

Shortfall-Probability-Optimized Portfolios in the Case of Riskless Borrowing and Lending

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Abstract

This paper analyses different types of shortfall-probability-based diagrams of efficient frontiers and compares them with each other in terms of their advantages and disadvantages. Expanding the existing literature, the alternative concepts are then generalized for the case of riskless borrowing and lending. In addition, a new type of diagram is developed, which is able to overcome the shortcomings of existing approaches. A scenario with assets having normally distributed returns is used throughout to illustrate the mathematical analysis. Furthermore, two methods of solving the portfolio selection problem under shortfall risk for the case of arbitrary return distributions are presented.

Keywords

Portfolio Optimization, Shortfall Risk, Efficient Frontier, Lower Partial Moment

1. Introduction

It is well known, that under particular assumptions referring to the shape of asset return distributions, portfolio optimization based on shortfall probabilities according to ROY (1952), leads to portfolios, which are positioned on the classical Efficient Frontier in the Mean-Variance-World of MARKOWITZ (1952).¹ This is especially true in the case of normally distributed returns. Therefore, a section of the μ - σ -Efficient-Frontier is also efficient in a *Mean-Shortfall-Probability-World*. To take not only analytically but also graphically account of this fact, several modified representations of the classical Efficient Frontier have been proposed in the literature. These diagrams may be used by institutional investment committees as well as in consulting the private banker's client and lead to a better understanding of the shortfall risk concept, in general. Moreover, performance results can be communicated ex post using these graphics.

This paper has three main goals: First of all, the most important, already known types of graphical representations are discussed and compared which each other in terms of their advantages and disadvantages.² Secondly, expanding the existing literatur, the different concepts are generalized for the (more practical) case of riskless lending and borrowing. Thirdly, a new type of diagramm is developed, which is able to overcome the massive shortcomings of the existing approaches.

The paper is structured as follows: At first, the conventional approaches are explained analytically and graphically in detail. This is the so-called *Mean-LPM₀-Efficient-Frontier* in the second section, the *Efficient Shortfall Frontier* in the third section and the *L-Efficient-Frontier* in the fourth section. Two illustrative scenarios, one for the sole risky portfolio selection and one for the possibility of investing in a riskless asset, may help to understand the different concepts. In the fifth section a new type of diagram, the *Efficient Shortfall Surface* is introduced and explained. Whereas for the first five sections the existence of normally distributed returns is assumed, in the sixth section the consequences of a relaxation of this assumption are discussed. Two different methods of solving the generalized portfolio optimization problem under shortfall risk are shown. The final appendix contains the data, which are used for the scenarios.

2. The Mean-LPM₀-Efficient-Frontier

The first type of shortfall-probability-based representation of portfolios presented here is the so-called *Mean-LPM₀-Efficient-Frontier*³ by HARLOW (1991), who positions all feasible portfolios in a Mean-Shortfall-Probability-World⁴ and derives a corresponding efficient frontier

there. To illustrate the approach, consider a portfolio with a normally distributed return $R \sim N(\mu, \sigma)$. Specifying a target return t^* , the probability $p(t^*)$ of this portfolio of returning less than this threshold is

$$p(t^*) = \frac{1}{\sqrt{2\pi}\sigma} \cdot \int_{-\infty}^{t^*} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx. \quad (1)$$

Substituting $x \equiv \sigma \cdot z + \mu$, which represents the transition to a random variable $Z \sim N(0,1)$, equation (1) can be modified to

$$p(t^*) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\frac{t^*-\mu}{\sigma}} e^{-\frac{1}{2}z^2} dz. \quad (2)$$

Together with corresponding expected returns μ , all feasible portfolios can be positioned now in the μ - $p(t^*)$ -World. HARLOW (1991) introduced the name *Mean-LPM₀-Efficient-Frontier* for the resulting efficient frontier.⁵ Analytically this alternative efficient frontier can be derived by substituting the equation of the classical Mean-Variance Efficient Frontier^{6,7}

$$\sigma = \sqrt{\frac{c\mu^2 - 2b\mu + a}{ac - b^2}} \quad (3)$$

into the equation of the shortfall probability (2):

$$p(t^*) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{-\sqrt{\frac{(ac-b^2)(\mu-t^*)^2}{c\mu^2-2b\mu+a}}} e^{-\frac{1}{2}z^2} dz. \quad (4)$$

Equation (4) shows for all portfolios positioned on the Markowitz Efficient Frontier the relationship between their shortfall probability $p(t^*)$ and their expected return μ , given the fixed target return t^* . Figure 1 displays the relationship graphically:⁸

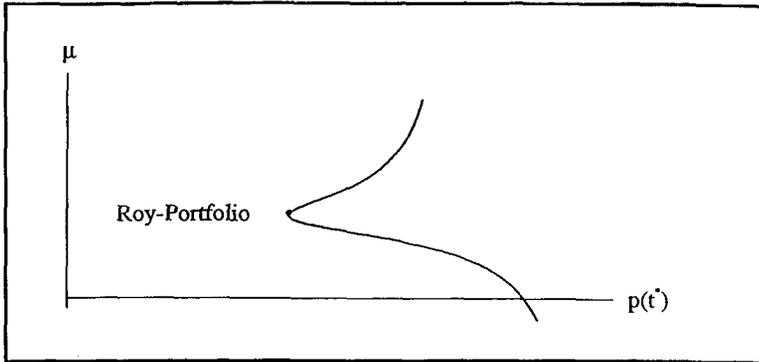


Figure 1: The Mean-LPM₀(t^*)-Efficient-Frontier.

Looking at Figure 1, several points deserve special attention: The global Minimum-Shortfall-Probability-Portfolio within the μ - $p(t^*)$ -World is given by the so-called *Roy-Portfolio* or *Safety-First-Portfolio*. This portfolio is found by drawing a tangent to the classical Efficient Frontier [see Figure 12] through the target return t^* on the vertical axis. The tangent is termed *Shortfall Line*, because it connects all portfolios with equal shortfall probability referring to t^* . All portfolios, positioned in the μ - σ -World on the Efficient Frontier *above* the Roy-Portfolio, are efficient in the μ - $p(t^*)$ -World, too. They can be found on the Mean-LPM₀-Efficient-Frontier *above* the Roy-Portfolio [see Figure 1]. Portfolios, which are positioned in the μ - σ -World on the Efficient Frontier but *under* the Roy-Portfolio, are inefficient in the μ - $p(t^*)$ -World, because for each of these portfolios another portfolio with identical shortfall probability referring to t^* exists, but with a higher expected return. This is especially the case for the so-called Minimum-Standard-Deviation-Portfolio, which is the portfolio with global minimum standard deviation in the μ - σ -World [see for example Figure 11]. Only in the (for practical purposes irrelevant) case of $t^* \rightarrow -\infty$, in which the Roy-Portfolio approaches the Minimum-Standard-Deviation-Portfolio, this portfolio becomes efficient in the μ - $p(t^*)$ -World. Please note, that all other feasible portfolios in the μ - $p(t^*)$ -World are positioned *on the right-hand-side* of the Mean-LPM₀(t^*)-Efficient-Frontier in Figure 1.

