this equation are continuous $\widetilde{\pi}$-martingales the drift term on the right-hand side of the equation must vanish and we have the linear partial differential equation for the portfolio value function the equation must vanish and we have the linear partial differential equation for the portfolio value function

$$\frac{\partial V}{\partial t} + A^T\nabla V + \frac{1}{2}tr(BB^T\nabla^2 V) - \nabla V^T B\alpha - Vr + I^c(t,X) = 0 \qquad (2.68)$$

with boundary conditions $V(0,P,X) = v$ and $V(H,P,X) = I^V(X)$. Furthermore, the optimal trading strategy is

$$\theta_v = [B\sigma^{-1}]^T\nabla V = I_P\nabla_P V - \kappa^{-1}[\mu - r1_N]X\nabla_X V \qquad (2.69)$$

where

$$I_P = \begin{bmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & P_N \end{bmatrix}. \qquad (2.70)$$

Note also that on the interval $[\tau, H]$ where $\tau = \inf\{t: V_v^{c_v\theta_v}(t) = 0\}$ we have

$$V_v^{c_v\theta_v}(t) = 0,\ \theta_v(t) = 0 \text{ and } c_v(t) = 0. \qquad (2.71)$$

Moreover, if $\frac{\partial V}{\partial X} \le 0$, we can write

$$X(t) = X(t, P(t), V_v^{c_v\theta_v}(t)) \qquad (2.72)$$

where the function $X(t,P,V)$ is twice continuously differentiable in the state variables P and V and once continuously differentiable with respect to time t which means that the optimal fund withdrawal rate and the optimal trading strategy are continuously differentiable feedback controls, i.e.,

$$c_v(t) = c(t, P(t), V_v^{c_v\theta_v}(t)) \text{ and } \theta_v(t) = \theta(t, P(t), V_v^{c_v\theta_v}(t)). \qquad (2.73)$$

In a corresponding HJB dynamic programming (control theory) framework

$$\overline{V}(t,P(t),V(t)) = \sup_{(c,\theta)\in A(t,V(t))} E[\int_t^H U^c(s,c(s))ds + U^V(V_{V(t)}^{c\theta}(H)) \mid P(t),V(t)]$$

$$\sup_{(c,\theta)\in A(t,V(t))} [U^c(t,c(t)) + \frac{\partial \overline{V}}{\partial t} + A^T\nabla\overline{V} + \frac{1}{2}tr(BB^T\nabla^2\overline{V})] = 0 \qquad \overline{V}(H,P,V) = U^V(V) \qquad (2.74)$$

with a state space that [instead of the investor's marginal utility of wealth

$$X(t) = y_v \exp(-\int_0^t \alpha(s)^T dW(s) - \int_0^t [r(s) + \frac{1}{2}\|\alpha(s)\|^2]ds) \qquad (2.75)$$

has the portfolio value process

$$V(t) = \frac{1}{D(t)}\left[v + \int_0^t D(s)[\theta(s)^T[\mu(s) - r(s)1_N] - c(s)]ds + \int_0^t D(s)\theta(s)^T\sigma(s)dW(s)\right] \qquad (2.76)$$

as an additional variable and] is characterized by

$$A=\begin{bmatrix} P_1\mu_1 \\ \vdots \\ P_N\mu_N \\ \theta^T[\mu - r1_N] \\ +Vr-c \end{bmatrix},\ B=\begin{bmatrix} P_1\sigma_{11} & \cdots & P_1\sigma_{1N} \\ \vdots & & \vdots \\ P_N\sigma_{N1} & \cdots & P_N\sigma_{NN} \\ & \theta^T\sigma & \end{bmatrix} \text{ and } \nabla=\begin{bmatrix} \frac{\partial}{\partial P_1} \\ \vdots \\ \frac{\partial}{\partial P_N} \\ \frac{\partial}{\partial V} \end{bmatrix}=\begin{bmatrix} \nabla_P \\ \nabla_V \end{bmatrix} \qquad (2.77)$$

the optimal trading strategy is

$$\theta_v = -I_P\frac{\overline{V}_{VP}}{\overline{V}_{VV}} - \kappa^{-1}[\mu - r1_N]\frac{\overline{V}_V}{\overline{V}_{VV}} \qquad \begin{aligned} -\frac{\overline{V}_{VP}}{\overline{V}_{VV}} &= V_P = \nabla_P V \\ \frac{\overline{V}_V}{\overline{V}_{VV}} &= XV_X = X\nabla_X V \end{aligned} \qquad (2.78)$$

and the optimal fund withdrawal rate and final wealth have the properties

$$\begin{aligned} \frac{\partial}{\partial c}U^c(t,c_v(t)) &= \frac{\partial}{\partial V}\overline{V}(t,P(t),V_v^{c_v\theta_v}(t)),\ c_v(t)>0 \\ \frac{d}{dV}U^V(V_H^v) &= \frac{\partial}{\partial V}\overline{V}(H,P(H),V_H^v),\ V_H^v>0. \end{aligned} \qquad (2.79)$$

Incomplete Securities Markets. The above portfolio analysis can be extended to an incomplete securities market (i.e., a financial market with genuinely unhedgeable risk) where the number N of exogenous sources of market uncertainty is greater than the number M of traded assets spanning the market. Following Karatzas, Lehoczky, Shreve and Xu [29] and He and Pearson [30] we assume for this purpose that appropriate integrability conditions ensure unique strong existence of the portfolio value processes $V_v^{c\theta}(t)$ associated with fund withdrawal rates $c(t)$ and trading strategies $\theta(t)$ and that the $M\times N$ volatility matrix $\sigma(t)$ has full rank which allows us to define the market prices of risk

$$\alpha(t)=\sigma(t)^T\kappa(t)^{-1}[\mu(t)-r(t)1_M] \qquad (2.80)$$

associated with the exogenous sources $W(t)$ of securities market uncertainty, the exponential local martingale

$$\beta(t)=\exp\left(-\int_0^t\alpha(s)^T dW(s)-\frac{1}{2}\int_0^t\|\alpha(s)\|^2 ds\right) \qquad (2.81)$$

and the stochastic process

$$\overline{W}(t)=W(t)+\int_0^t\alpha(s)ds \qquad (2.82)$$

that (although not necessarily of the Wiener type) represents security prices, portfolio value and gains from trade in the form

$$dP_i(t)=P_i(t)[r(t)dt+\sum_{j=1}^N\sigma_{ij}(t)d\overline{W}_j(t)], \qquad (2.83)$$

$$\begin{aligned} dV(t) &= \theta(t)^T[\mu(t)dt+\sigma(t)dW(t)]+[V(t)-\sum_{i=1}^M\theta_i(t)]r(t)dt-c(t)dt \\ &= [\theta(t)^T[\mu(t)-r(t)1_M]+V(t)r(t)-c(t)]dt+\theta(t)^T\sigma(t)dW(t) \\ &= [V(t)r(t)-c(t)]dt+\theta(t)^T\sigma(t)d\overline{W}(t) \end{aligned} \qquad (2.84)$$

and

$$\begin{aligned} dG(t) &= \theta(t)^T[\mu(t)dt + \sigma(t)dW(t)] + [G(t) - \sum_{i=1}^{M}\theta_i(t)]r(t)dt \\ &= [\theta(t)^T[\mu(t) - r(t)1_M] + G(t)r(t)]dt + \theta(t)^T\sigma(t)dW(t) \qquad (2.85) \\ &= G(t)r(t)dt + \theta(t)^T\sigma(t)d\overline{W}(t). \end{aligned}$$

The discounted portfolio value process corresponding to a fund withdrawal rate $c(t)$ and a trading strategy $\theta(t)$ is then

$$D(t)V_v^{c\theta}(t) = v - \int_0^t D(s)c(s)ds + \int_0^t D(s)\theta(s)^T\sigma(s)d\overline{W}(s). \qquad (2.86)$$

