# INEQUALITIES FOR THE EXPECTED VALUE OF AN EXCHANGE OPTION STRATEGY

Werner Hürlimann
Allgemeine Mathematik
Konzernbereich Leben WGR
Paulstrasse 9
CH-8401 Winterthur - SWITZERLAND
Telephone : +41 52 261 58 61
and
Schönholzweg 24
CH-8409 Winterthur - SWITZERLAND

**Abstract.**

Two exchange option strategies are considered, one of which does only depend on the marginal distributions of the underlying sequence of random variables. Their expected total financial payoffs are compared under various assumptions, including stochastic ordering ones. While some results are closely related to earlier prophet/gambler comparisons in gambling theory, a lot of new inequalities are presented. In particular some new sharp inequalities by known marginal means and variances are formulated.

**Keywords** : exchange option, bivariate dependence, Hoeffding-Fréchet extremal distributions, stop-loss transform, stochastic orders

## 1. Introduction.

Let the random variables $X_0, X_1, ..., X_n$ represent the unknwon future prices at future times $t_0, t_1, ..., t_n$ of some specific goods trated in a financial market. Suppose an agent operates according to the following *exchange option strategy*. At present time he agrees that at each future time $t_i, i = 0,...,n-1$, he writes an option to exchange at time $t_{i+1}$ a specific good at price $X_i$ for another specific good at price $X_{i+1}$, whose financial payoff at time $t_{i+1}$ equals $(X_{i+1} - X_i)_+$. Neglect or suppose the $X_i$'s are interest adjusted quantities. Then the agent wants to know as much as possible about the *total financial payoff* $S_n = \sum_{i=1}^{n}(X_i - X_{i-1})_+$ as viewed from present time, say about the expected value $E[S_n]$ of this exchange option strategy. As alternative exchange option strategy the agent could agree at present time that at each future date $t_i$, he writes an option to exchange at time $t_{i+1}$ a good at the price $X_i$ for another good at the expected price $\mu_{i+1} = E[X_{i+1}]$, whose financial payoff at time $t_{i+1}$ equals $(\mu_{i+1} - X_i)_+$. Let the total financial payoff of this alternative strategy be $T_n = \sum_{i=1}^{n}(\mu_i - X_{i-1})_+$. From a statistical point of view, the expected value $E[T_n]$ depends only on the *marginal* distributions of $X_0, X_1, ..., X_n$, while $E[S_n]$ depends additionally on the *dependence* between $X_{i-1}$ and $X_i$. It is thus of general interest to study the variation of $E[S_n]$ given the value of $E[T_n]$.

The considered exchange option strategies have a nice interpretation in game and gambling theory, a topic developed by Dubins and Savage(1965) and for which one can recommend the recent textbook by Maitra and Sudderth(1996) for background. Let us follow Krengel and Sucheston(1987), from which we have extracted the general inequalities (2.4) and (2.12) in Theorems 2.1 and 2.2. By definition a *prophet* is a player with complete foresight and a *gambler* knows only the past and the present, but not the future. Suppose both bet on differences of consecutive non-negative random variables $X_i$ such that $E[X_i|X_{i-1}] = E[X_i]$, where the players are supposed to multiply their stakes by uniformly bounded random variables. Then the expected payoff $E[S_n]$ corresponds to the expected gain of the prophet while $E[T_n]$ is the optimal expected reward of the gambler. In this situation Theorem 2.1 says that the expected gain of the prophet is at most three times that of the gambler, where the constant three is optimal.

The main point, where we differ from Krengel and Sucheston(1987), is in the innovative application of their results to Finance. Furthermore, in our context, we work under less restrictive assumptions and we formulate some new properties like (2.5), (2.13) and (2.14). Potential applications lie in Asset and Liability Management, including those suggested in Hürlimann(1997a), Section 5.

## 2. Inequalities by known means.

By knowledge of only the means, we investigate the relationship between the expected values $E[S_n]$ and $E[T_n]$ as defined in Section 1. For a random variable $X$ with survival function $\overline{F}(x)$, the stop-loss transform of $X$ is denoted by $\pi(x) = E[(X-x)_+] = \int_x^{\infty}\overline{F}(t)dt$, where the last equality is obtained through partial integration.

**Lemma 2.1.** If a<b then one has the inequalities

(2.1) $$\overline{F}(b) \le \frac{\pi(a)-\pi(b)}{b-a} \le \overline{F}(a).$$

**Proof.** This follows directly from the stop-loss transform definition :

$$\pi(a)-\pi(b) = \int_a^b (x-a)dF(x) + \int_b^\infty \{(x-a)-(x-b)\}dF(x) \ge \int_b^\infty (b-a)dF(x) = (b-a)\overline{F}(b).$$

and

$$\pi(a)-\pi(b) = \int_a^b (x-a)dF(x) + \int_b^\infty (b-a)dF(x) \le \int_a^\infty (b-a)dF(x) = (b-a)\overline{F}(a). \ \Diamond$$

**Lemma 2.2.** If $E[X]=\mu$, then the following inequalities are satisfied :

(2.2) $$\pi(\mu) \le (1+F(\mu))\cdot\pi(c)-(\mu-c), \text{ for all } c<\mu$$

(2.3) $$\pi(\mu) \le \pi(c)+(c-\mu), \text{ for all } c\ge\mu$$

**Proof.** If $\mu<c$ one has by the second inequality in (2.1) that $\pi(\mu)-\pi(c) \le \overline{F}(c)\cdot(c-\mu)$, hence (2.3). If $c<\mu$ the first inequality in (2.1) implies $\pi(\mu) \le \pi(c)-\overline{F}(\mu)\cdot(\mu-c) = \pi(c)+F(\mu)\cdot(\mu-c)-(\mu-c) \le (1+F(\mu))\cdot\pi(c)-(\mu-c)$. ◊

The following terminology is borrowed from the theory of stochastic orders (e.g. Kaas et al.(1994), Shaked and Shanthikumar(1994)).