Has an investor in addition the opportunity to borrow or lend capital risklessly at a rate R_f , then the set of efficient portfolios in the μ - σ -World is represented by the Capital Market Line [see Figure 12 in the appendix]. Depending on the choice of the target return t^* , three different situations ($t^* < R_f$, $t^* > R_f$ and $t^* = R_f$) result. But it is sufficient here, to analyse only the case $t^* < R_f$ in detail, because this case is the most relevant one for practical purposes.⁹

Analytically, the relationship between expected return and shortfall probability of all Capital Market Line portfolios can be derived by substituting the equation of the Capital Market Line¹⁰

$$\mu = \frac{\mu_M - R_f}{\sigma_M} \cdot \sigma + R_f \Leftrightarrow \tag{5a}$$

$$\sigma = \frac{(\mu - R_f) \cdot \sqrt{c \cdot R_f^2 - 2bR_f + a}}{a - R_f \cdot (2b - cR_f)} \tag{5b}$$

in the equation

$$s = \frac{\mu - t}{\sigma} \tag{6}$$

for the slope of the Shortfall Line. Therewith, the negative slopes of the corresponding, now parameterized Shortfall Lines result in

$$s = \frac{(\mu - t^*) \cdot (a - R_f \cdot (2b - cR_f))}{(\mu - R_f) \cdot \sqrt{c \cdot R_f^2 - 2bR_f + a}} \tag{7}$$

Substituting (7) in (2)¹¹ delivers the sought-after relationship

$$p(t^*) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\frac{(\mu - t^*) \cdot (a - R_f \cdot (2b - cR_f))}{(\mu - R_f) \cdot \sqrt{c \cdot R_f^2 - 2bR_f + a}}} e^{-\frac{1}{2}z^2} dz \tag{8}$$

Graphically, the capital Market Line in the μ - $p(t^*)$ -World is presented in Figure 2:¹²

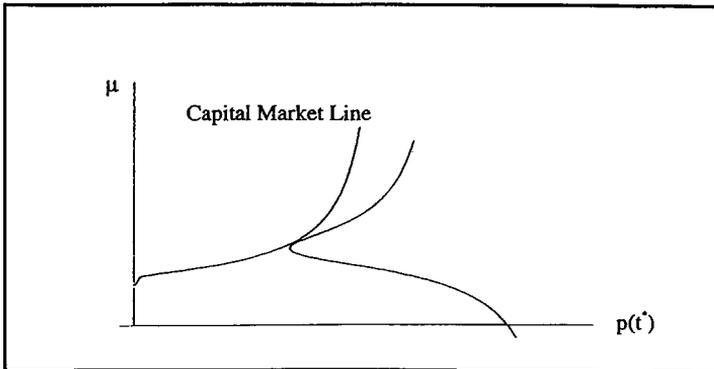


Figure 2: The Capital Market Line in the μ - $p(t^*)$ -World.

Two points in Figure 2 are worth noting: Firstly, because of the form of equation (8), the Capital Market Line in the μ - $p(t^*)$ -World is not a straight line, as one might expect from the classical Mean-Variance-World.¹³ But, as expected, the Capital Market Line is tangent to the Mean-LPM₀-Efficient-Frontier, with the Market Portfolio as tangent point, as this is the case in the μ - σ -World.¹⁴

The approach of HARLOW (1991) described so far, has a significant disadvantage: In order to derive a Mean-LPM₀(t^*)-Efficient-Frontier, a *fixed* target return t^* must be specified *in advance*. This requires, that the investor using Harlow's methodology must be able to specify a single, most relevant threshold for himself. But in practice, often the opposite may be observed: At the beginning of the asset allocation process, most investors¹⁵ do not know what is a suitable target for them and therefore want to consider several or even better *all* feasible alternatives, first.¹⁶ To implement such a procedure within the framework of HARLOW (1991), for each possible target return a *separate* efficient frontier would have to be calculated and plotted. Subsequently, all the resulting curves, which can *not* be drawn in one diagram, must be compared with each other.

3. The Efficient Shortfall Frontier

Is the investor unsure about the suitable target return he should employ, he alternatively can position all Safety-First-Portfolios in a *Threshold-Shortfall-Probability-World*¹⁷. In this world, for each feasible target return t the corresponding minimal shortfall probability $p(t)$, which can be achieved, is displayed. The resulting efficient frontier is termed *Efficient Shortfall Frontier*.¹⁸ We firstly analyse the situation where there is no riskless asset. Here, the minimal shortfall probability which can be achieved, can be derived analytically by using the equation of a tangent

$$\mu = \frac{(ac - b^2) \cdot \sigma_P}{c \cdot \mu_P - b} \cdot \sigma + \frac{b \cdot \mu_P - a}{c \cdot \mu_P - b} \quad (9)$$

to the Mean-Variance Efficient Frontier in point (σ_P, μ_P) . Please note, that the Mean-Variance Efficient Frontier is a hyperbola.¹⁹ Now, for each point (σ_P, μ_P) on the efficient section of the hyperbola, intersection and slope of the corresponding tangent have to be calculated. The intersection equals the target return t , and with the slope of the tangent, the shortfall probability $p(t)$ can be derived. Substituting these results in (2), the sought-after functional relationship between target return t and corresponding minimal shortfall probability $p(t)$ is:

$$p(t) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{-\sqrt{c \cdot t^2 - 2bt + a}} e^{-\frac{1}{2}z^2} dz. \quad (10)$$

Please note also, that in equation (10) the restriction, that only target returns strictly below the expected return $\frac{b}{c}$ of the Minimum-Standard-Deviation-Portfolio are allowed, have to be observed.²⁰ The following Figure 3 represents equation (10) graphically, using the example from the appendix as underlying data:

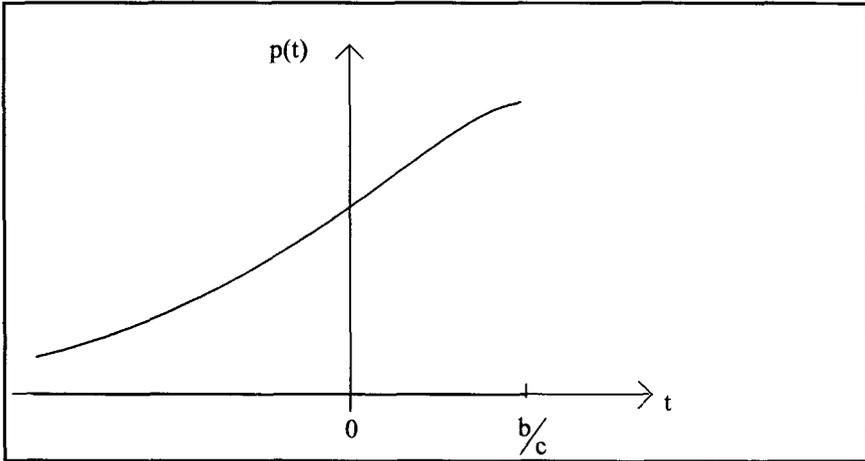


Figure 3: The Efficient Shortfall Frontier in the t - $p(t)$ -World.

In addition, Figure 3 illustrates the situation for $t \rightarrow -\infty$. In this case $p(t)$ asymptotically approaches zero, as it also may be seen analytically from (10). Interpreted economically this means, that an investor is able to meet his decreasing target return with increasing probability. Graphical correspondence of the process $t \rightarrow -\infty$ is the shift of the intercept of the Shortfall Line in the μ - σ -World downwards. In the limiting case, the Shortfall Line equals the vertical axis. Note also, that the set of all feasible portfolios in Figure 3 is positioned *above* the Efficient Shortfall Frontier.

The concept of the Efficient Shortfall Frontier can also be generalized for the case of borrowing and lending risklessly at rate R_f . As it is well known, in this situation the Capital Market Line represents all efficient portfolios in the μ - σ -World. Again, three cases have to be considered: $t < R_f$, $t = R_f$ and $t > R_f$. For the case of $t > R_f$, no Safety-First-Portfolio exists.²¹ Therefore, no Efficient Shortfall Frontier can be generated. The two cases $t < R_f$ and $t = R_f$ can be

analysed together, but a refined distinction of the term „target achievement“ is in order now: Either the investor interprets risk only as the strict underperformance of the target in the sense of $R < t$ or he wants to outperform the target strictly. The latter can be represented mathematically by the formulation $R \leq t$ for the shortfall risk. Of course, this refined distinction has just to be done, if there are discontinuous distributions under consideration, because the value of an integral stays unchanged, if only a single point within the integration intervall is changed.²² Therefore, the generalization for the case of riskless lending and borrowing with a discrete (because deterministic) distribution, forces the above distinction. Figure 4 displays the situation, in which the investor interprets risk only as strict underperformance of the specified target return:

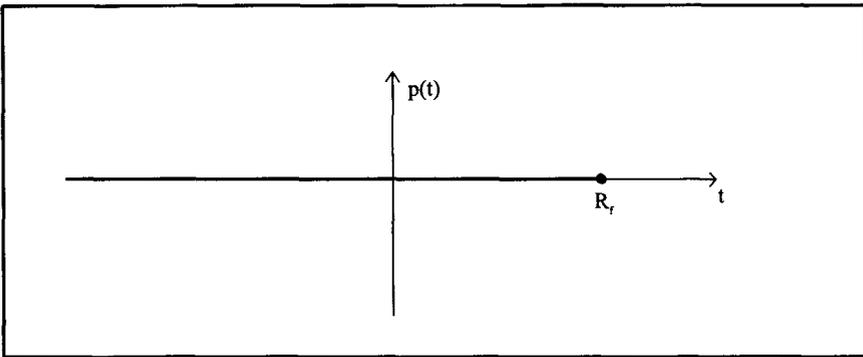


Figure 4: The Efficient Shortfall Frontier with riskless borrowing and lending and strict underperformance of the target return as interpretation of risk.

In this situation, the portfolio invested completely in the riskless asset generates the unambiguous Safety-First-Portfolio. The shortfall probability of this portfolio, interpreted in the described manner, is for all $t \leq R_f$ zero. Geometrically, this means that each point on the Efficient Shortfall Frontier, which is given by a right side enclosed, horizontal half-straight-line in Figure 4, represents the same portfolio: complete investment in the riskless asset.

If, on the opposite the investor is interested in strict outperformance of the target, the Efficient Shortfall Frontier exhibits a discontinuity: Now, for $t = R_f$ the riskless portfolio has a shortfall probability of 100% and is therefore not safety first anymore. Instead of this portfolio all portfolios positioned on the Capital Market Line are safety first.²³ The shortfall probability of these portfolios can be calculated using (2), and is

$$p(R_f) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\frac{R_f - \mu_M}{\sigma_M}} e^{-\frac{1}{2}z^2} dz. \quad (11)$$

Figure 5 illustrates the situation:

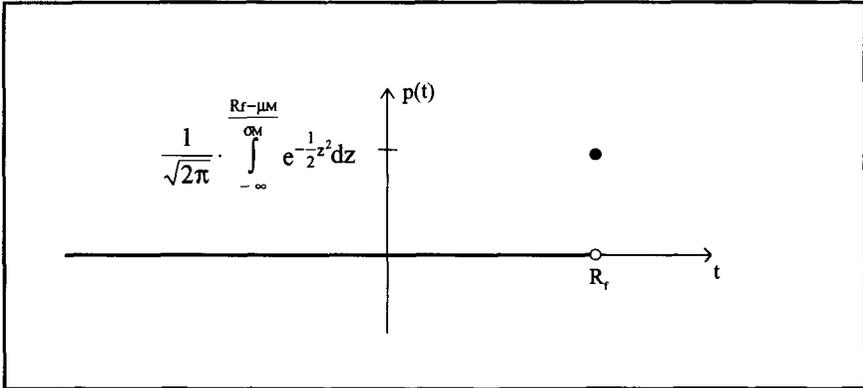


Figure 5: The Efficient Shortfall Frontier with riskless borrowing and lending and strict out-performance of the target return as interpretation of risk.

Again, all points on the now right side opened, horizontal half-straight-line represent the riskless portfolio. But *all* Capital Market Line portfolios, except the riskless portfolio, are plotted

on the point $(R_f, \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\frac{R_f - \mu_M}{\sigma_M}} e^{-\frac{1}{2}z^2} dz)$.