Using the Arrow-Debreu price system $\xi(t) = \beta(t)D(t)$ which in this case is not necessarily associated with a martingale [i.e., later on we shall have to assume that in addition to the above standard integrability requirements for the incomplete securities market model a Novikov condition for the market prices of risk $\alpha(t)$ is satisfied] and applying Ito's formula

$$d(\xi V) = [A^T\nabla(\xi V) + \frac{1}{2}tr(BB^T\nabla^2(\xi V))]dt + \nabla(\xi V)^T B d\overline{W} \qquad (2.87)$$

where

$$A = \begin{bmatrix} Vr - c \\ -\xi(r - \|\alpha\|^2) \end{bmatrix},\ B = \begin{bmatrix} \theta^T\sigma \\ -\xi\alpha^T \end{bmatrix},\ \nabla(\xi V) = \begin{bmatrix} \xi \\ V \end{bmatrix} \text{ and } \nabla^2(\xi V) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad (2.88)$$

we find

$$\xi(t)V_v^{c\theta}(t) + \int_0^t \xi(s)c(s)ds = v + \int_0^t \xi(s)[\theta(s)^T\sigma(s) - V_v^{c\theta}(s)\alpha(s)^T]dW(s) \qquad (2.89)$$

with the right-hand side of the equation a non-negative local martingale, hence a supermartingale, and therefore the natural budget constraint

$$E[\xi(H)V_v^{c\theta}(H) + \int_0^H \xi(t)c(t)dt] \le v. \qquad (2.90)$$

The general investment management and asset allocation problem in the incomplete securities market under consideration is thus once again

$$\begin{aligned} &\max_{(c,\theta)\in A(v)} E[\int_0^H U^c(t,c(t))dt + U^V(V_v^{c\theta}(H))] \\ &E[\int_0^H \xi(t)c(t)dt + \xi(H)V_v^{c\theta}(H)] \le v. \end{aligned} \qquad (2.91)$$

As in the complete market case it follows immediately that the continuous and strictly decreasing function

$$H(y) = E[\int_0^H \xi(t)I^c(t, y\xi(t))dt + \xi(H)I^V(y\xi(H))] \qquad (2.92)$$

has an inverse $[H(y)]^{-1}$ that maps $v \xrightarrow[1-1]{} y_v$. The corresponding fund withdrawal rate $c_v(t) = I^c(t, y_v\xi(t))$ and final wealth $V_H^v = I^V(y_v\xi(H))$ then have the property

$$E[\int_0^H \xi(t)c_v(t)dt + \xi(H)V_H^v] = v. \qquad (2.93)$$

Furthermore, the inequality

$$E[\int_0^H U^c(t,c(t))dt + U^V(V_v^{c\theta}(H))] \le E[\int_0^H U^c(t,c_v(t))dt + U^V(V_H^v)] \qquad (2.94)$$

holds for every admissible pair $(c,\theta) \in A(v)$.

Securities Market Completions. Since in an incomplete securities market we cannot hope to directly find a trading strategy $\theta_v(t)$ such that $(c_v,\theta_v) \in A(v)$ and $V_v^{c_v\theta_v}(H) = V_H^v$ the main idea therefore is to vary the Arrow-Debreu price system $\xi(t)$ in such a way that the right-hand side of the above inequality is minimized. To this end we introduce $N-M$ additional market state variables

$$dQ_i(t) = Q_i(t)[a_i(t)dt + \sum_{j=1}^{N} b_{ij}(t)dW_j(t)],\ Q_i(0) = q_i, \quad (2.95)$$

which are not prices of traded assets. We choose the associated volatility matrix $b(t)$ with orthonormal rows $b_{i.}(t)$ such that $b_{i.}(t)^T \in \ker\sigma(t)$, i.e., $\sigma(t)b_{i.}(t)^T = 0$, and consider the drift rate vector $a(t)$ as the relevant variation parameter in what follows. This choice is essentially motivated by the existence of an orthogonal decomposition

$$\begin{gathered}\Omega = K(\sigma) \oplus K^{\perp}(\sigma) \\ K(\sigma) = \{\omega : \forall t[\omega(t) \in \ker\sigma(t)]\} \qquad K^{\perp}(\sigma) = \{\omega : \forall t[\omega(t) \in \operatorname{ran}\sigma(t)^T]\}\end{gathered} \quad (2.96)$$

of the linear space Ω of progressively measurable stochastic processes

$$\omega(t) = [\omega_1(t) \ \ \dots \ \ \omega_N(t)]^T \text{ with } \int_0^H \|\omega(t)\|^2 dt < \infty. \quad (2.97)$$

[Later on we shall see that not attainable - and thus totally non-tradable - contingent claims written on the bonds and stocks in the market have a value process of the form

$$dV(t) = \omega_1(t)^T \omega_2(t)dt + \omega_1(t)^T dW(t) \qquad \omega_1, \omega_2 \in K(\sigma)$$

and can therefore be chosen to complete the securities market in the above outlined way.] It is then clear that in such a securities market completion where we define the stochastic processes

$$\nu_a(t) = b(t)^T[a(t) - r(t)1_{N-M}],\ \alpha(t)^T \nu_a(t) = 0, \text{ and } \hat{\alpha}_a(t) = \alpha(t) + \nu_a(t), \quad (2.98)$$

the exponential local martingale

$$\begin{aligned}\hat{\beta}_a(t) &= \exp(-\int_0^t \hat{\alpha}_a(s)^T dW(s) - \frac{1}{2}\int_0^t \|\hat{\alpha}_a(s)\|^2 ds) \\ &= 1 - \int_0^t \hat{\beta}_a(s)\hat{\alpha}_a(s)^T dW(s)\end{aligned} \quad (2.99)$$

and the Arrow-Debreu price system $\hat{\xi}_a(t) = \hat{\beta}_a(t)D(t)$ the discounted asset price processes $\hat{\xi}_a(t)P_i(t)$ are local martingales and for every admissible pair $(c,\theta) \in A(v)$ the natural budget constraint

$$E[\hat{\xi}_a(H)V_v^{c\theta}(H) + \int_0^H \hat{\xi}_a(t)c(t)dt] \le v \quad (2.100)$$

applies.

<u>Maximum Principle</u>. Given a fund withdrawal rate $c(t)$ and a positive final wealth target V_H let us for the moment assume that some market completion parameter $\hat{a}(t)$ satisfies

$$\max_{a(t)} E[\int_0^H \hat{\xi}_a(t)c(t)dt + \hat{\xi}_a(H)V_H] = E[\int_0^H \hat{\xi}_{\hat{a}}(t)c(t)dt + \hat{\xi}_{\hat{a}}(H)V_H] = v. \quad (2.101)$$

We then consider the positive, continuous martingale

$$\begin{aligned}M(t) &= E[\int_0^H \hat{\xi}_{\hat{a}}(t)c(t)dt + \hat{\xi}_{\hat{a}}(H)V_H \big| F_t] \\ &= v + \int_0^t \psi(s)^T dW(s)\end{aligned} \quad (2.102)$$

which we represent in the form

$$M(t) = v\exp(-\int_0^t \varphi(s)^T dW(s) - \frac{1}{2}\int_0^t \|\varphi(s)\|^2 ds) \\ = v - \int_0^t M(s)\varphi(s)^T dW(s) \qquad (2.103)$$

with

$$\varphi(t) = -\frac{\psi(t)}{M(t)} = \varphi_{\hat{a}}(t) + \frac{M(t) - \int_0^t \hat{\xi}_{\hat{a}}(s)c(s)ds}{M(t)} \nu_{\hat{a}}(t) \text{ and } \varphi_{\hat{a}}(t)^T \nu_{\hat{a}}(t) = 0. \qquad (2.104)$$

If we now define the stochastic processes

$$V_v^{c\cdot}(t) = \frac{M(t) - \int_0^t \hat{\xi}_{\hat{a}}(s)c(s)ds}{\hat{\xi}_{\hat{a}}(t)} \text{ and } \theta(t),\ \sigma(t)^T\theta(t) = V_v^{c\cdot}(t)\alpha(t) - \frac{M(t)}{\hat{\xi}_{\hat{a}}(t)}\varphi_{\hat{a}}(t), \qquad (2.105)$$

we find $V_v^{c\cdot}(H) = V_H$ and

$$\hat{\xi}_{\hat{a}}(t)V_v^{c\cdot}(t) + \int_0^t \hat{\xi}_{\hat{a}}(s)c(s)ds = v + \int_0^t \hat{\xi}_{\hat{a}}(s)[\theta(s)^T\sigma(s) - V_v^{c\cdot}(s)\hat{\alpha}_{\hat{a}}(s)^T]dW(s) \qquad (2.106)$$

which means that $\theta(t)$ is a trading strategy in the given M bonds and stocks, $(c,\theta) \in A(v)$, and $V_v^{c\cdot}(t) = V_v^{c\theta}(t)$ is the value process of the corresponding asset allocation which is optimal if $c(t) = I^c(t, y_v^a \hat{\xi}_a(t))$ and $V_H = I^V(y_v^a \hat{\xi}_a(H))$ holds for some market completion parameter $a(t)$.