**Definitions 2.1.** A random variable $X$ precedes another random variable $Y$ in the usual *stochastic order*, written $X \le_{st} Y$, if the corresponding survival functions satisfy the inequalities $\overline{F}_X(x) \le \overline{F}_Y(x)$ uniformly for all $x$. One says that $X$ precedes $Y$ in *stop-loss order* or in *increasing convex order*, written $X \le_{sl} Y$ or $X \le_{icx} Y$, if the corresponding stop-loss transforms satisfy the inequalities $\pi_X(x) \le \pi_Y(x)$ uniformly for all $x$. One says that $X$ precedes $Y$ in *increasing concave order*, written $X \le_{icv} Y$, if for all $x$ the inequalities $E[(X-x)_-] \le E[(Y-x)_-]$ or equivalently $\int_{-\infty}^x F_X(t)dt \ge \int_{-\infty}^x F_Y(t)dt$ hold. By equal means $E[X]=E[Y]$, the stop-loss order relation is written $X \le_{sl,=} Y$. In this case the increasing convex order by equal means is called *convex order*, denoted by $\le_{cx}$, and the increasing concave order by equal means is called *concave order*, denoted by $\le_{cv}$. Furthermore all three partial orders $\le_{sl,=}$, $\le_{cx}$ and $\ge_{cv}$ are equivalent partial orders.

In the following suppose $\mu_i = E[X_i]$, i=0,1,...,n, is finite, and let $S_n, T_n$ be the random sums defined in Section 1. The distribution of $X_i$ is denoted by $F_i(x)$ and its stop-loss transform by $\pi_i(x) = E[(X_i - x)_+]$. The smallest closed interval containing the support of $X_i$ is denoted by $[a_i, b_i]$, i=0,1,...,n, where $a_i = \inf\{x : F_i(x) > 0\}$ and $b_i = \sup\{x : F_i(x) < 1\}$, with $-\infty \le a_i \le b_i \le \infty$.

**Theorem 2.1.** Let $X_0, X_1, \ldots, X_n$ be random variables with finite means $\mu_i = E[X_i]$, i=0,1,...,n, and let $S_n = \sum_{i=1}^{n}(X_i - X_{i-1})_+$ and $T_n = \sum_{i=1}^{n}(\mu_i - X_{i-1})_+$. First, one has the following inequality :

(2.4) $$E[S_n] \le 3 \cdot E[T_n] - \{\pi_0(\mu_0) - \pi_n(\mu_n) - (\mu_0 - \mu_n)\}.$$

Second, if either $X_0 \le_{st} X_n$ (usual stochastic order) or $X_0 \le_{icv} X_n$ (increasing concave order), then one has

(2.5) $$E[S_n] \le 3 \cdot E[T_n].$$

Third, one has equality in (2.5) if, and only if, $X_i \equiv \mu_0$ with probability one, i=0,1,...,n.

**Proof.** One has the inequality

(2.6) $$\begin{aligned} E[S_n] &= E\left[\sum_{i=1}^{n}(X_i - X_{i-1})_+\right] \le \sum_{i=1}^{n}\left\{E[(X_i - \mu_i)_+] + E\left[(\mu_i - X_{i-1})_+\right]\right\} \\ &= E[T_n] + \sum_{i=1}^{n}\pi_i(\mu_i) \end{aligned}$$

On the other side, the identity $(\mu_i - X_{i-1})_+ = (X_{i-1} - \mu_i)_+ + (\mu_i - X_{i-1})$ implies that

(2.7) $$E[T_n] = \sum_{i=0}^{n-1}\pi_i(\mu_{i+1}) + \mu_n - \mu_0.$$

From Lemma 2.2 one knows that

(2.8) $$\pi_i(\mu_i) \le (1 + F_i(\mu_i)) \cdot \pi_i(\mu_{i+1}) + (\mu_{i+1} - \mu_i) \le 2 \cdot \pi_i(\mu_{i+1}) + (\mu_{i+1} - \mu_i), \quad i=0,1,\ldots,n-1$$

Using (2.7) and (2.8) it follows that

(2.9) $$\begin{aligned} \sum_{i=1}^{n}\pi_i(\mu_i) &= \pi_n(\mu_n) - \pi_0(\mu_0) + \sum_{i=0}^{n-1}\pi_i(\mu_i) \\ &\le \pi_n(\mu_n) - \pi_0(\mu_0) + 2 \cdot \left\{\sum_{i=0}^{n-1}\pi_i(\mu_{i+1}) + \mu_n - \mu_0\right\} - (\mu_n - \mu_0) \\ &= 2 \cdot E[T_n] - \{\pi_0(\mu_0) - \pi_n(\mu_n) - (\mu_0 - \mu_n)\}, \end{aligned}$$

which shows (2.4) by means of (2.6). Now, if either $X_0 \le_{st} X_n$ or $X_0 \le_{icv} X_n$, then by Lemma 2.3 below, the expression in curly bracket in (2.4) is non-negative, and (2.5) follows. Assume equality holds in (2.5). Then one must have $\mu_n - \pi_n(\mu_n) = \mu_0 - \pi_0(\mu_0)$. If $\pi_i(\mu_{i+1}) > 0$ then equality cannot hold because of $1 + F_i(\mu_i) < 2$ and (2.8). Thus each $\pi_i(\mu_{i+1}) = 0$, hence $X_i \le \mu_{i+1}$. It follows that $\pi_i(\mu_i) \le \mu_{i+1} - \mu_i$, i=0,1,...,n-1. The last inequality is strict unless $\mu_{i+1} = \mu_i$ and $X_i \equiv \mu_i$ with probability one. Suppose now $\sum_{i=0}^{n-1}\pi_i(\mu_i) < \mu_n - \mu_0$. By (2.7) one must have $E[T_n] = \mu_n - \mu_0 = \pi_n(\mu_n) - \pi_0(\mu_0)$. From (2.6) one obtains the strict inequality

$$E[S_n] \le E[T_n] + \sum_{i=1}^{n} \pi_i(\mu_i) = E[T_n] + \pi_n(\mu_n) - \pi_0(\mu_0) + \sum_{i=0}^{n-1} \pi_i(\mu_i)$$
$$< 2 \cdot E[T_n] + \mu_n - \mu_0 = 3 \cdot E[T_n].$$

Therefore one has $E[S_n] < 3 \cdot E[T_n]$ unless $X_i \equiv \mu_0$ with probability one, i=0,1,...,n. ◊

**Lemma 2.3.** Let X, Y be random variables with survival functions $\overline{F}(x), \overline{G}(x)$ and finite means $\mu_X, \mu_Y$. Denote by $I_X = [a_X, b_X], I_Y = [a_Y, b_Y]$, *the smallest closed intervals* containing the supports of X, Y, such that $a_X = \inf\{x : F(x) > 0\}$, $b_X = \sup\{x : F(x) < 1\}$, $a_Y = \inf\{x : G(x) > 0\}$, $b_Y = \sup\{x : G(x) < 1\}$. If either $X \le_{st} Y$ or $X \le_{icv} Y$, then the following inequality holds :

$$\mu_X - \pi_X(\mu_X) \le \mu_Y - \pi_Y(\mu_Y). \tag{2.10}$$

**Proof.** Since $\mu_X = a_X + \int_{a_X}^{b_X} \overline{F}(x)dx$, $\pi_X(\mu_X) = \int_{\mu_X}^{b_X} \overline{F}(x)dx$, and similar expressions for $\mu_Y, \pi_Y(\mu_Y)$, one sees that (2.10) is equivalent with the inequality

$$a_X + \int_{a_X}^{\mu_X} \overline{F}(x)dx \le a_Y + \int_{a_Y}^{\mu_Y} \overline{G}(x)dx. \tag{2.11}$$