This approach shows to those investors, who are still unsure about the target return they should use, *all* alternatives summarized in *one* diagram. So in this respect, the Efficient Shortfall Frontier has a clear advantage compared to Harlow's Mean-LPM₀-Efficient-Frontier discussed earlier. But on the other hand, an important disadvantage of this type of graph must be considered: The t-p(t)-diagram does *not* provide any information about the expected return of the portfolios, which is an absolute essential characteristic of any portfolio. Therefore, this approach is also not able to meet our aim of representing portfolios in a Mean-Shortfall-Probability-World analogue to the Mean-Variance-World.

4. The L-Efficient-Frontier

The above discussed type of representation for portfolios in a t - $p(t)$ -World is able to show *shortfall-minimal portfolios for all feasible target returns*, but offers no information about the corresponding expected returns of these portfolios. An improved approach by KADUFF/SPREMANN (1996), the so-called *L-Efficient-Frontier* merges both advantages: Without the need of specifying a fixed target return, all Safety-First-Portfolios are positioned in the μ - $p(t)$ -World. Therefore, the expected return of each portfolio can be directly read from such a diagram. One should notice carefully, that the shortfall probability $p(t)$ displayed here on the horizontal axis, is calculated for *varying* target returns, which separates the μ - $p(t)$ -World strictly from Harlow's μ - $p(t^*)$ -World discussed in the second section of this paper.

Analytically, the functional relationship between expected returns and shortfall probabilities for varying targets of all Safety-First-Portfolios, can be found by substituting the equation for the hyperbola tangent

$$\mu(t) = \frac{a - b \cdot t}{b - c \cdot t} \quad (12)$$

in (6) and (2)^{24,25}:

$$p(t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{\left(\frac{b\mu(t)-a}{c\mu(t)-b} \mu(t) \right) \sqrt{ac-b^2}}{\sqrt{c\mu(t)^2 - 2b\mu(t) + a}} \int_{-\infty}^{-\frac{1}{2}z^2} e^{-\frac{1}{2}z^2} dz. \quad (13)$$

The following Figure 6 shows this relationship graphically. Again, in order to calculate a concrete L-Efficient-Frontier, the scenario from the appendix has been used. Several remarks to Figure 6 are necessary:

1. For $\mu \rightarrow \frac{b}{c}$ the L-Efficient-Frontier converges on the vertical axis.²⁶ This equals the limiting case $t \rightarrow -\infty$. In this situation, only relevant for theoretical considerations, the Minimum-Standard-Deviation-Portfolio is safety first, with a shortfall probability of zero.

2. For $p(t) \rightarrow \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{-\frac{1}{2}z^2} e^{-\frac{1}{2}z^2} dz$ the L-Efficient-Frontier converges on infinity. In the

μ - σ -World, this situation is represented by shifting the intersection t of the Shortfall Line

to $\frac{b}{c}$ from below on the vertical axis. $\frac{b}{c}$ itself is the origin of the asymptotes of the hyperbola on the vertical axis. The slope of the Shortfall Line therefore approaches then the slope of the upper asymptote of the hyperbola, which equals $\sqrt{\frac{ac-b^2}{c}}$. Economically interpreted this means, that a portfolio can be identified, which offers an unlimited expected return, combined with a shortfall probability of at most $\frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\frac{\sqrt{ac-b^2}}{c}} e^{-\frac{1}{2}z^2} dz$, referring to a target return of $\frac{b}{c}$.

3. The set of all feasible portfolios in Figure 6 is limited upwards by the L-Efficient-Frontier.

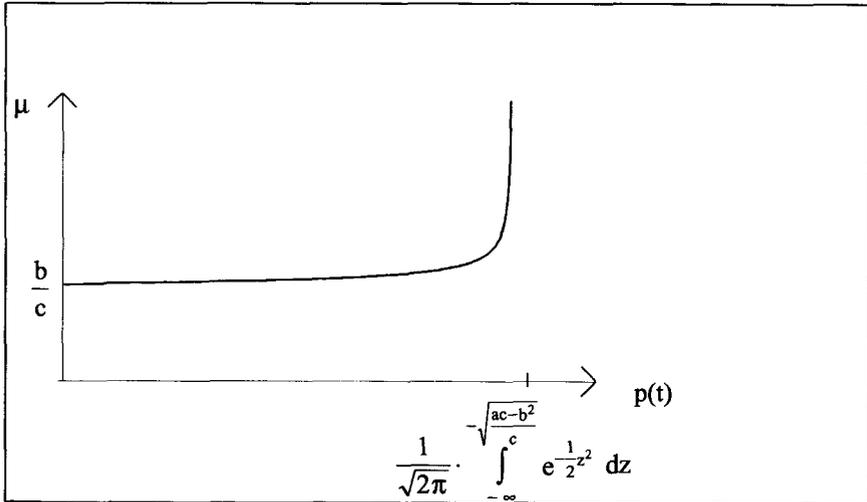


Figure 6: The L-Efficient-Frontier in the Mean-Shortfall-Probability-World.

Now, we generalize the L-Efficient-Frontier for the case of riskless lending and borrowing. Again, the situations $t > R_f$, $t = R_f$ and $t < R_f$ have to be differentiated: The situation $t > R_f$ is easy to handle, because no Safety-First-Portfolio exists here.²⁷ The situations $t < R_f$ and $t = R_f$ can be analysed together, but as in the case of the Efficient Shortfall Frontier a distinction between the alternative interpretations of risk has to be made.

In a situation of avoiding strict underperformance of the target return, only the riskless portfolio is safety first. The point $(0, R_f)$ in Figure 7 therefore represents only this portfolio, but it

should be stressed that the target return t varies in the background for $-\infty < t \leq R_f$. Because of this, the entire L-Efficient-Frontier is plotted on this single point:

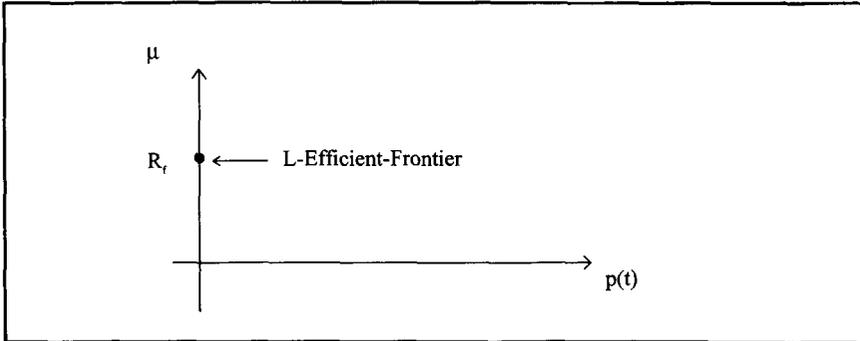


Figure 7: The L-Efficient-Frontier with riskless lending and borrowing and strict underperformance of the target return as interpretation of risk.