<u>Convex Duality</u>. In order to guarantee the existence of a parameter $\hat{a}(t)$ which satisfies the above maximum principle we make the further assumption that $I^c(t,\alpha c) \le \beta I^c(t,c)$, $c > 0$, and $I^V(\gamma V) \le \delta I^V(V)$, $V > 0$, for some $0 < \alpha, \gamma < 1$ and $1 < \beta, \delta$. Moreover, that the corresponding coefficients of relative risk-aversion

$$R^c(t,c) = -\frac{c\frac{\partial^2 U^c(t,c)}{\partial c^2}}{\frac{\partial U^c(t,c)}{\partial c}} \text{ and } R^V(V) = -\frac{V\frac{d^2 U^V(V)}{dV^2}}{\frac{dU^V(V)}{dV}} \qquad (2.107)$$

have the properties $R^c(t,c) \le 1$ and $R^V(V) \le 1$. Then given any $y > 0$ the convex minimization program

$$\min_{a(t)} E[\int_0^H U_\cdot^c(t, y\hat{\xi}_a(t))dt + U_\cdot^V(y\hat{\xi}_a(H))] \qquad (2.108)$$

with the continuous, strictly decreasing and continuously differentiable duals

$$U_\cdot^c(t,c) = U^c(t, I^c(t,c)) - cI^c(t,c) \text{ and } U_\cdot^V(V) = U^V(I^V(V)) - VI^V(V) \qquad (2.109)$$

of the investor's utility functions has a unique solution $\hat{a}_y(t)$. With this [in terms of $U^c(t,c)$ and $U^V(V)$ least favourable] securities market completion parameter we define

$$v_y = H_{\hat{a}_y}(y),\ c_y(t) = I^c(t, y\hat{\xi}_{\hat{a}_y}(t)) \text{ and } V_H^y = I^V(y\hat{\xi}_{\hat{a}_y}(H)) \qquad (2.110)$$

and have

$$\max_{a(t)} E[\int_0^H \hat{\xi}_a(t)c_y(t)dt + \hat{\xi}_a(H)V_H^y] = E[\int_0^H \hat{\xi}_{\hat{a}_y}(t)c_y(t)dt + \hat{\xi}_{\hat{a}_y}(H)V_H^y] = v_y \qquad (2.111)$$

and therefore a solution of the general investment management and asset allocation problem related to the initial wealth v_y. Under the additional assumptions $\lim_{c\downarrow 0} U^c(t,c)$, $\lim_{V\downarrow 0} U^V(V) > -\infty$ and $\lim_{c\uparrow\infty} U^c(t,c) = \lim_{V\uparrow\infty} U^V(V) = +\infty$ any initial wealth $v > 0$

can be reached with the mapping $y \to v_y$. Furthermore, under appropriate integrability conditions (see He and Pearson [30]) the stochastic process $\hat{\beta}_{\hat{a}_{y_v}}(t)$ is actually a martingale which allows the definition

$$\tilde{\pi}_{\hat{a}_{y_v}}(\varphi) = E[\hat{\beta}_{\hat{a}_{y_v}}(H)1_\varphi],\ \varphi \in \Phi, \text{ and } \tilde{W}_{\hat{a}_{y_v}}(t) = W(t) + \int_0^t \hat{\alpha}_{\hat{a}_{y_v}}(s)ds \qquad (2.112)$$

of the associated minimax local martingale measure. Moreover, the value function of the optimal portfolio has the representation

$$V_v^{c_{y_v},\theta_{y_v}}(t) = \frac{1}{\hat{\xi}_{\hat{a}_{y_v}}(t)} E[\int_t^H \hat{\xi}_{\hat{a}_{y_v}}(s)c_{y_v}(s)ds + \hat{\xi}_{\hat{a}_{y_v}}(H)V_H^{y_v} \,|\, F_t]. \qquad (2.113)$$

Markovian Characterization. With $X(t) = y_v\hat{\xi}_{\hat{a}_{y_v}}(t)$ (i.e., the investor's marginal utility of wealth) we can thus as in the complete securities market case write

$$V_v^{c_{y_v},\theta_{y_v}}(t) = V(t, P(t), Q(t), X(t)) \qquad (2.114)$$

where the function $V(t, P, Q, X)$ is twice continuously differentiable in the state variables P, Q and X and once continuously differentiable in t if the coefficients of the securities market model (including those of the additional market state variables) are of the form

$$\begin{aligned} r(t) = r(t,P(t),Q(t)) \quad \mu(t) = \mu(t,P(t),Q(t)) \quad \sigma(t) = \sigma(t,P(t),Q(t)) \\ a(t) = a(t,P(t),Q(t)) \quad b(t) = b(t,P(t),Q(t)) \end{aligned} \qquad (2.115)$$

and satisfy appropriate differentiability, growth and Lipschitz conditions in the state variables P and Q (see [30]). Applying Ito's formula we have then

$$dV = [\frac{\partial V}{\partial t} + A^T\nabla V + \frac{1}{2}tr(BB^T\nabla^2 V) - \nabla V^T B\hat{\alpha}_{\hat{a}_{y_v}}]dt + \nabla V^T B d\tilde{W}_{\hat{a}_{y_v}} \qquad (2.116)$$

where

$$A = \begin{bmatrix} P_1\mu_1 \\ \vdots \\ P_M\mu_M \\ Q_1\hat{a}_1^{y_v} \\ \vdots \\ Q_{N-M}\hat{a}_{N-M}^{y_v} \\ -Xr \end{bmatrix}, \quad B = \begin{bmatrix} P_1\sigma_{11} & \cdots & P_1\sigma_{1N} \\ \vdots & & \vdots \\ P_M\sigma_{M1} & \cdots & P_M\sigma_{MN} \\ Q_1 b_{11} & \cdots & Q_1 b_{1N} \\ \vdots & & \vdots \\ Q_{N-M}b_{N-M1} & \cdots & Q_{N-M}b_{N-MN} \\ -X\hat{\alpha}_1^{\hat{a}_{y_v}} & \cdots & -X\hat{\alpha}_N^{\hat{a}_{y_v}} \end{bmatrix} \text{ and } \nabla = \begin{bmatrix} \nabla_P \\ \nabla_Q \\ \nabla_X \end{bmatrix} \qquad (2.117)$$

and therefore

$$\begin{aligned} \int_0^t D(s)\theta_{y_v}(s)^T\sigma(s)d\tilde{W}_{\hat{a}_{y_v}}(s) = \int_0^t D(s)c_{y_v}(s)ds + \int_0^t d[D(s)V(s,P(s),Q(s),X(s))] = \\ \int_0^t D(s)[\frac{\partial V}{\partial s} + A^T\nabla V + \frac{1}{2}tr(BB^T\nabla^2 V) - \nabla V^T B\hat{\alpha}_{\hat{a}_{y_v}} - Vr + c_{y_v}](s)ds + \\ \int_0^t D(s)[\nabla V^T B](s)d\tilde{W}_{\hat{a}_{y_v}}(s). \end{aligned} \qquad (2.118)$$

Since the two stochastic integrals with respect to the Wiener process in this equation are continuous $\tilde{\pi}_{\hat{a}_{y_v}}$-martingales the drift term on the right-hand side of the equation must vanish and we have the quasi-linear partial differential equation for the portfolio value function

$$\frac{\partial V}{\partial t} + A^T\nabla V + \frac{1}{2}tr(BB^T\nabla^2 V) - \nabla V^T B\hat{\alpha}_{\hat{a}_{y_v}} - Vr + I^c(t,X) = 0 \qquad (2.119)$$

with the boundary conditions $V(0,P,Q,X)=v$ and $V(H,P,Q,X)=I^V(X)$. In addition, we find that $\theta_{y_v}{}^T\sigma = \nabla V^T B$ and therefore

$$\nu_{\hat{a}_{y_v}} = \frac{B_Q{}^T\nabla_Q V}{X\nabla_X V} = b^T \frac{I_Q \nabla_Q V}{X\nabla_X V} \qquad (2.120)$$

holds and the optimal trading strategy is

$$\theta_{y_v} = [B\sigma^T\kappa^{-1}]^T \nabla V = I_P \nabla_P V - \kappa^{-1}[\mu - r1_M] X\nabla_X V. \qquad (2.121)$$