If $X \le_{st} Y$ then one has necessarily $a_X \le a_Y$, $\overline{F}(x) \le \overline{G}(x)$, $\mu_X \le \mu_Y$, and thus (2.11) holds. Similarly, if $X \le_{icv} Y$ then one has necessarily $a_X \le a_Y$, $\int_{a_X}^{z} \overline{F}(x)dx \le \int_{a_Y}^{z} \overline{G}(x)dx$ for all $z \ge a_X$, and $\mu_X \le \mu_Y$, hence (2.11) follows. ◊

**Remark 2.1.** In case the $X_i$'s are non-negative random variables, there exist an absolute upper bound for $E[T_n]$ in terms of the means. For each i=0,...,n−1, suppose $X_i$ has support $[0, b_i]$, $b_i \ge \mu_{i+1}$. It is not difficult to see that $X_i \le_{sl,=} Z_i$, where $Z_i$ is diatomic with support $\{0, b_i\}$ and probabilities $\left\{1 - \frac{\mu_i}{b_i}, \frac{\mu_i}{b_i}\right\}$. Using (2.7) one sees that

$$E[T_n] \le \sum_{i=0}^{n-1} E[(Z_i - \mu_{i+1})_+] + \mu_n - \mu_0 = \sum_{i=0}^{n-1} (1 - \frac{\mu_{i+1}}{b_i}) \cdot \mu_i + \mu_n - \mu_0 \le \sum_{i=1}^{n} \mu_i.$$

It would be helpful to know under which assumptions the ratio of $E[S_n]$ to $E[T_n]$ can be lowered. A simple answer, which will be precised in Corollary 3.1, is as follows.

**Theorem 2.2.** Let $X_0, X_1, \ldots, X_n$ be random variables with finite means $\mu_i = E[X_i]$, i=0,1,...,n, and suppose that $\mu_0 \le \mu_1 \le \ldots \le \mu_n$. Let $S_n = \sum_{i=1}^{n} (X_i - X_{i-1})_+$ and $T_n = \sum_{i=1}^{n} (\mu_i - X_{i-1})_+$. First, one has the inequality

$$E[S_n] \le 2 \cdot E[T_n] + \pi_n(\mu_n) - \pi_0(\mu_0). \tag{2.12}$$

Second, if either $X_0 \le_{st} X_n$ or $X_0 \le_{icv} X_n$, then one has

$$(2.13) \qquad E[S_n] \le 2 \cdot E[T_n] + \mu_n - \mu_0.$$

Third, if $\mu_0 = \mu_1 = \ldots = \mu_n$ and $X_n \le_{sl,=} X_0$, then one has

$$(2.14) \qquad E[S_n] \le 2 \cdot E[T_n].$$

**Proof.** The proof of Theorem 2.1 remains valid under the following modifications. The inequality (2.3) of Lemma 2.2 shows that (2.8) can be replaced by the sharper one

$$(2.8') \qquad \pi_i(\mu_i) \le \pi_i(\mu_{i+1}) + (\mu_{i+1} - \mu_i), \ i=0,1,\ldots,n-1.$$

Then (2.9) reads

$$(2.9') \qquad \sum_{i=1}^{n} \pi_i(\mu_i) \le E[T_n] + \pi_n(\mu_n) - \pi_0(\mu_0),$$

which implies (2.12) by means of (2.6). Clearly (2.13) follows from Lemma 2.3, and (2.14) follows from (2.13) if one notes that by equal means the increasing concave order reduces to the concave order, which turns out to be equivalent to a reversed convex order or stop-loss order as stated in Definitions 2.1. ◊

On the other side, one knows that the constant three is optimal in (2.4) and (2.5), in the sense that there exist random variables for which the bounds are nearly attained. With a slight simplification, recall the argument by Krengel and Sucheston(1987), Section 4.

**Theorem 2.3.** Suppose that $0 < \varepsilon < \frac{1}{2}$, $1 - \frac{\varepsilon}{2} < p < 1$, $\mu_1 > 0$, and set $\mu_{i+1} = \frac{\mu_i}{p}$, i=1,...,n−2. Let $Z_0, Z_1, \ldots, Z_n$ be independent random variables such that $Z_0 = Z_n = 0$, $Z_1 = \mu_1$, $Z_i$ is diatomic with support $\{0, \mu_{i+1}\}$ and probabilities $\{1-p, p\}$, i=2,...,n−2, and $Z_{n-1}$ is diatomic with support $\left\{0, \frac{\mu_{n-1}}{1-p}\right\}$ and probabilities $\{p, 1-p\}$. Define the following sequence $X_0, X_1, \ldots, X_{2n}$ by setting $X_{2i} = Z_{2i}$, i=0,...,n, and $X_{2i-1} = \mu_i$, i=1,...,n. Then, for each $n \ge 2 + \frac{\ln\{\varepsilon - 2(1-p)\}}{\ln\{p\}}$, one has the inequality

$$(2.15) \qquad E[S_{2n}] \ge (3 - \varepsilon) \cdot E[T_{2n}].$$

**Proof.** Using (2.7) one obtains

$$E[T_{2n}] = \sum_{i=1}^{n} E[(\mu_i - Z_{i-1})_+] = \sum_{i=0}^{n-1} E[(Z_i - \mu_{i+1})_+] + \mu_n - \mu_0 = E[(Z_{n-1})_+] = \mu_{n-1} = \frac{\mu_1}{p^{n-2}}.$$

Similarly one gets

$$E[S_{2n}] = \sum_{i=1}^{n} E[(\mu_i - Z_{i-1})_+] + \sum_{i=1}^{n} E[(Z_i - \mu_i)_+]$$
$$= E[T_{2n}] + \sum_{i=2}^{n-2} E[(Z_i - \mu_i)_+] + E[(Z_{n-1} - \mu_{n-1})_+]$$
$$= E[T_{2n}] + \sum_{i=2}^{n-2} p \cdot (\mu_{i+1} - \mu_i) + (1-p) \cdot (\frac{\mu_{n-1}}{1-p} - \mu_{n-1})$$
$$= E[T_{2n}] + 2p \cdot \mu_{n-1} - \mu_1 = E[T_{2n}] + \left\{ \frac{2}{p^{n-3}} - 1 \right\} \cdot \mu_1 .$$