With the interpretation of risk as strict outperformance of the target, the diagram changes completely [see Figure 8]. Now, the resulting curve has a discontinuity. As before, for all $t < R_f$, the riskless portfolio is safety first, with a shortfall probability of zero. But if the investor wants to employ a target return which equals exactly the riskless return, *all* portfolios on the Capital Market Line, *except* the riskless portfolio, are safety first. Their shortfall probability is given by equation (11).

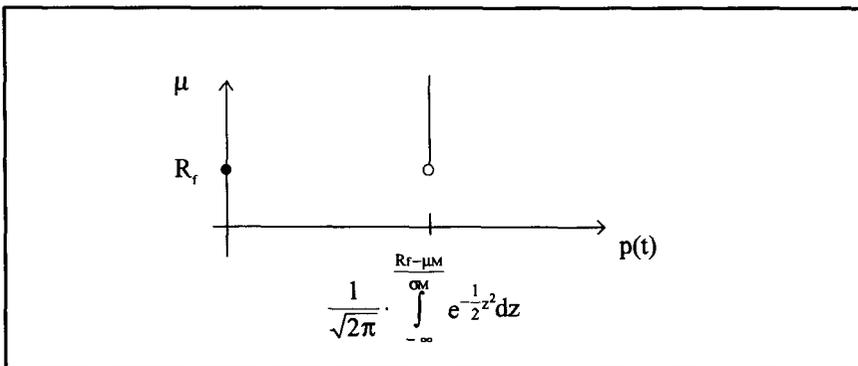


Figure 8: The case with riskless lending and borrowing and strict outperformance of the target return as interpretation of risk.

Figure 8 deserves an additional explaining remark: In their paper KADUFF/SPREMANN (1996) denote a curve *L-efficient*, if it displays for each target t the corresponding Safety-First-Portfolio with *maximum expected return*. This distinction was not necessary for the previously discussed case without riskless lending and borrowing, because the assumption of normally distributed returns and other usually employed assumptions²⁸ for the optimization problem guarantee unambiguity. But in the case with riskless lending and borrowing discussed now, this distinction is necessary. For $t = R_f$ combined with the second interpretation of risk (i. e. strict outperformance of the target return) no Safety-Mean-Efficient-Portfolio²⁹ in the sense of KADUFF/SPREMANN (1996) can be found. In order to keep consistent with the notation in KADUFF/SPREMANN (1996) the curve in Figure 8 should not be denoted as L-Efficient-Frontier in the sense of KADUFF/SPREMANN (1996).

Summarizing the results, the L-Efficient-Frontier allows the presentation of the expected returns of *all* Safety-First-Portfolios and their shortfall probabilities referring to *all* permissible target returns. But these target returns vary along the L-Efficient-Frontier, without being displayed directly in the μ - $p(t)$ -World. Ultimately, an investor, who has decided that shortfall probability is a suitable risk measure for his purposes, has to add not only one but *two additional dimensions* to the expected return, that is the shortfall probability *and* the corresponding target return, for which the shortfall probability is calculated. The logic consequence is the extension of the two-dimensional diagrams in a *third dimension*.

5. The Efficient Shortfall Surface

To proceed in such a manner, in analogy to the μ - σ -World each feasible portfolio is to be positioned in a μ - t - $p(t)$ -World, a procedure, which can be illustrated graphically very easily. Again, firstly the case without riskless lending and borrowing is discussed. Of special interest now are those portfolios, which yield the *maximum* expected return for a given target return and shortfall probability combination. This criteria of optimization equals the method of TELSER (1955)³⁰, with the difference, that not only one but all feasible t - $p(t)$ -combinations are considered. Within the μ - t - $p(t)$ -World, these portfolios generate a surface for which we introduce the name *Efficient Shortfall Surface*. For the corresponding efficiency property the term *Mean-Shortfall-Constraint-Efficiency*³¹ will be used.

Analytically, the *Mean-Shortfall-Constraint-Efficient* Portfolios can be derived by intersecting the so-called Telser-Line with the classical Efficient Frontier. The Telser-Line is the line identified in the Mean-Variance-World by selecting a target return and a shortfall probability. The upper intersection point between the Telser-Line and the classical Efficient Frontier iden-

tifies - if existent - the portfolio, which meets the restrictions referring to target return and shortfall probability and yields the maximum expected return. We will represent this maximum expected return through μ_s . Combined with the corresponding standard deviation σ_s , which can be easily calculated from equation (3) for all portfolios on the Efficient Frontier, μ_s is substituted in (2). This results in the equation³²:

$$p(t) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{-\frac{\sqrt{(ac-b^2)(\mu_s-t)^2}}{c\mu_s^2-2b\mu_s+a}} e^{-\frac{1}{2}z^2} dz, \quad (14)$$

which can be interpreted in the following way: Any two of the three parameters μ_s , t and $p(t)$ can be chosen arbitrarily³³, fixing the third one unambiguously. This identifies a point on the three-dimensional Efficient Shortfall Surface in the μ - t - $p(t)$ -World.³⁴ Figure 9 represents equation (14) graphically. Again, the example of the appendix has been used to generate the surface here.³⁵

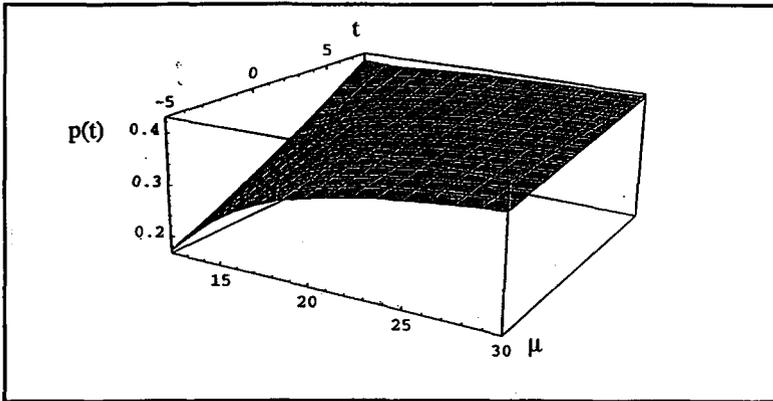


Figure 9: The Efficient Shortfall Surface in the μ - t - $p(t)$ -World.