Note that on the interval $[\tau, H]$ where $\tau = \inf\{t: V_v^{c_{y_v},\theta_{y_v}}(t)=0\}$ we have

$$V_v^{c_{y_v},\theta_{y_v}}(t)=0,\ \theta_{y_v}(t)=0 \text{ and } c_{y_v}(t)=0. \qquad (2.122)$$

Furthermore, if $\frac{\partial V}{\partial X} \le 0$, we can write

$$X(t) = X(t, P(t), Q(t), V_v^{c_{y_v},\theta_{y_v}}(t)) \qquad (2.123)$$

where the function $X(t,P,Q,V)$ is twice continuously differentiable in the state variables P, Q and V and once continuously differentiable in t which means that the optimal fund withdrawal rate and the optimal trading strategy are again continuously differentiable feedback controls, i.e.,

$$c_{y_v}(t) = c(t, P(t), Q(t), V_v^{c_{y_v},\theta_{y_v}}(t)) \text{ and } \theta_{y_v}(t) = \theta(t, P(t), Q(t), V_v^{c_{y_v},\theta_{y_v}}(t)). \qquad (2.124)$$

Investment management decisions concerning both liability funding and asset allocation problems in the securities or traditional financial markets - the bond and stock markets - can therefore be based on numerical solutions of the above linear partial differential equations, i.e., on precise quantitative concepts and guidelines. Investment risk in these markets is characterized by the stochastic variation of the optimal fund withdrawal rate $c(t)$ and final wealth $V_v^{c\theta}(H)$ about their mathematical expectations. Investors manage this risk exposure by a specific choice of their utility functions. However, on a single-instrument basis claims contingent on these securities are highly geared investments. A small relative change in the value of the underlying securities usually leads to a large relative change in the value of the corresponding contingent claims (leverage). Derivatives therefore introduce an additional element of investment risk described by their first order $[\Delta(t)]$ and second order $[\Gamma(t)]$ sensitivities with respect to changes in the value of the underlying securities (or market variables in general). Typically, investors try to eliminate or reduce this derivatives risk exposure by entering into securities (futures) market transactions that hedge their options books on an instrument-by-instrument basis (by full or partial replication with the underlying, i.e., by creating an equal or approximately equal but opposite synthetic options position).

3. Contingent Claims and Hedging Strategies

Hedging Strategies. A European-style contingent claim (derivative security) with maturity $T \le H$ is a non-negative, F_T-measurable random variable C that satisfies $E[C^n] < \infty$ for some $n > 1$. A hedging strategy for claim C is an admissible pair $(c,\theta) \in A(v)$ with $V_v^{c\theta}(T) = C$. Let H_v^C denote the set of hedging strategies for the contingent claim C which require an initial investment $v > 0$. The fair price of claim C is then

$$p(C) = \inf\{v > 0: H_v^C \ne \varnothing\}. \qquad (3.1)$$

In a complete underlying securities market we have

$$p(C)=\widetilde{E}[D(T)C] \quad (3.2)$$

and there exists a (up to stochastic equivalence) unique hedging strategy $(c_C,\theta_C)\in H^C_{p(C)}$ with $c_C=0$ and associated value process

$$V^{\theta_C}_{p(C)}(t)=V^{c_C\theta_C}_{p(C)}(t)=\frac{1}{D(t)}\widetilde{E}[D(T)C|F_t]. \quad (3.3)$$

An American-style derivative security with maturity $T\le H$ is characterized by a stopping time $\tau\in[0,T]$ (exercise date) and a non-negative, continuous and adapted process $C(t)$ that satisfies $E[(\sup_{0\le t\le T} C(t))^n]<\infty$ for some $n>1$ (payoff function). A hedging strategy for claim $C(t)$ is an admissible pair $(c,\theta)\in A(v)$ with $V^{c\theta}_v(t)\ge C(t)$, $0\le t<T$, and $V^{c\theta}_v(T)=C(T)$. Let again H^C_v denote the set of hedging strategies for the contingent claim $C(t)$ which require an initial investment $v>0$. The fair price of claim $C(t)$ is then as above

$$p(C)=\inf\{v>0: H^C_v\ne\varnothing\}. \quad (3.4)$$

In a complete underlying securities market we define the functions

$$u(t)=\sup_{t\le\tau\le T}\widetilde{E}[D(\tau)C(\tau)] \text{ and } Y(t)=\operatorname{ess\,sup}_{t\le\tau\le T}\widetilde{E}[D(\tau)C(\tau)|F_t] \quad (3.5)$$

and find $u(0)\le p(C)$. Furthermore, since the RCLL $\widetilde{\pi}$-supermartingale $Y(t)$ is the regular and uniformly integrable Snell envelope (i.e., the smallest dominating RCLL $\widetilde{\pi}$-supermartingale) of the discounted contingent claim $D(t)C(t)$ we have

$$u(t)=\widetilde{E}[Y(t)]=\widetilde{E}[D(\tau_t)C(\tau_t)] \text{ with } \tau_t=\inf\{t\le s\le T: D(s)C(s)=Y(s)\} \quad (3.6)$$

and the Doob-Meyer decomposition

$$Y(t)=u(0)+M(t)-A(t),\ M(0)=A(0)=0, \quad (3.7)$$

where the $\widetilde{\pi}$-martingale $M(t)$ can be represented in the form

$$M(t)=\int_0^t D(s)\theta_C(s)^T\sigma(s)d\widetilde{W}(s). \quad (3.8)$$

With

$$V(t)=\frac{Y(t)}{D(t)} \text{ and } c_C(t)=\int_0^t\frac{dA(s)}{D(s)} \quad (3.9)$$

we find

$$dV(t)=V(t)r(t)dt-dc_C(t)+\theta_C(t)^T\sigma(t)d\widetilde{W}(t) \qquad \begin{array}{c}\int_0^T 1_{V(t)>C(t)}dc_C(t)=0\\ \tau_t=\inf\{t\le s\le T: V(s)=C(s)\}\end{array} \quad (3.10)$$

and therefore that $(c_C,\theta_C)\in H^C_{u(0)}$ is a hedging strategy for claim $C(t)$ with associated value process $V(t)=V^{c_C\theta_C}_{u(0)}(t)$, that $p(C)=u(0)=\widetilde{E}[D(\tau_0)C(\tau_0)]$ and thus τ_0 is the optimal option exercise date. Furthermore, the value process has the representation

$$V^{\theta_C}_{p(C)}(t)=V^{c_C\theta_C}_{p(C)}(t)=\frac{1}{D(t)}\widetilde{E}[D(\tau_0)C(\tau_0)|F_t]. \quad (3.11)$$

In a Markovian setting

$$C=h(P(T)) \text{ (European options)} \quad (3.12a)$$

$$C(t)=h(t,P(t)) \text{ (American options)} \quad (3.12b)$$

$$V^{\theta_C}_{p(C)}(t)=V(t,P(t)) \quad (3.12c)$$

(i.e., under our additional assumptions about the coefficients of the complete securities market model) the above option value functions are solutions of the linear partial differential equation

$$\frac{\partial V}{\partial t}+A^T\nabla V+\frac{1}{2}\mathrm{tr}(BB^T\nabla^2 V)-\nabla V^T B\alpha - Vr=0 \qquad B\alpha = A-rP$$
$$\frac{\partial V}{\partial t}+\frac{1}{2}\mathrm{tr}(BB^T\nabla^2 V)+r(P^T\nabla V - V)=0 \qquad (3.13)$$

with boundary conditions

$$V(T,P)=h(P) \text{ (European options)} \quad (3.14a)$$
$$V(\tau_0,P)=h(\tau_0,P) \text{ (American options)} \quad (3.14b)$$

where

$$A=\begin{bmatrix} P_1\mu_1 \\ \vdots \\ P_N\mu_N \end{bmatrix},\ B=\begin{bmatrix} P_1\sigma_{11} & \cdots & P_1\sigma_{1N} \\ \vdots & & \vdots \\ P_N\sigma_{N1} & \cdots & P_N\sigma_{NN} \end{bmatrix} \text{ and } \nabla=\begin{bmatrix} \frac{\partial}{\partial P_1} \\ \vdots \\ \frac{\partial}{\partial P_N} \end{bmatrix}. \quad (3.15)$$