By assumption on $\varepsilon$, p, n, this implies $\frac{E[S_{2n}]}{E[T_{2n}]} = 1 + 2p - p^{n-2} \geq 3 - \varepsilon$. ◊

### 3. Inequalities by known means and variances.

What is the maximal value of $E[S_n]$ when only the means $\mu_0, \mu_1, \ldots, \mu_n$ and standard deviations $\sigma_0, \sigma_1, \ldots, \sigma_n$ of $X_0, X_1, \ldots, X_n$ are known ? And what are the possible extremal random variables for which the maximum is attained ? To solve these questions, it suffices to apply the bivariate version of the inequality of Bowers(1969) presented in Hürlimann(1993), Theorem 2.

The solution depends on the following quantities, defined for each i=1,...,n :

(3.1)
$$\alpha_i = \mu_i - \mu_{i-1},$$
$$\beta_i = \sigma_i + \sigma_{i-1},$$
$$z_i = \gamma_i + \sqrt{1+\gamma_i^2}, \quad \gamma_i = \frac{\alpha_i}{\beta_i}.$$

From the mentioned result, one knows that

(3.2)
$$E[S_n] = \sum_{i=1}^{n} E[(X_i - X_{i-1})_+] \leq \frac{1}{2} \cdot \sum_{i=1}^{n} \beta_i \cdot z_i .$$

The equality is attained provided equality is attained for each summand. The latter holds for a bivariate diatomic extremal random pair $(X_i, X_{i-1})$ with support

$$\{x_i, y_i\} x \{x_{i-1}, y_{i-1}\} = \left\{ \mu_i - \sigma_i z_i, \mu_i + \frac{\sigma_i}{z_i} \right\} x \left\{ \mu_{i-1} - \frac{\sigma_{i-1}}{z_i}, \mu_{i-1} + \sigma_{i-1} z_i \right\}$$

and joint probabilities determined by the 2x2-contingency table

| | $x_{i-1}$ | $y_{i-1}$ |
|---|---|---|
| $x_i$ | 0 | $\frac{1}{1+z_i^2}$ |
| $y_i$ | $\frac{z_i^2}{1+z_i^2}$ | 0 |

Looking at consecutive extremal pairs, one must satisfy the relations $z_{i+1} = \frac{1}{z_i}$, i=1,...,n−1, which are equivalent to the conditions

$$(3.3) \qquad \gamma_{i+1} + \gamma_i = 0, \quad i = 1,\dots,n-1.$$

This implies that

$$(3.4) \qquad \gamma_i = (-1)^{i-1}\gamma_1, \quad z_i = (-1)^{i-1}\gamma_1 + \sqrt{1+\gamma_1^2}.$$

The following result has been shown.

**Theorem 3.1.** Let $X_0, X_1, \dots, X_n$ be random variables with means $\mu_0, \mu_1, \dots, \mu_n$ and standard deviations $\sigma_0, \sigma_1, \dots, \sigma_n$, and set $\beta_i = \sigma_i + \sigma_{i-1}$, i=1,...,n, $\gamma = \frac{\mu_1 - \mu_0}{\sigma_1 + \sigma_0}$. Then the following inequality holds :

$$(3.5) \qquad E[S_n] = \sum_{i=1}^n E[(X_i - X_{i-1})_+] \le \frac{1}{2}\gamma \cdot \sum_{i=1}^n (-1)^{i-1}\beta_i + \frac{1}{2}\sqrt{1+\gamma^2} \cdot \sum_{i=1}^n \beta_i.$$

Furthermore the inequality is *sharp* and attained by diatomic random variables with the following properties. For each i=0,...,n, the support $\{x_i, y_i\}$ of $X_i$ is given by

$$(3.6) \qquad \begin{aligned} x_i &= \mu_i - \sigma_i \cdot (\sqrt{1+\gamma^2} + (-1)^{i+1}\gamma), \\ y_i &= \mu_i + \sigma_i \cdot (\sqrt{1+\gamma^2} + (-1)^{i}\gamma), \end{aligned}$$

and the joint probabilities of the pair $(X_i, X_{i-1})$, i=1,...,n, are determined by

$$(3.7) \qquad \begin{aligned} &\Pr(X_i = x_i, X_{i-1} = x_{i-1}) = 0, \\ &\Pr(X_i = x_i, X_{i-1} = y_{i-1}) = \frac{1}{2}\left(1 + (-1)^i \frac{\gamma}{\sqrt{1+\gamma^2}}\right), \\ &\Pr(X_i = y_i, X_{i-1} = x_{i-1}) = \frac{1}{2}\left(1 + (-1)^{i-1} \frac{\gamma}{\sqrt{1+\gamma^2}}\right), \\ &\Pr(X_i = y_i, X_{i-1} = y_{i-1}) = 0. \end{aligned}$$

As a consequence, this best upper bound allows us to show that the constant two in Theorem 2.2 is optimal in case of equal means, and attained by diatomic random variables. This result contrasts with the optimal constant three in Theorem 2.3, which is only nearly attained by a non-degenerate random vector.

**Corollary 3.1.** Let $X_0, X_1, \dots, X_n$ be random variables with equal means $\mu_0 = \mu_1 = \mu_n = \mu$ and standard deviations $\sigma_0, \sigma_1, \dots, \sigma_n$. For i=1,...,n let $(X_i, X_{i-1})$ be the bivariate diatomic couple with support

$\{x_i, y_i\} x \{x_{i-1}, y_{i-1}\} = \{\mu_i - \sigma_i, \mu_i + \sigma_i\} x \{\mu_{i-1} - \sigma_{i-1}, \mu_{i-1} + \sigma_{i-1}\}$ and joint probabilities $\Pr(X_i = x_i, X_{i-1} = x_{i-1}) = 0$, $\Pr(X_i = x_i, X_{i-1} = y_{i-1}) = \frac{1}{2}$, $\Pr(X_i = y_i, X_{i-1} = x_{i-1}) = \frac{1}{2}$, $\Pr(X_i = y_i, X_{i-1} = y_{i-1}) = 0$. Then one has equality in (2.12) :

$$E[S_n] = 2 \cdot E[T_n] + \pi_n(\mu) - \pi_0(\mu). \quad (3.8)$$

Suppose further that $X_n = X_0$ with probability one, then one has equality in (2.14), that is

$$E[S_n] = 2 \cdot E[T_n]. \quad (3.9)$$

Moreover $E[S_n]$, $E[T_n]$ are maximal among all random variables with equal means and known variances.