Several remarks to Figure 9 are in order now: In contrast to all previous diagrams, the scale of the axes have been plotted here, in order to ease the understanding of the diagram. Special attention should be paid to the scale of the z-axis, which shows the shortfall probabilities of the portfolios. The scale increases upwards, which means that the set of all feasible portfolios is positioned *above* the Efficient Shortfall Surface. In other words, the Efficient Shortfall Surface restricts the set of feasible portfolios *downwards*. Furthermore, the shape of the Efficient Shortfall Surface coincides with our intuition: The acceptance of a decreasing target return is *ceteris paribus*³⁶ compensated with a decreasing probability of failing to meet this target. It should be also stressed, that under the assumptions made, each point on the Efficient Shortfall

Surface identifies a portfolio unambiguously [see endnote 34], but this relationship is not injective in a mathematical sense. This means, that the same portfolio can be plotted on different points on the Efficient Shortfall Frontier through different t - $p(t)$ -combinations.

Figure 9 can also be employed to visualise the three classical portfolio optimization strategies based on shortfall probabilities according to ROY (1952), KATAOKA (1963) and TELSER (1955):

1. The Safety-First-Rule of ROY can be illustrated by fitting in a vertical surface, parallel to the surface spanned by the x - and z -axis, running through the prespecified target return. Intersecting this surface with the Efficient Shortfall Surface yields an intersection line in the three-dimensional space. The point on this line with the minimum height referring to the z -axis is the Roy-Portfolio.
2. The Kataoka-Portfolio³⁷ can be identified by intersecting a horizontal surface through the prespecified shortfall probability with the Efficient Shortfall Surface. Again, an intersection line in the three-dimensional space results. The point with the maximum y -scale value on this line is the Kataoka-Portfolio.
3. The method of TELSER equals graphically the intersection of a line, generated through the specification of a target return and corresponding shortfall probability³⁸, with the Efficient Shortfall Surface. The resulting intersection point represents the Telser-Portfolio.

In the case of riskless borrowing and lending at rate R_f , the efficient portfolios are represented in the μ - σ -World by the Capital Market Line. Dependent on the choice of the intersection point t_{Telser} and the slope $-s_{\text{Telser}}$ of the Telser-Line different situations may occur. But relevant for the generation of an Efficient Shortfall Surface for practical purposes are only those situations, in which Telser-Portfolios exist. This is true for $t_{\text{Telser}} < R_f$ combined with $-s_{\text{Telser}} > \frac{\mu_M - R_f}{\sigma_M}$. The corresponding expected return μ_{Telser} can be calculated through:³⁹

$$\mu_{\text{Telser}} = \frac{(t_{\text{Telser}} - R_f) \cdot \sigma_M}{\mu_M - R_f + \sigma_M \cdot s_{\text{Telser}}} \quad (15)$$

For this area in the μ - t - $p(t)$ -World, the following mathematical relationship for Mean-Shortfall-Constraint-Efficient portfolios holds:⁴⁰

$$p(t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{\frac{(tT_{\text{telsr}} - R_f) \sigma_M}{\mu_{\text{Telsr}}} + R_f - \mu_M}{\sigma_M} \int_{-\infty}^{\sigma_M} e^{-\frac{1}{2}z^2} dz, \quad (16)$$

Again, these portfolios are represented in the three-dimensional space by a surface, to which we will refer as Capital Market Surface. Figure (10) shows equation (16) graphically:

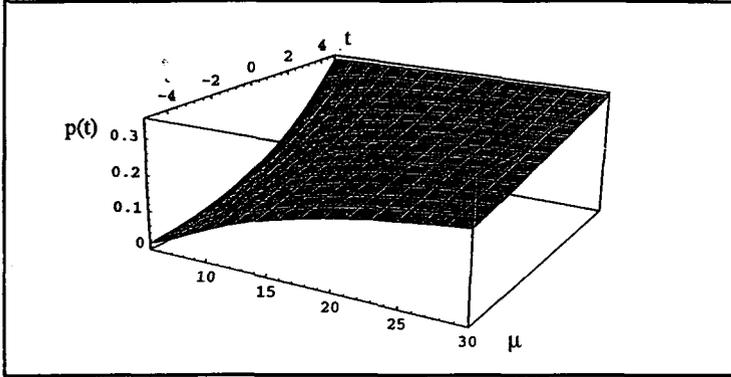


Figure 10: The Capital Market Surface for the case of riskless borrowing and lending.

Several remarks are necessary: Figure 10 is also based on the scenario of the appendix.⁴¹ By comparing Figure 9 and Figure 10, one can see that the Capital Market Surface is positioned - as expected - completely below the Efficient Shortfall Surface. In other words, the Capital Market Surface dominates the Efficient Shortfall Surface. Not that obviously to see but also expected is the fact, that both surfaces are tangent to each other along a curve. This curve represents all the points in the μ - t - $p(t)$ -World, on which the Market Portfolio is plotted. For these points, according to equation (16) the following relationship holds:

$$p(t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{\frac{(tT_{\text{telsr}} - R_f) \sigma_M}{\mu_M} + R_f - \mu_M}{\sigma_M} \int_{-\infty}^{\sigma_M} e^{-\frac{1}{2}z^2} dz. \quad (17)$$

Therefore, we now have developed a theorem of separation for the three-dimensional space: Independently of his preferences, each investor structures the risky fraction of his portfolio identically (exactly according to the allocation of risky assets in the Market Portfolio). Dependent on his preferences, either capital remains, which is then invested in the riskless asset, or

additional capital is required, which is then borrowed risklessly. The border between borrowing and lending is given by the curve on the Capital Market Surface, which represents the Market Portfolio. Portfolios, which are positioned on the left-hand-side of this curve contain a long position in the riskless asset. Portfolios, which are positioned on the right-hand-side of the curve contain a short position.