Furthermore, the associated trading strategies that hedge (or replicate) these options are

$$\theta_C=[B\sigma^{-1}]^T\nabla V = I_P\nabla V. \quad (3.16)$$

<u>Mean Self-Financing Replication</u>. Returning to the general setting we note that the trading strategies $\nu_C(t)$,

$$\nu_C(t)=\nabla V(t,P(t)), \quad (3.17)$$

$[\theta_i(t)=\nu_i(t)P_i(t)]$ derived above are predictable stochastic processes and the associated discounted derivatives value functions $D(t)V^{\nu_C}_{p(C)}(t)=D(t)V^{\theta_C}_{p(C)}(t)$ are $\tilde{\pi}$-martingales with the representation

$$D(t)V^{\nu_C}_{p(C)}(t)=p(C)+\int_0^t d[D(s)G^{\nu_C}(s)]. \quad (3.18)$$

That is, these option replication (or hedging) strategies are self-financing and the exercise payoffs of the discounted contingent claims can be written in the form

$$D(T)C=p(C)+\int_0^T d[D(t)G^{\nu_C}(t)] \text{ (European options)} \quad (3.19a)$$

$$D(\tau_0)C(\tau_0)=p(C)+\int_0^{\tau_0} d[D(t)G^{\nu_C}(t)] \text{ (American options).} \quad (3.19b)$$

In an incomplete underlying securities market we first consider the case of European options. Following Jacka [31] we define attainability of contingent claim C in a securities market completion $\hat{\xi}_a(t)$ by (the required existence of a representation)

$$\hat{\xi}_a(T)C=p(C)+\int_0^T d[\hat{\xi}_a(t)G^{\nu_C}(t)] \quad (3.20)$$

where the trading strategy $\nu_C(t)$ is a predictable process and the discounted financial gains process $\hat{\xi}_a(t)G^{\nu_C}(t)$ a martingale. We then consider probability measures $\hat{\pi}_a$ such that the discounted security price processes $\hat{\xi}_a(t)P_i(t)$ are local martingales under $\hat{\pi}_a$ (local martingale measures) and define the sets

$$\Pi_a=\{\hat{\pi}_a:\hat{\pi}_a\cong\pi\} \qquad \hat{\Pi}_a=\{\hat{\pi}_a:\hat{\pi}_a<<\pi\}$$
$$\tilde{\Pi}_a=\left\{\hat{\pi}_a:\hat{\pi}_a\cong\pi,\left\|d\hat{\pi}_a/d\pi\right\|_\infty\vee\left\|d\pi/d\hat{\pi}_a\right\|_\infty<\infty\right\} \quad (3.21)$$

of equivalent, absolutely continuous and uniformly equivalent [a(t)-] local martingale measures on (Ω,Φ,F). Attainability of a European option C with respect to the given probability measure π (reference probability) means that the associated value process has the form

$$V^{v_C}_{p(C)}(t) = \frac{1}{\hat{\xi}_a(t)} E[\hat{\xi}_a(T)C|F_t] \quad (3.22)$$

and can now be characterized as follows. The statements (1) contingent claim C is attainable, (2) $E[\hat{\xi}_a(T)C] = \hat{E}_a[\hat{\xi}_a(T)C]$ holds, $\tilde{\pi}_a \in \tilde{\Pi}_a$, and (3) the stochastic process $E[\hat{\xi}_a(T)C|F_t]$ is a $\tilde{\pi}_a$-martingale, $\tilde{\pi}_a \in \tilde{\Pi}_a$, are equivalent. Furthermore, claim C is π-attainable if and only if it is $\tilde{\pi}_a$-attainable for all local martingale measures $\tilde{\pi}_a \in \Pi_a$ [or all measures $\tilde{\pi}_a \in \hat{\Pi}_a$] and therefore for these measures we have a similar representation

$$V^{v_C}_{p(C)}(t) = \frac{1}{\hat{\xi}_a(t)} \hat{E}_a[\hat{\xi}_a(T)C|F_t] \quad (3.23)$$

of its value process. (This representation of the option value can be extended to non-attainable contingent claims with the definition of a unique minimal equivalent martingale measure $\tilde{\pi}_a$ under which the discounted security price processes are square-integrable martingales.) In addition, attainability of contingent claim C is equivalent to the fact that $E[\hat{\xi}_a(T)C] = \max_{\tilde{\pi}_a \in \Pi_a} \hat{E}_a[\hat{\xi}_a(T)C]$. If however a given European option C is not attainable, then following Schweizer [36] (see also Foellmer and Sondermann [32], Schweizer [33], Elliott and Foellmer [34], Foellmer and Schweizer [35], Schweizer [37], Ansel and Stricker [38], Hofmann, Platen and Schweizer [39] and Colwell and Elliott [40]) we can consider trading strategies $v(t)$ that are predictable stochastic processes but not necessarily self-financing and define the associated discounted (cumulative) cost process

$$C^v(t) = \hat{\xi}_a(t)[V^v(t) - G^v(t)]. \quad (3.24)$$

A trading strategy $v(t)$ is self-financing if $C^v(t) = c$ is constant and it is mean self-financing if $C^v(t)$ is a martingale. Note that the corresponding discounted financial gains process

$$\hat{\xi}_a(t)G^v(t) = \int_0^t \hat{\xi}_a(s)[\theta(s)^T\sigma(s) - G^v(s)\hat{\alpha}_a(s)^T]dW(s) \quad (3.25)$$

$[\theta_i(s) = v_i(s)P_i(s)]$ is a local martingale but not necessarily a martingale. The risk process (financing cost uncertainty) associated with such a generalized trading strategy $v(t)$ is then

$$R^v_T(t) = E[(C^v(T) - C^v(t))^2|F_t]. \quad (3.26)$$

A trading strategy $v(t)$ is risk minimizing if for any $0 \le s < T$ and any admissible variation $\delta(t)$ of $v(t)$ from s on $R^{v+\delta}_T(s) \ge R^v_T(s)$ holds. Given a small perturbation $\delta(t)$ of trading strategy $v(t)$ and a partition $0 = \tau_0 <..< \tau_k <..< \tau_n = T$ of the relevant replication (hedge) horizon the associated local risk coefficient is

$$r^\tau(v,\delta) = \sum_{k=0}^{n-1} \frac{R^{v+\delta|(\tau_k,\tau_{k+1}]}_T(\tau_k) - R^v_T(\tau_k)}{E[\langle M\rangle_{\tau_{k+1}} - \langle M\rangle_{\tau_k}|F_{\tau_k}]} 1_{(\tau_k,\tau_{k+1}]} \quad (3.27)$$

where $M(t)$ is the square-integrable martingale part (in the Doob-Meyer decomposition) of the continuous supermartingale $\hat{\xi}_a(t)P(t)$. Trading strategy $v(t)$ is locally risk minimizing if $\underline{\lim}_{m\to\infty} r^{\tau_m}(v,\delta) \ge 0$ for every small perturbation $\delta(t)$ and every increasing sequence (τ_m) of partitions with $|\tau_m| \downarrow 0$. Note that any European option C (which is not necessarily attainable) can now be replicated with such generalized trading strategies, i.e., $C = V^v(T)$ holds for some $v(t)$ with associated (discounted) replication costs $C^v(T)$. The following statements are equivalent (characterization of attainability): (1) Contingent claim C is attainable; (2) there is a self-financing replication strategy $v(t)$ for claim C; (3) there exists a replication strategy $v(t)$ for claim C with $R_T^v(0) = 0$; (4) there exists a replication strategy $v(t)$ for claim C with $R_T^v(t) = 0$, $0 \le t \le T$. Furthermore, if $v(t)$ is any trading strategy and $0 \le s \le T$ is given, then there exists a trading strategy $\bar{v}(t)$ with the properties (1) $V^{\bar{v}}(T) = V^v(T)$, (2) $C^{\bar{v}}(t) = E[C^{\bar{v}}(T)|F_t]$, $s \le t \le T$, and (3) $R_T^{\bar{v}}(t) \le R_T^v(t)$, $s \le t \le T$. [Note that choosing $s = 0$ implies that the strategy $\bar{v}(t)$ is mean self-financing.] In addition, the following statements are equivalent: (1) Replication strategy $v(t)$ for claim C is locally risk minimizing; (2) $v(t)$ *is mean self-financing and the associated (discounted) replication cost process* $C^v(t)$ is orthogonal to $M(t)$. If $A(t)$ is the bounded variation part of the discounted securities price process $\hat{\xi}_a(t)P(t)$, then under appropriate integrability assumptions (see [36], [37], [38] and [40]) the density