**Proof.** This follows from the special case $\gamma = 0$ in Theorem 3.1 by noting that $\pi_i(\mu) = \frac{1}{2}\sigma_i$ and making the necessary comparisons. ◊

## 4. Inequalities by known means, medians and variances.

Let X, Y be *non-negative* random variables with marginals F(x), G(x) and joint probability function H(x,y). The indicator function of a set $\{\cdot\}$ is denoted by $I\{\cdot\}$. From the identity

$$(X - Y)_+ = \int_0^\infty I\{X \ge u, Y \le u\} du = \int_0^\infty \left(I\{Y \le u\} - I\{X \le u, Y \le u\}\right) du, \quad (4.1)$$

one derives, by taking expectations, the expected positive difference formula

$$E[(X - Y)_+] = \int_0^\infty (\overline{H}(u,u) - \overline{G}(u)) du = \int_0^\infty \overline{H}(u,u) du - \mu_Y, \quad (4.2)$$

where the bar denotes survival functions. By Hoeffding(1940) and Fréchet(1951), the following extremal bounds

$$H_*(x,y) = (F(x) + G(y) - 1)_+ \le H(x,y) \le H^*(x,y) = \min\{F(x), G(y)\} \quad (4.3)$$

hold, where $H_*, H^*$ are themselves distribution functions, and H(x,y) belongs to the space BD(F,G) of all bivariate pairs (X,Y) with given marginals F(x), G(x). It follows that

$$\overline{H}^*(x,x) = \max\{\overline{F}(x), \overline{G}(x)\} \le \overline{H}(x,x) \le \overline{H}_*(x,x) = \min\{\overline{F}(x) + \overline{G}(x), 1\}, \quad (4.4)$$

from which divers bounds for $E[(X - Y)_+]$ can be obtained. In fact, the applied method allows to determine bounds for expected values of the more general type $E[f(X - Y)]$, where f(x) is any convex non-negative function, as observed by Tchen(1980), Corollary 2.3. As a general result, the same method allows to determine, under some regularity assumptions, bounds for expected values of the general type $E[f(X,Y)]$, where f(x,y) is either a quasimonotone (sometimes called superadditive) or a quasi-antitone right-continuous function (cf. Lorentz(1953), Whitt(1976), Cambanis, Simons and Stout(1976), Tchen(1980), Cambanis and Simons(1982)). First, some immediate but practical results are presented. A more detailed analysis is postponed to Sections 5 and 6.

**Proposition 4.1.** (*Upper bounds by known medians and mean*) Let X, Y be non-negative random variables with medians $m_X$, $m_Y$, and let the mean $\mu_Y$ of Y exists. Then one has the inequality

$$(4.5) \qquad E[(X-Y)_+] \le \max\{m_X, m_Y\} + \pi_X(m_X) + \pi_Y(m_Y) - \mu_Y.$$

If $Y \le_{st} X$ then one has

$$(4.6) \qquad E[(X-Y)_+] \le m_X + 2\cdot\pi_X(m_X) - \mu_Y.$$

If $X \le_{st} Y$ then one has

$$(4.7) \qquad E[(X-Y)_+] \le m_Y + 2\cdot\pi_Y(m_Y) - \mu_Y.$$

**Proof.** If $x \ge \max\{m_X, m_Y\}$ then $\overline{F}(x)+\overline{G}(x) \le 1$. By (4.4) one has

$$\int_0^\infty \overline{H}_*(x,x)dx \le \int_0^{\max\{m_X,m_Y\}} dx + \int_{\max\{m_X,m_Y\}}^\infty (\overline{F}(x)+\overline{G}(x))dx$$
$$\le \max\{m_X,m_Y\} + \int_{m_X}^\infty \overline{F}(x)dx + \int_{m_Y}^\infty \overline{G}(x)dx = \max\{m_X,m_Y\} + \pi_X(m_X) + \pi_Y(m_Y),$$

and (4.5) follows by (4.2). If $Y \le_{st} X$ then $\overline{G}(x) \le \overline{F}(x)$ for all $x \ge 0$. If $x \ge m_X$ then $\overline{F}(x)+\overline{G}(x) \le 2\cdot\overline{F}(x) \le 1$, hence

$$\int_0^\infty \overline{H}_*(x,x)dx \le \int_0^{m_X} dx + \int_{m_X}^\infty (\overline{F}(x)+\overline{G}(x))dx \le m_X + 2\cdot\pi_X(m_X),$$

and (4.6) follows from (4.2). If $X \le_{st} Y$ then (4.7) follows similarly. ◊

**Proposition 4.2.** (*Lower bounds by known means*) Let X, Y be non-negative random variables with finite means. Then one has the inequality

$$(4.8) \qquad E[(X-Y)_+] \ge \max\{\mu_X, \mu_Y\} - \mu_Y.$$

If $Y \le_{st} X$ then one has the best lower bound

$$(4.9) \qquad E[(X-Y)_+] \ge \mu_X - \mu_Y \ge 0.$$

If $X \le_{st} Y$ then one has the best lower bound

$$(4.10) \qquad E[(X-Y)_+] \ge 0.$$

**Proof.** The inequality (4.8) follows from (4.2) and (4.4) using that

$$\int_0^\infty \overline{H}^*(x,x)dx=\int_0^\infty \max\{\overline{F}(x),\overline{G}(x)\}dx \ge \max\left\{\int_0^\infty \overline{F}(x)dx, \int_0^\infty \overline{G}(x)dx\right\} = \max\{\mu_X,\mu_Y\}.$$

If $Y \le_{st} X$ then one has $\int_0^\infty \overline{H}^*(x,x)dx = \int_0^\infty \overline{F}(x)dx = \mu_X$, and (4.9) follows. If $X \le_{st} Y$ then $\int_0^\infty \overline{H}^*(x,x)dx = \int_0^\infty \overline{G}(x)dx = \mu_Y$ implies (4.10). The lower bounds in (4.9) and (4.10) are attained at the corresponding Hoeffding-Fréchet extremal distributions. ◊