6. Extensions

The previous analysis is, as mentioned, based on the assumption of normally distributed portfolio returns.⁴² In general, there are two possibilities of treating situations, in which this assumption is violated:

1. The first possibility is to use the inequality of CHEBYSHEV, which requires no information about the shape of the probability distributions at all. According to this inequality, the following relationship for the shortfall probability $p(t)$ referring to a target return t holds:

$$p(t) \leq \frac{\sigma(R)^2}{[\mu(t) - t]^2} . \quad (18)$$

Of course, with this approach no statements about the actual shortfall probabilities of portfolios can be made, but at least *upper bounds* for the shortfall probabilities can be calculated. In this way, all previous derived relationships can be generalized very easily. Also, the corresponding diagrams may be modified in this manner: In the μ - $p(t^*)$ -World the shortfall-probability-scale on the x -axis is substituted by an upper-bound-estimation-scale according to CHEBYSHEV. Therefore, all portfolios are now positioned in a world, which could be denoted by *Mean-Shortfall-Probability-Bound-World*, with a *Mean-Shortfall-Probability-Bound-Efficient-Frontier*. In the t - $p(t)$ -World the y -axis has to be transformed, yielding a *Threshold-Shortfall-Probability-Bound-World*, with an *Efficient-Shortfall-Bound-Frontier*. The L -Efficient-Frontier of the μ - $p(t)$ -World can be transformed into a *Generalized-Distribution-L-Efficient-Frontier*. The three-dimensional μ - t - $p(t)$ -World can be modified to a *Mean-Threshold-Shortfall-Probability-Bound-World*, with an *Efficient-Shortfall-Probability-Bound-Surface*.

2. Instead of using no information about the probability distributions of portfolios⁴³, the investor may decide - probably because of empirical investigations - to assume a concrete type of not normally distributed asset return. This could be an alternative *continuous distribution*, for example the lognormal distribution, or a *discontinuous one*, generated out of the

histogram of historical return realisations. In both cases, no complete equation for the shortfall probabilities of efficient portfolios can be derived in general, but nevertheless, shortfall probabilities can be calculated using numerical approximations. The numerical methods that are available and may be employed can guarantee a sufficient precise analysis for practical purposes. Because of this, the use of alternative distribution assumptions admittedly involves extended computing time, but is no problem from a mathematical point of view.⁴⁴

Appendix

In order to generate the alternative representations of efficient frontiers displayed in this article, a concrete scenario has been used. To focus attention on the main ideas behind the diagrams, it has been refrained from plotting the scale of axes in most of the diagrams. Nevertheless, the underlying data of the example will be given here.

Imagine an investor with two investment opportunities with stochastic returns. Mean and standard deviation of those assets are given by the following table. All data refer to a holding period of one year and are chosen to have practical relevance:

Asset	A	B
	9%	12%
	17%	18%

Table 1: Annual expected returns and standard deviations of the available risky assets.

In addition, the investor knows the covariance matrix between these two assets:

	A	B
A	289	162
B	162	324

Table 2: Covariance matrix of the risky assets.

The investor can invest any amount of his capital in both assets and is even allowed to sell them short, if he wants to. Therefore, we have an optimization problem according to BLACK, as it was stated earlier several times. With the data given above, the classical Efficient Frontier of the μ - σ -World according to equation (3) can be generated [see Figure 11]. In addition, for each type of diagramm the case with riskless borrowing and lending is analysed. For the scenario here, a riskless rate of return of 5% p. a. is assumed. With this rate, the Capital Market Line can be generated, which represents all efficient portfolios [see Figure 12].

With the data given, the Minimum-Standard-Deviation-Portfolio has the coordinates (15.27 , 10.31), and the Market Portfolio the coordinates (17.12 , 11.68). To generate the Efficient Frontiers in Figure 11 and 12 the data given so far are sufficient, but in order to calculate the shortfall probabilities additional information about the shape of the probability distributions is required. As assumed for the analytical analysis, we use normal-distributions for our diagrams. Further, a target return has to be prespecified, corresponding to which the shortfall

probabilities are calculated. Here, a target return of 0% has been employed, expressing the idea of nominal capital preservation. Of course, other target returns are possible.

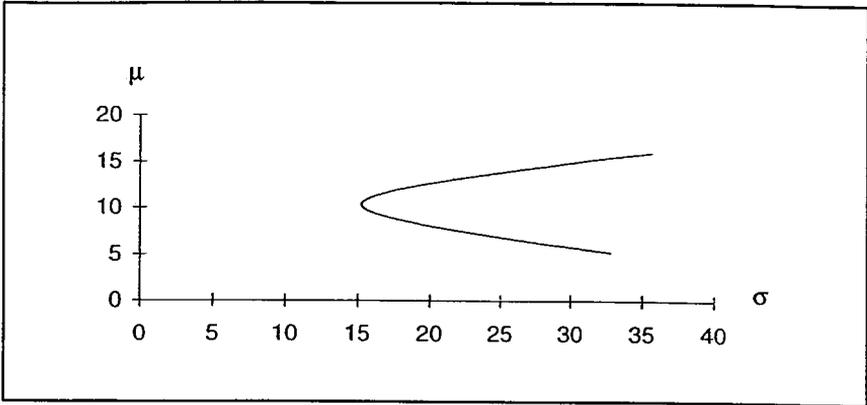


Figure 11: The Efficient Frontier in the μ - σ -World.

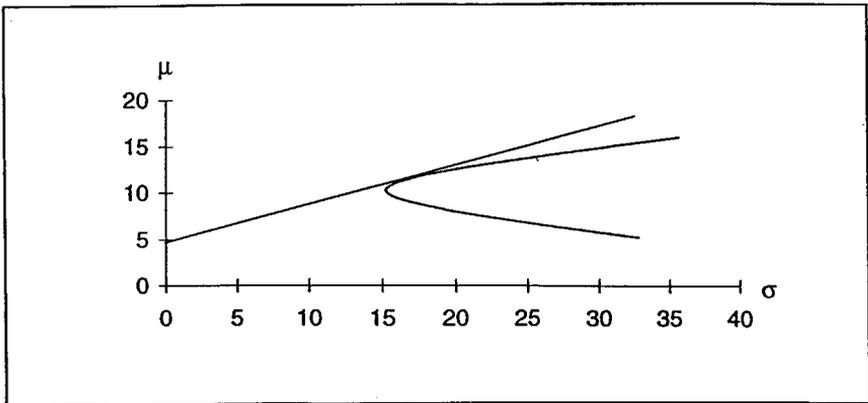


Figure 12: The Capital Market Line in the μ - σ -World.

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¹ See for example KADUFF/SPREMANN (1996).

² For the case without the possibility of riskless borrowing and lending see also KADUFF (1996a).

³ The name explains itself from the fact, that the shortfall probability is exactly the zero-degree Lower Partial Moment denoted by LPM_0 .

⁴ Hereafter, also the term $\mu-p(t^*)$ -World is used. The „ t^* “-Index at t indicates, that the target return t , referring to which the shortfall probability is calculated, has to be fixed.

⁵ HARLOW (1991) analysed n -degree Lower Partial Moments in general. He especially calculated efficient frontiers for first-degree and second-degree Lower Partial Moments, but subordinated zero-degree Lower Partial Moment, because of its inability to yield information about how severe a possible shortfall of the target might be.