$$\alpha_a(t) = \frac{dA(t)}{d\langle M\rangle_t} \qquad (3.28)$$

exists and the process

$$\begin{aligned} \beta_a(t) &= \exp(-\int_0^t \alpha_a(s)^T dM(s) - \frac{1}{2}\int_0^t \|\alpha_a(s)\|^2 d\langle M\rangle_s) \\ &= 1 - \int_0^t \beta_a(s)\alpha_a(s)^T dM(s) \end{aligned} \qquad (3.29)$$

is a martingale which means that a unique minimal equivalent martingale measure

$$\tilde{\pi}_a(\varphi) = E[\beta_a(H)1_\varphi], \ \varphi \in \Phi, \quad (3.30)$$

for the securities market completion $\hat{\xi}_a(t)$ can be defined. The discounted asset price process $\hat{\xi}_a(t)P(t)$ is a square-integrable martingale under this measure and any square-integrable π-martingale orthogonal to $M(t)$ is a martingale under $\tilde{\pi}_a$. Note in this context (see Schachermayer [41]) that the construction of an equivalent martingale measure with the Girsanov theorem in general only succeeds if the discounted equilibrium securities price process is continuous and that the existence of such an equivalent martingale measure - even if it is of uniformly bounded density with respect to the reference probability π - does not imply that there is also a minimal equivalent martingale measure (as defined in [36]). Using a Kunita-Watanabe decomposition we can now represent a European contingent claim C in the form

$$\hat{\xi}_a(T)C = \tilde{E}_a[\hat{\xi}_a(T)C] + \int_0^T v_C(t)^T d[\hat{\xi}_a(t)P(t)] + L_C(T) \qquad (3.31)$$

where the process $L_C(t)$, $L_C(0) = 0$, is a square-integrable $\tilde{\pi}_a$-martingale that is orthogonal to $\hat{\xi}_a(t)P(t)$ under $\tilde{\pi}_a$. $v_C(t)$ is the unique risk minimizing replication strategy for claim C under the minimal equivalent martingale measure, it is mean self-financing under $\tilde{\pi}_a$ and the associated value process has the representation

$$V^{v_C}(t) = \frac{1}{\hat{\xi}_a(t)}\tilde{E}_a[\hat{\xi}_a(T)C|F_t]. \qquad (3.32)$$

Furthermore, $\nu_C(t)$ is locally risk minimizing and therefore the associated (discounted) replication cost process $C^{\nu_C}(t)$ a martingale orthogonal to M(t) under the initially given probability measure π. Note that in such a general securities market completion

$$d[\hat{\xi}_a(t)G^{\nu_C}(t)] = \hat{\xi}_a(t)[\theta_C(t)^T\sigma(t) - G^{\nu_C}(t)\hat{\alpha}_a(t)^T]dW(t) \quad (3.33)$$

whereas

$$\nu_C(t)^T d[\hat{\xi}_a(t)P(t)] = \hat{\xi}_a(t)[\theta_C(t)^T\sigma(t) - (G^{\nu_C}(t) - \rho_C(t))\hat{\alpha}_a(t)^T]dW(t) \quad (3.34)$$

[$\theta_i(t) = \nu_i(t)P_i(t)$ and $\rho(t) = G(t) - \sum_{i=1}^{M}\theta_i(t)$]. Moreover,

$$dC^{\nu_C}(t) = dL_C(t) + \hat{\xi}_a(t)\rho_C(t)\hat{\alpha}_a(t)^T dW(t),\ C^{\nu_C}(0) = V^{\nu_C}(0). \quad (3.35)$$

In the securities market completion $\hat{\xi}_{\hat{a}_{y_v}}(t)$ [defined by the investor's overall risk management objectives $U^c(t,c)$ and $U^V(V)$] the minimax local martingale measure $\tilde{\pi}_{\hat{a}_{y_v}}$ makes the discounted security price processes $D(t)P_i(t)$ square-integrable martingales

$$D(t)P_i(t) = p_i \exp(\int_0^t \sigma_{i\cdot}(s)d\tilde{W}_{\hat{a}_{y_v}}(s) - \frac{1}{2}\int_0^t \|\sigma_{i\cdot}(s)\|^2 ds) \quad \tilde{E}_{\hat{a}_{y_v}}[\exp(\frac{1}{2}\int_0^t \|\sigma_{i\cdot}(s)\|^2 ds)] < \infty \quad (3.36)$$

and therefore the Kunita-Watanabe decomposition technique can be applied to a European option C with respect to this probability measure, i.e.,

$$D(T)C = \tilde{E}_{\hat{a}_{y_v}}[D(T)C] + \int_0^T \nu_C(t)^T d[D(t)P(t)] + L_C(T), \quad (3.37)$$

where $L_C(t)$, $L_C(0) = 0$, is a square-integrable $\tilde{\pi}_{\hat{a}_{y_v}}$-martingale orthogonal to D(t)P(t) under $\tilde{\pi}_{\hat{a}_{y_v}}$. $\nu_C(t)$ is the unique risk minimizing replication strategy for claim C under the minimax local martingale measure, furthermore it is mean self-financing under $\tilde{\pi}_{\hat{a}_{y_v}}$ and the associated value process has the representation

$$V^{\nu_C}(t) = \frac{1}{D(t)}\tilde{E}_{\hat{a}_{y_v}}[D(T)C|F_t]. \quad (3.38)$$

Note that in this (in terms of the underlying general investment management and asset allocation problem relevant) securities market completion we have the representation

$$d[D(t)G^{\nu_C}(t)] = \nu_C(t)^T d[D(t)P(t)] = D(t)\theta_C(t)^T\sigma(t)d\tilde{W}_{\hat{a}_{y_v}}(t) \quad (3.39)$$

[$\theta_i(t) = \nu_i(t)P_i(t)$] of the discounted financial gains process. Moreover,

$$dC^{\nu_C}(t) = dL_C(t),\ C^{\nu_C}(0) = V^{\nu_C}(0), \quad (3.40)$$

holds for the corresponding (discounted) replication costs.

Partial Replication. Bouleau and Lamberton [42] point out an alternative to the above approach in a Markovian framework

$$D(t)P(t) = g(t,P(t),Q(t)) \qquad D(T)C = h(D(T)P(T))$$
$$\tilde{E}_{\hat{a}_{y_v}}[h(D(T)P(T))|F_t] = f(t,P(t),Q(t)) \quad (3.41)$$

(i.e., under our additional assumptions about the coefficients of the incomplete securities market model). They consider the (discounted) residual at maturity

$$R_T^{cv} = h(D(T)P(T)) - D(T)V_c^v(T) \quad (3.42)$$

associated with a self-financing trading strategy $\nu(t)$ and an initial capital $c > 0$ (lack of an appropriate hedging strategy for the European option C) and find that the inequality

$$\widetilde{E}_{\hat{a}_{y_v}}[(R_T^{cv})^2] \geq \widetilde{E}_{\hat{a}_{y_v}}[(c - \widetilde{E}_{\hat{a}_{y_v}}[h(D(T)P(T))])^2] + \widetilde{E}_{\hat{a}_{y_v}}\left[\int_0^T [\Gamma(f,f) - \lim_{\varepsilon\to 0}\Gamma(f,g^T)[\Gamma(g,g^T)+\varepsilon I_M]^{-1}\Gamma(g,f)](t)dt\right] \quad (3.43)$$

holds where

$$\Gamma(u,v) = A(uv) - uAv - vAu \qquad A = A^T\nabla + \frac{1}{2}\mathrm{tr}(BB^T\nabla^2) \quad (3.44)$$

with the state space characteristics

$$A = \begin{bmatrix} P_1\mu_1 \\ \vdots \\ P_M\mu_M \\ Q_1\hat{a}_1^{y_v} \\ \vdots \\ Q_{N-M}\hat{a}_{N-M}^{y_v} \end{bmatrix} \qquad B = \begin{bmatrix} P_1\sigma_{11} & \cdots & P_1\sigma_{1N} \\ \vdots & & \vdots \\ P_M\sigma_{M1} & \cdots & P_M\sigma_{MN} \\ Q_1b_{11} & \cdots & Q_1b_{1N} \\ \vdots & & \vdots \\ Q_{N-M}b_{N-M1} & \cdots & Q_{N-M}b_{N-MN} \end{bmatrix} \qquad \nabla = \begin{bmatrix} \nabla_P \\ \nabla_Q \end{bmatrix} \quad (3.45)$$

is the carre-du-champ operator of the incomplete securities market model, i.e., of $(t, P(t), Q(t))$. The unique trading strategy $\nu_C(t)$ and initial investment c_C that minimize this residual risk (i.e., for which equality holds above) are then given by

$$\nu_C = \lim_{\varepsilon\to 0}\Gamma(f,g^T)[\Gamma(g,g^T)+\varepsilon I_M]^{-1} \text{ and } c_C = \widetilde{E}_{\hat{a}_{y_v}}[h(D(T)P(T))]. \quad (3.46)$$