We apply these elementary results to bound the expected total financial payoff of our exchange option strategy $S_n = \sum_{i=1}^n (X_i - X_{i-1})_+$. Let $m_i, \mu_i, \sigma_i$ denote the median, mean and standard deviation of $X_i$. In general, (4.5) and (4.8) yield the bounds

$$\begin{aligned} &\sum_{i=1}^n (\max\{\mu_{i-1},\mu_i\} - \mu_{i-1}) \le E[S_n] \\ &\le \sum_{i=1}^n (\max\{m_{i-1},m_i\} + \pi_{i-1}(m_{i-1}) + \pi_i(m_i) - \mu_{i-1}). \end{aligned} \tag{4.11}$$

If there is an increasing uncertainty reflected by an increasing riskiness in the $X_i$'s, for example $X_0 \le_{st} \le X_1 \le_{st} \dots \le_{st} X_n$, then (4.6), (4.9) imply the simpler bounds

$$\mu_n - \mu_0 \le E[S_n] \le \sum_{i=1}^n (m_i + 2\pi_i(m_i) - \mu_{i-1}) \le \mu_n - \mu_0 + \sum_{i=1}^n \sqrt{\sigma_i^2 + (m_i - \mu_i)^2}, \tag{4.12}$$

where the last inequality follows by the inequality of Bowers(1969). Similarly, if there is a decreasing uncertainty reflected by a decreasing riskiness in the $X_i$'s, for example $X_0 \ge_{st} \ge X_1 \ge_{st} \dots \ge_{st} X_n$, then (4.7) and (4.10) imply the bounds

$$0 \le E[S_n] \le \sum_{i=0}^{n-1} (m_i + 2\pi_i(m_i) - \mu_i) \le \sum_{i=0}^{n-1} \sqrt{\sigma_i^2 + (m_i - \mu_i)^2}. \tag{4.13}$$

## 5. Inequalities by known marginals.

Let X, Y be *non-negative* random variables with marginals F(x), G(x), and joint probability function H(x,y). Our aim is a detailed analysis of the upper bound

$$E[(X-Y)_+] \le U := \int_0^\infty \min\{\overline{F}(x) + \overline{G}(x), 1\}dx - \mu_Y, \tag{5.1}$$

which follows from (4.2) and (4.4). To evaluate (5.1) two cases are distinguised.

Case (I) : $\overline{F}(x) + \overline{G}(x) \le 1$ for all $x \in (0,\infty)$

From (5.1) one obtains immediately

$$U = \int_0^\infty \{\overline{F}(x) + \overline{G}(x)\}dx - \mu_Y = \mu_X \tag{5.2}$$

Case (II) : there exists a unique $x_0 \in (0,\infty)$ such that $\overline{F}(x) + \overline{G}(x) \ge 1$ for $x \le x_0$ and

$\overline{F}(x)+\overline{G}(x)\le 1$ for $x \ge x_0$

From (5.1) one obtains

(5.3) $$U=\int_0^{x_0} dx+\int_{x_0}^{\infty}\{\overline{F}(x)+\overline{G}(x)\}dx-\mu_Y = x_0-\mu_Y+\pi_X(x_0)+\pi_Y(x_0).$$

There is an alternative way to derive an upper bound for the expected positive difference, which is based on the inequality

(5.4) $$(X-Y)_+ \le (X-\alpha)_+ + (\alpha-Y)_+ \text{ for all real } \alpha,$$

which implies that

(5.5) $$E[(X-Y)_+] \le \min_{\alpha}\{\alpha-\mu_Y+\pi_X(\alpha)+\pi_Y(\alpha)\}.$$

It is remarkable that both upper bounds coincide.

**Theorem 5.1.** (*Minimax property of the Hoeffding-Fréchet upper bound*) Let X, Y be non-negative random variables with distributions F(x), G(x), and suppose the means $\mu_X, \mu_Y$ of X, Y exists. Then one has

(5.6) $$\max_{(X,Y)\in BD(F,G)} E[(X-Y)_+] = \min_{\alpha}\{\alpha-\mu_Y+\pi_X(\alpha)+\pi_Y(\alpha)\}.$$

**Proof.** Set $\varphi(\alpha)=\alpha-\mu_Y+\pi_X(\alpha)+\pi_Y(\alpha)$. For $\alpha<0$ one has $\varphi(\alpha)=\pi_X(\alpha)>\pi_X(0)=\varphi(0)$. Therefore it suffices to consider the minimum over the interval $[0,\infty)$. The result depends upon the sign change of $\varphi'(\alpha)=1-\overline{F}(\alpha)-\overline{G}(\alpha)$.

Case (I) : $\overline{F}(x)+\overline{G}(x)\le 1$ for all $x\in(0,\infty)$

Since $\varphi(\alpha)$ is monotone increasing on $(0,\infty)$, its minimum is attained at $\alpha=0$, and it coincides with the maximum (5.2).

Case (II) : there exists a unique $x_0\in(0,\infty)$ such that $\overline{F}(x)+\overline{G}(x)\ge 1$ for $x\le x_0$ and $\overline{F}(x)+\overline{G}(x)\le 1$ for $x\ge x_0$

Since $\varphi(\alpha)$ is monotone decreasing on $(0,x_0)$ and increasing on $(x_0,\infty)$, its minimum is attained at $\alpha=x_0$, and it coincides with the maximum (5.3). ◊

**Example 5.1** : maximal price of an exchange option by known marginal option prices

Let v be the risk-free discount rate over some fixed period, say one year, and let $C_X(\alpha)=v\cdot E^*[(X-\alpha)_+]$, $P_Y(\alpha)=v\cdot E^*[(\alpha-Y)_+]$ be the known marginal call- and put-option prices taken with respect to risk-neutral probability distributions $F^*(x), G^*(x)$. Then a discounted version of (5.6) is

(5.7) $$\max_{(X,Y)\in BD(F^*,G^*)} v\cdot E^*[(X-Y)_+] = \min_{\alpha}\{C_X(\alpha)+P_Y(\alpha)\}.$$

For the analytical models by Black-Scholes(1973), Vasicek(1977), or Cox-Ingersoll-Ross(1985), the maximum can be calculated explicitely, at least numerically.

## 6. A further distribution-free upper bound.

In many practical situations, only incomplete information about the marginals is available, for example the means and variances. In case the range is $(-\infty,\infty)$, a best upper bound for the expected positive difference follows from the bivariate inequality of Bowers-Hürlimann(1993), as already seen in Section 3. In the present Section, we derive a distribution-free version of (5.6) valid for the non-negative range $[0,\infty)$ often encountered in Insurance and Finance.