⁶ This assumes implicitly the presence of an optimization problem according to the Black-Model, i. e. short selling of all investment opportunities is allowed [see MARKOWITZ (1987)]. But other modellings of the restrictions in the optimization program can be treated analogously by calculating the shortfall probability for all portfolios on the referring efficient frontier.

⁷ σ is the minimal standard deviation of all portfolios with parametric mean μ . V denotes the covariance matrix of the assets, and V^{-1} the inverse to it. In addition there is $a = \mu^T V^{-1} \mu$, $b = \mu^T V^{-1} e$ and $c = e^T V^{-1} e$, with the unity vector e of corresponding dimension.

⁸ In order to draw this concrete efficient frontier, a simple but realistic scenario has been used. The underlying assumptions and data can be found in the appendix.

⁹ For the other cases ($t^* > R_f$ and $t^* = R_f$), different scenarios have been analysed, too. Other referring analytical and graphical results were achieved, which can not all be discussed here. However, some of the results are presented below.

¹⁰ The following modification (5b) of the equation of the Capital Market Line (5a) is derived by substituting the relationships

$$\mu_M = \frac{a - b \cdot R_f}{b - c \cdot R_f} \quad \text{and}$$

$$\sigma_M = \frac{\sqrt{c \cdot R_f^2 - 2bR_f + a}}{b - c \cdot R_f}$$

for the expected return μ_M and the standard deviation of return σ_M of the Market Portfolio in (5a). Afterwards, the equation has to be solved for σ .

¹¹ Equation (7) specifies the upper integration limit for (2).

¹² Again, in order to generate Figure 2, the scenario described in the appendix has been used.

¹³ It has to be mentioned, that this is not true, if $t^* = R_f$. In this special case, the Capital Market Line is represented by a vertical half-line in the $\mu-p(t^*)$ -World.

¹⁴ Therefore, the theorems of separation for the $\mu - \sqrt[3]{LPM_n(R_f)}$ -World, $n = 1, 2$, proven in KADUFF (1996b) can be complemented with this case here.

¹⁵ This is the case for private as well as institutional investors.

¹⁶ Theoretically, the investor has an unlimited number of alternatives to select his target return from. For practical applications this is of course not true, but still the number of alternatives can be high.

¹⁷ In order to short notation the name *t-p(t)-World* is going to be used hereafter.

¹⁸ See JAEGER/RUDOLF/ZIMMERMANN (1995).

¹⁹ This assumes implicitly that an optimization problem according to the Black-Model is used. Other formulations can be treated analogously.

²⁰ Only this guarantees the existence of tangential points on the *efficient section* of the hyperbola.

²¹ For a proof see KADUFF (1996b).

²² The reason is, that for a continuous distribution the probability mass of a single point is defined as zero.

²³ Except the riskless portfolio, which is of course also positioned on the Capital Market Line.

²⁴ Again, this assumes implicitly, that an optimization problem according to the Black-Model is used.

²⁵ Under the assumption of normally distributed returns, Safety-First-Portfolios can be found only on the upper section of the hyperbola. Therefore, μ has to satisfy the condition: $\mu > \frac{b}{c}$ in equation (13).

²⁶ The term $\frac{b}{c}$ equals the expected return of the Minimum-Standard-Deviation-Portfolio.

²⁷ See KADUFF (1996b).

²⁸ Arbitrary divisibility of all assets is assumed. In addition, the expected returns are not allowed to be equal for all assets, and their variances must be finite. Furthermore, it is not possible to represent the random return of any single asset by a linear combination of the random return of other assets. This ensures the regularity of the covariance matrix, an assumption which guarantees the solvability of the problem.

²⁹ A portfolio is denoted *safety mean efficient*, if it is safety first referring to a target t and has the maximum expected return of all portfolios, which are safety first referring to this target t [see also Definition 3 in KADUFF/SPREMANN (1996)].

³⁰ According to TELSER (1955), an investor should specify in advance an individual target return as well as an acceptable probability of failing to reach that target, and then select from the set of feasible portfolios meeting both conditions simultaneously the portfolio with the maximum expected return.

³¹ This term expresses the fact, that now both risk dimensions, target return and corresponding shortfall probability are integrated parametrically.

³² Again, we implicitly assume an optimization problem according to the Black-Model here.

³³ In choosing μ_s the restriction $\mu_s > \frac{b}{c}$, in choosing t the restriction $t < \frac{b}{c}$ and in choosing $p(t)$ the restriction $0 < p(t) < 1$ has to be respected.

³⁴ The selection of two parameters in the μ - σ -World is nothing more than the identification of a Shortfall Line, because a line can be identified unambiguously by either fixing two points (this means choosing μ_s and t) or fixing one point and the slope (this means choosing t and $p(t)$ or μ_s and $p(t)$).

³⁵ For the target return the interval [-5% ; 8%] and for the expected return the interval [12% ; 30%] has been used. This is, because the expected return of the Minimum-Standard-Deviation-Portfolio is 10,32% in our example, which means that the intervals for the target return and the expected return must lie below and above, respectively.

³⁶ *Ceteris paribus* means here to keep the expected return constantly.

³⁷ According to KATAOKA (1964) an investor should first, in contrast to the approach of ROY (1952), specify an acceptable shortfall probability, with which the target return, which is to maximise subsequently, can be failed to meet. The portfolio solving this optimization problem is the so-called Kataoka-Portfolio.

³⁸ This line runs parallel to the x-axis in three-dimensional space.

³⁹ Equation (15) is obtained by equating the formulas of the Telser Line and the Capital Market Line.

⁴⁰ Equation (16) is obtained by solving (15) for s_{Telser} and substituting the result as upper integration limit in (2).

⁴¹ For the target returns, the interval [-5% ; 4,9%] and for the expected returns the interval [5.1% ; 30%] have been employed. The rate for riskless borrowing and lending is 5%.

⁴² This is the case for all the analytical as well as graphical relationships derived so far.

⁴³ „No information“ refers only to the *shape* of the probability distributions of the returns. Of course, also the CHEBYSHEV-estimations require at least the knowledge of the distribution parameters mean and standard deviation of all assets [see Equation (18)]. In addition, one has to know the correlations between all asset returns.

⁴⁴ In this context the paper of KALIN/ZAGST (1995) should be mentioned. They investigate under which generalized distributions Mean-Variance portfolio optimization and shortfall-risk-based portfolio optimization coincide. They prove the connection for the wide class of two-parameter-distributions. This class contains besides the normal-distribution, other in the context of Portfolio Theory commonly used distributions, like the two-point-distribution, the triangular-distribution and the exponential-distribution.

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