The fair price of contingent claim C is therefore $p(C) = c_C = \widetilde{E}_{\hat{a}_{y_v}}[D(T)C]$ and its value process (which is the same as the one obtained above by applying the methods of Schweizer [36]) is given by

$$p_t(C) = \frac{1}{D(t)}\widetilde{E}_{\hat{a}_{y_v}}[D(T)C|F_t]. \quad (3.47)$$

The corresponding (discounted) time t residual is

$$R_C(t) = D(t)[p_t(C) - V_{p(C)}^{\nu_C}(t)] \quad (3.48)$$

and thus by Doob's inequality the maximum residual risk over the entire hedge horizon satisfies

$$\widetilde{E}_{\hat{a}_{y_v}}[(\sup_{0\leq t\leq T}|R_C(t)|)^2] \leq 4\int_0^T \Pi(t)[\Gamma(f,f) - \lim_{\varepsilon\to 0}\Gamma(f,g^T)[\Gamma(g,g^T)+\varepsilon I_M]^{-1}\Gamma(g,f)](t)dt \quad (3.49)$$

where $\Pi(t)$, $f(t,P,Q) = \Pi(T-t)h(g(T,P,Q))$, is the transition semigroup associated with the securities market variables, i.e., with $(P(t),Q(t))$.

American Options. We now consider an American option $C(t)$ which we analyze in the securities market completion $\hat{\xi}_{\hat{a}_{y_v}}(t)$ with the techniques used in Schweizer [36]. To this end we define the two functions

$$u(t) = \sup_{t\leq\tau\leq T}\widetilde{E}_{\hat{a}_{y_v}}[D(\tau)C(\tau)] \text{ and } Y(t) = \operatorname*{ess\,sup}_{t\leq\tau\leq T}\widetilde{E}_{\hat{a}_{y_v}}[D(\tau)C(\tau)|F_t] \quad (3.50)$$

where (as in the case of a complete underlying securities market)

$$Y(t) = u(0) + M(t) - A(t),\ M(0) = A(0) = 0, \quad (3.51)$$

is the regular and uniformly integrable Snell envelope (i.e., the smallest dominating RCLL $\widetilde{\pi}_{\hat{a}_{y_v}}$-supermartingale) of the discounted contingent claim $D(t)C(t)$. Then we have

$$u(t) = \widetilde{E}_{\hat{a}_{y_v}}[Y(t)] = \widetilde{E}_{\hat{a}_{y_v}}[D(\tau_t)C(\tau_t)] \text{ with } \tau_t = \inf\{t\leq s\leq T: D(s)C(s) = Y(s)\}. \quad (3.52)$$

If we now define

$$V(t) = \frac{Y(t)}{D(t)}, \quad (3.53)$$

we find

$$\int_0^T 1_{V(t)>C(t)} dA(t) = 0 \text{ and } \tau_t = \inf\{t \le s \le T: V(s) = C(s)\} \quad (3.54)$$

and therefore that if we require the discounted value processes $D(t)V^{\nu}(t)$ of replication strategies $\nu(t)$ for claim $C(t)$ to be RCLL $\tilde{\pi}_{\hat{a}_{y_\nu}}$-supermartingales [i.e., the American option $C(t)$ to be compatible with the given equilibrium bond and stock prices $P(t)$], then $V^{\nu c}(t) \ge V(t)$, $0 \le t < \tau_0$, and $V^{\nu c}(\tau_0) = V(\tau_0)$ holds for the risk minimizing replication strategy $\nu_C(t)$ for claim $C(t)$ and τ_0 is the optimal option exercise date. This is the case because on the time interval $[0, \tau_0]$ the Snell envelope $Y(t)$ is actually a $\tilde{\pi}_{\hat{a}_{y_\nu}}$-martingale and consequently any replication strategy $\nu(t)$ with $V^{\nu}(t) \ge V(t)$ can be replaced by a trading strategy $\bar{\nu}(t)$ that given $0 \le s \le \tau_0$ satisfies (1) $V^{\bar{\nu}}(t) \ge V(t)$, $s \le t < \tau_0$, and $V^{\bar{\nu}}(\tau_0) = V(\tau_0)$, (2) $C^{\bar{\nu}}(t) = \tilde{E}_{\hat{a}_{y_\nu}}[C^{\bar{\nu}}(\tau_0)|F_t]$, $s \le t \le \tau_0$, and (3) $R^{\bar{\nu}}_{\tau_0}(t) \le R^{\nu}_{\tau_0}(t)$, $s \le t \le \tau_0$. As in the European option case the Kunita-Watanabe decomposition technique can then be applied to the exercise payoff $C(\tau_0)$ of contingent claim $C(t)$, i.e.,

$$D(\tau_0)C(\tau_0) = \tilde{E}_{\hat{a}_{y_\nu}}[D(\tau_0)C(\tau_0)] + \int_0^{\tau_0} \nu_C(t)^T d[D(t)P(t)] + L_C(\tau_0), \quad (3.55)$$

where $L_C(t)$, $L_C(0) = 0$, is a square-integrable $\tilde{\pi}_{\hat{a}_{y_\nu}}$-martingale orthogonal to $D(t)P(t)$ under $\tilde{\pi}_{\hat{a}_{y_\nu}}$. $\nu_C(t)$ is the unique risk minimizing replication strategy for claim $C(t)$ under the minimax local martingale measure, furthermore it is mean self-financing under $\tilde{\pi}_{\hat{a}_{y_\nu}}$ and the associated value process has the representation

$$V^{\nu c}(t) = V(t) = \frac{1}{D(t)} \tilde{E}_{\hat{a}_{y_\nu}}[D(\tau_0)C(\tau_0)|F_t]. \quad (3.56)$$

In a Markovian setting

$$C = h(P(T), Q(T)) \text{ (European options)} \quad (3.57a)$$

$$C(t) = h(t, P(t), Q(t)) \text{ (American options)} \quad (3.57b)$$

$$V^{\nu c}(t) = V(t, P(t), Q(t)) \quad (3.57c)$$

(i.e., under our additional assumptions about the coefficients of the incomplete securities market model) the above option value functions are solutions of the linear partial differential equation

$$\frac{\partial V}{\partial t} + A^T \nabla V + \frac{1}{2} \mathrm{tr}(BB^T \nabla^2 V) - \nabla V^T B \hat{\alpha}_{\hat{a}_{y_\nu}} - Vr = 0 \qquad B\hat{\alpha}_{\hat{a}_{y_\nu}} = A - r\begin{bmatrix} P \\ Q \end{bmatrix}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2} \mathrm{tr}(BB^T \nabla^2 V) + r(\begin{bmatrix} P \\ Q \end{bmatrix}^T \nabla V - V) = 0 \quad (3.58)$$

with boundary conditions

$$V(T, P, Q) = h(P, Q) \text{ (European options)} \quad (3.59a)$$

$$V(\tau_0, P, Q) = h(\tau_0, P, Q) \text{ (American options)} \quad (3.59b)$$

where

$$A = \begin{bmatrix} P_1\mu_1 \\ \vdots \\ P_M\mu_M \\ Q_1\hat{a}_1^{y_v} \\ \vdots \\ Q_{N-M}\hat{a}_{N-M}^{y_v} \end{bmatrix}, \quad B = \begin{bmatrix} P_1\sigma_{11} & \cdots & P_1\sigma_{1N} \\ \vdots & & \vdots \\ P_M\sigma_{M1} & \cdots & P_M\sigma_{MN} \\ Q_1b_{11} & \cdots & Q_1b_{1N} \\ \vdots & & \vdots \\ Q_{N-M}b_{N-M1} & \cdots & Q_{N-M}b_{N-MN} \end{bmatrix} \text{ and } \nabla = \begin{bmatrix} \nabla_P \\ \nabla_Q \end{bmatrix}. \quad (3.60)$$