Let $D_X = D([0,\infty);\mu_X,\sigma_X)$, $D_Y = D([0,\infty);\mu_Y,\sigma_Y)$ be the sets of all random variables X, Y with range $[0,\infty)$, finite means $\mu_X, \mu_Y$, and standard deviations $\sigma_X, \sigma_Y$. The coefficients of variation are denoted by $k_X = \frac{\sigma_X}{\mu_X}$, $k_Y = \frac{\sigma_Y}{\mu_Y}$. The set of all bivariate pairs (X,Y) such that $X \in D_X, Y \in D_Y$ is denoted by $BD = BD([0,\infty)x[0,\infty);\mu_X,\sigma_X,\mu_Y,\sigma_Y)$. The maximal stop-loss transforms over $D_X, D_Y$ are denoted by $\pi_X^*(\alpha) = \max_{X\in D_X}\{\pi_X(\alpha)\}$, $\pi_Y^*(\alpha) = \max_{Y\in D_Y}\{\pi_Y(\alpha)\}$. Using the one-to-one correspondence between a distribution function and its stop-loss transform, it is possible to determine stop-loss ordered maximal random variables $X^*, Y^*$ with survival functions $\overline{F}^*(x) = -\frac{d}{dx}\pi_X^*(x)$, $\overline{G}^*(x) = -\frac{d}{dx}\pi_Y^*(x)$ such that $\pi_{X^*}(\alpha) = \pi_X^*(\alpha)$, $\pi_{Y^*}(\alpha) = \pi_Y^*(\alpha)$ for all $\alpha$ (e.g. Hürlimann(1995)). Setting $\varphi(\alpha;X,Y) = \alpha - \mu_Y + \pi_X(\alpha) + \pi_Y(\alpha)$, it follows that

$$\varphi(\alpha;X,Y) \le \varphi(\alpha;X^*,Y^*) \text{ uniformly for all } \alpha, \text{ all } X \in D_X, Y \in D_Y. \tag{6.1}$$

This uniform property implies that

$$\min_\alpha \varphi(\alpha;X,Y) \le \min_\alpha \varphi(\alpha;X^*,Y^*) \text{ uniformly for all } X \in D_X, Y \in D_Y. \tag{6.2}$$

By the minimax Theorem 5.1, the following distribution-free upper bound has been found :

$$\begin{aligned} &\max_{(X,Y)\in BD} E[(X-Y)_+] \\ &\le U^* := \max_{(X,Y)\in BD(F^*,G^*)} E[(X-Y)_+] = \min_\alpha\{\alpha - \mu_Y + \pi_X^*(\alpha) + \pi_Y^*(\alpha)\}. \end{aligned} \tag{6.3}$$

Once the right-hand side has been determined, it remains, in order to obtain possibly a *best* upper bound, to analyze under which conditions the equality is attained.

To obtain $U^*$ one simplifies calculation by *reduction* to the case of stop-loss ordered *standard* maxima $Z(X^*) = \frac{X^* - \mu_X}{\sigma_X}$, $Z(Y^*) = \frac{Y^* - \mu_Y}{\sigma_Y}$ with distributions F(x), G(x), and ranges $[\overline{k}_X,\infty), [\overline{k}_Y,\infty)$, $\overline{k}_X = -k_X^{-1}$, $\overline{k}_Y = -k_Y^{-1}$. Then one has the relation

$$(6.4) \qquad \varphi(\alpha; X^*, Y^*) = \sigma_X \cdot \pi_{Z(X^*)}\left(\frac{\alpha-\mu_X}{\sigma_X}\right) + \sigma_Y \cdot \left\{\frac{\alpha-\mu_Y}{\sigma_Y} + \pi_{Z(Y^*)}\left(\frac{\alpha-\mu_Y}{\sigma_Y}\right)\right\}.$$

As seen in Section 5, the value of $U^*$ depends upon the sign change of $\overline{F}^*(\alpha)+\overline{G}^*(\alpha)-1=\overline{F}\left(\frac{\alpha-\mu_X}{\sigma_X}\right)+\overline{G}\left(\frac{\alpha-\mu_Y}{\sigma_Y}\right)-1$. For mathematical convenience set $\alpha_X=\frac{\alpha-\mu_X}{\sigma_X}, \alpha_Y=\frac{\alpha-\mu_Y}{\sigma_Y}$, where both quantities are related by $\mu_Y-\mu_X=\alpha_X\sigma_X-\alpha_Y\sigma_Y$. One knows that $\overline{F}(x)$ consists of three pieces described as follows :

$$(6.5) \qquad \overline{F}(x)=\begin{cases} \overline{F}_0(x)=1, \quad x<\overline{k}_X, \\ \overline{F}_1(x)=\dfrac{1}{1+k_X^2}, \quad \overline{k}_X \le x \le \dfrac{1}{2}(k_X+\overline{k}_X), \\ \overline{F}_2(x)=\dfrac{\psi(x)^2}{1+\psi(x)^2}, \quad x \ge \dfrac{1}{2}(k_X+\overline{k}_X). \end{cases}$$

In this formula the function $\psi(x)=x-\sqrt{1+x^2}<0$ is the inverse of the function $\omega(x)=\frac{1}{2}(x+\overline{x}), \overline{x}=-x^{-1}$. The distribution $\overline{G}(x)$ is defined similarly. We distinguish between several cases as in the proof of Theorem 5.1.

<u>Case (I)</u> : $\overline{F}^*(\alpha)+\overline{G}^*(\alpha)\le 1$ for all $\alpha \in (0,\infty)$

This occurs when $\overline{F}_1(\alpha_X)+\overline{G}_1(\alpha_Y)\le 1$, that is $k_Xk_Y \ge 1$. The upper bound is $U^*=\mu_X$.

<u>Case (II)</u> : there exists a unique $\alpha_0 \in (0,\infty)$ such that $\overline{F}^*(\alpha)+\overline{G}^*(\alpha)\ge 1$ for $\alpha \le \alpha_0$ and $\overline{F}^*(\alpha)+\overline{G}^*(\alpha)\le 1$ for $\alpha \ge \alpha_0$

This occurs if $k_Xk_Y \le 1$. The equation $\overline{F}(\alpha_X)+\overline{G}(\alpha_Y)=1$ consists of three pieces $\overline{F}_i(\alpha_X)+\overline{G}_j(\alpha_Y)=1$, i,j=1,2, leading to three subcases.