In addition, we find that

$$D(t)[\nabla V^T B - \theta_C{}^T\sigma](t)d\widetilde{W}_{\hat{a}_{y_v}}(t) = dL_C(t) \quad (3.61)$$

and therefore

$$D[\nabla V^T B - \theta_C{}^T\sigma]dt = \widetilde{E}_{\hat{a}_{y_v}}[dL_C d\widetilde{W}_{\hat{a}_{y_v}}{}^T | F_t] \qquad \sigma\widetilde{E}_{\hat{a}_{y_v}}[dL_C d\widetilde{W}_{\hat{a}_{y_v}} | F_t] = 0$$

$$\sigma[B^T\nabla V - \sigma^T\theta_C] = 0 \quad (3.62)$$

$$\theta_C = [B\sigma^T\kappa^{-1}]^T\nabla V = I_P\nabla_P V$$

$[\theta(t) = I_P(t)\nu(t)]$ holds and the associated unique risk minimizing replication strategies are

$$\nu_C = \nabla_P V. \quad (3.63)$$

Market Completion With Options. Within the class of general assets [i.e., assets with RCLL semimartingale price processes $dV(t) = \omega(t)dW(t) + dA(t)$ where $\omega \in \Omega$, $\Omega = K(\sigma) \oplus K^{\perp}(\sigma)$ and $A(t)$ is of class VF (variation finie)] in the incomplete securities market model

$$\alpha(t) = \sigma(t)^T\kappa(t)^{-1}[\mu(t) - r(t)1_M]$$

$$\beta(t) = \exp(-\int_0^t \alpha(s)^T dW(s) - \frac{1}{2}\int_0^t \|\alpha(s)\|^2 ds) \qquad E[\exp(\frac{1}{2}\int_0^t \|\alpha(s)\|^2 ds)] < \infty \quad (3.64)$$

$$\xi(t) = \beta(t)D(t)$$

considered here contingent claims are compatible with the equilibrium bond and stock prices [i.e., after discounting with the risk-neutral discount factor $D(t)$ their value processes are continuous martingales under the minimax local martingale measure $\widetilde{\pi}_{\hat{a}_{y_v}}$] and can therefore be written in the form

$$dV(t) = \begin{matrix} & [\theta(t)^T[\mu(t) - r(t)1_M] + V(t)r(t)]dt + \theta(t)^T\sigma(t)dW(t) \\ + & \omega_1(t)^T\omega_2(t)dt + \omega_1(t)^T dW(t) \end{matrix} \quad (3.65)$$

with the (partial) replication strategy $\theta \in K^{\perp}(\sigma)$ and the additional parameters $\omega_1, \omega_2 \in K(\sigma)$. The first term in this representation is the value process of a purely tradable (attainable) asset while the second term characterizes a totally non-tradable asset in this financial economy. The above decomposition is unique up to constants. Furthermore, every compatible asset (with non-constant associated value process) is either purely tradable (attainable) or totally non-tradable. It also follows that the minimal equivalent martingale measure is necessarily

$$\widetilde{\pi}(\varphi) = E[\beta(H)1_\varphi], \ \varphi \in \Phi, \text{ and } \widetilde{W}(t) = W(t) + \int_0^t \alpha(s)ds \quad (3.66)$$

and that any equivalent martingale measure $\widetilde{\psi}$ for the equilibrium security price processes is related to this minimal (in terms of its distance to the reference probability π measured in units of relative entropy) martingale measure via

$$E[\frac{d\tilde{\psi}}{d\pi}\Big|F_t] = \exp(-\int_0^t \gamma(s)^T dW(s) - \frac{1}{2}\int_0^t \|\gamma(s)\|^2 ds) \quad \gamma(t) = \alpha(t) + \omega(t),\ \omega \in K(\sigma). \quad (3.67)$$

[We therefore call the quantity

$$\nu_{\hat{a}_{y_v}} = \frac{B_Q{}^T \nabla_Q V}{X\nabla_X V} = b^T \frac{I_Q \nabla_Q V}{X\nabla_X V},\ \nu_{\hat{a}_{y_v}} \in K(\sigma)$$

$$E[\frac{d\tilde{\pi}_{\hat{a}_{y_v}}}{d\pi}\Big|F_t] = \exp(-\int_0^t \hat{\alpha}_{\hat{a}_{y_v}}(s)^T dW(s) - \frac{1}{2}\int_0^t \|\hat{\alpha}_{\hat{a}_{y_v}}(s)\|^2 ds) \quad \hat{\alpha}_{\hat{a}_{y_v}}(t) = \alpha(t) + \nu_{\hat{a}_{y_v}}(t)$$

a completion premium.] Moreover, among all equivalent martingale measures for the equilibrium bond and stock price processes the minimal equivalent martingale measure is characterized by the property that for every totally non-tradable asset the stochastic differential equation

$$dV(t) = \omega_1(t)^T \omega_2(t)dt + \omega_1(t)^T dB(t) \quad (3.68)$$

holds with the same coefficients $\omega_1, \omega_2 \in K(\sigma)$ and any (related to each other via the Girsanov theorem) π - and $\tilde{\pi}$ - Brownian motions $B(t)$. Note that all the results about option pricing and hedging derived above in the securities market completion $\hat{\xi}_{\hat{a}_{y_v}}(t)$ also apply in this originally given incomplete market model $\xi(t)$ [with respect to the minimal equivalent martingale measure $\tilde{\pi}$ and the corresponding Girsanov transform $\tilde{W}(t)$ of the exogenous sources $W(t)$ of securities market uncertainty], i.e., we have the derivatives value processes

$$V^{\nu_C}(t) = \frac{1}{D(t)}\tilde{E}[D(T)C|F_t] \text{ (European options)} \quad (3.69a)$$

$$V^{\nu_C}(t) = \frac{1}{D(t)}\tilde{E}[D(\tau_0)C(\tau_0)|F_t] \text{ (American options)} \quad (3.69b)$$

with associated risk minimizing replication costs

$$dC^{\nu_C}(t) = dL_C(t),\ C^{\nu_C}(0) = V^{\nu_C}(0), \quad (3.70)$$

and their Markovian characterization

$$\begin{aligned} &\frac{\partial V}{\partial t} + A^T \nabla V + \frac{1}{2}\mathrm{tr}(BB^T \nabla^2 V) - \nabla V^T B\alpha - Vr = 0 \qquad B\alpha = A - rP \\ &\frac{\partial V}{\partial t} + \frac{1}{2}\mathrm{tr}(BB^T \nabla^2 V) + r(P^T \nabla V - V) = 0 \end{aligned} \quad (3.71)$$

with corresponding boundary conditions $V(T,P) = h(P)$ for European options and $V(\tau_0, P) = h(\tau_0, P)$ for American options where

$$A = \begin{bmatrix} P_1\mu_1 \\ \vdots \\ P_M\mu_M \end{bmatrix} \quad B = \begin{bmatrix} P_1\sigma_{11} & \cdots & P_1\sigma_{1N} \\ \vdots & & \vdots \\ P_M\sigma_{M1} & \cdots & P_M\sigma_{MN} \end{bmatrix} \quad \nabla = \begin{bmatrix} \frac{\partial}{\partial P_1} \\ \vdots \\ \frac{\partial}{\partial P_M} \end{bmatrix} \quad (3.72)$$

as well as the associated unique risk minimizing replication strategies

$$\nu_C = \nabla V, \quad (3.73)$$

but the obtained results are not necessarily consistent with the investor's overall risk management objectives $U^c(t,c)$ and $U^V(V)$. Note also that not attainable (i.e., totally non-tradable with respect to a given incomplete set of underlying bonds and stocks) contingent claims are suitable candidates for a securities market completion in which subsequently an optimal (in the sense of Arrow and Debreu) allocation of investment risk is feasible.

Appendix: References

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