(1) $\quad \overline{F}_1(\alpha_X)+\overline{G}_2(\alpha_Y)=1 \;\Leftrightarrow\; \psi(\alpha_Y)=-k_X$

One has $\alpha_Y=(\omega\circ\psi)(\alpha_Y)=\omega(-k_X)=-\frac{1}{2}(k_X+\overline{k}_X)$. Furthermore the constraints $\alpha_Y \ge \frac{1}{2}(k_Y+\overline{k}_Y)$, $\overline{k}_X \le \alpha_X \ge \frac{1}{2}(k_X+\overline{k}_X)$ are equivalent to the conditions $k_Xk_Y \le 1$ and $\frac{\mu_Y-\mu_X}{\sigma_X+\sigma_Y}\le\frac{1}{2}(k_X+\overline{k}_X)$. A calculation using (6.4) shows that

$$U^*=\sigma_X(-\overline{k}_X)\frac{1+\overline{k}_X\alpha_X}{1+\overline{k}_X^2}+\frac{1}{2}\sigma_Y\cdot(\alpha_Y+\sqrt{1+\alpha_Y^2})=\mu_X-\left(\frac{1-k_Xk_Y}{1+k_X^2}\right)\mu_Y.$$

(2) $\quad \overline{F}_2(\alpha_X)+\overline{G}_1(\alpha_Y)=1 \;\Leftrightarrow\; \psi(\alpha_X)=-k_Y$

By symmetry to the subcase (1) one gets $\alpha_X=-\frac{1}{2}(k_Y+\overline{k}_Y)$, and the constraints are equivalent to the conditions $k_Xk_Y\le 1$ and $\frac{\mu_X-\mu_Y}{\sigma_X+\sigma_Y}\le\frac{1}{2}(k_Y+\overline{k}_Y)$. Moreover one has by the symmetry relation $(X-Y)_+=X-Y+(Y-X)_+$ with subcase (1) that

$$U^* = \mu_X - \mu_Y + \left\{ \mu_Y - \left( \frac{1-k_X k_Y}{1+k_Y^2} \right) \mu_X \right\} = \mu_X - \left( \frac{1-k_X k_Y}{1+k_Y^2} \right) \mu_X .$$

(3) $\quad \overline{F}_2(\alpha_X) + \overline{G}_2(\alpha_Y) = 1 \Leftrightarrow \psi(\alpha_X)\psi(\alpha_Y) = 1$

Using that $\omega\left(\frac{1}{\psi(x)}\right) = -x$ one has $\alpha_X + \alpha_Y = 0$, hence $\alpha_X = \frac{\mu_Y - \mu_X}{\sigma_X + \sigma_Y}$. The conditions under which (3) holds are $\frac{\mu_Y - \mu_X}{\sigma_X + \sigma_Y} \geq \frac{1}{2}(k_X + \overline{k}_X)$ and $\frac{\mu_X - \mu_Y}{\sigma_X + \sigma_Y} \geq \frac{1}{2}(k_Y + \overline{k}_Y)$. Through calculation one obtains

$$\begin{aligned} U^* &= \frac{1}{2}\sigma_X \cdot (\sqrt{1+\alpha_X^2} - \alpha_X) + \frac{1}{2}\sigma_Y \cdot (\sqrt{1+\alpha_Y^2} + \alpha_Y) \\ &= \frac{1}{2} \cdot \left\{ \sqrt{(\sigma_X + \sigma_Y)^2 + (\mu_X - \mu_Y)^2} + (\mu_X - \mu_Y) \right\}. \end{aligned}$$

The following result has been shown.

**Theorem 6.1.** Let X, Y be non-negative random variables with finite means $\mu_X, \mu_Y$, and standard deviations $\sigma_X, \sigma_Y$. Then the distribution-free upper bound for the expected positive difference $E[(X-Y)_+]$, given by the Hoeffding-Fréchet upper bound (6.3), is determined in tabular form as follows :

| case | conditions | Hoeffding-Fréchet upper bound |
|---|---|---|
| (I) | $k_X k_Y \geq 1$ | $\mu_X$ |
| (II) | $k_X k_Y \leq 1$ | |
| (1) | $\frac{\mu_Y - \mu_X}{\sigma_X + \sigma_Y} \leq \frac{1}{2}(\frac{k_X^2 - 1}{k_X})$ | $\mu_X - \left( \frac{1-k_X k_Y}{1+k_X^2} \right) \mu_Y$ |
| (2) | $\frac{\mu_X - \mu_Y}{\sigma_X + \sigma_Y} \leq \frac{1}{2}(\frac{k_Y^2 - 1}{k_Y})$ | $\mu_X - \left( \frac{1-k_X k_Y}{1+k_Y^2} \right) \mu_X$ |
| (3) | $\frac{\mu_Y - \mu_X}{\sigma_X + \sigma_Y} \geq \frac{1}{2}(\frac{k_X^2 - 1}{k_X})$, $\frac{\mu_X - \mu_Y}{\sigma_X + \sigma_Y} \geq \frac{1}{2}(\frac{k_Y^2 - 1}{k_Y})$ | $\frac{1}{2} \cdot \left\{ \sqrt{(k_X \mu_X + k_Y \mu_Y)^2 + (\mu_X - \mu_Y)^2} + (\mu_X - \mu_Y) \right\}$ |

**Remark 6.1.** As shown by the author(1997b), generalized versions of the minimax Theorem 5.1 as well as of the Theorem 6.1 can be formulated for random variables X, Y with arbitrary ranges. The similar proofs are a bit more technical and require the distinction between several more cases and subcases. For pedagogical reasons, only the simplest extension of the bivariate inequality of Bowers in Hürlimann(1993) has been presented here.

**References.**

*Black, F. and M. Scholes* (1973). The pricing of options and corporate liabilities. Journal of Political Economy 81, 637-59. Reprinted in Luskin(1988).
*Bowers, N.L.* (1969). An upper bound for the net stop-loss premium. Transactions of the Society of Actuaries XIX, 211-16.
*Cambanis, S., Simons, G. and W. Stout* (1976). Inequalities for $E[k(X,Y)]$ when the marginals are fixed. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete/ Probability Theory and related Fields 36, 285-94.
*Cambanis, S. and G. Simons* (1982). Probability and expectation inequalities. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete/ Probability Theory and related Fields 59, 1-25.
*Cox, J.C., Ingersoll, J. and S. Ross* (1985). A theory of the term structure of interest rates. Econometrica 53, 385-467.
*Dubins, L.E. and L.J. Savage* (1965). Inequalities for Stochastic Processes (How to Gamble if you must). McGraw-Hill, New York. 2nd edition (1976), Dover, New York.
*Fréchet, M.* (1951). Sur les tableaux de corrélation dont les marges sont données. Ann. Univ. Lyion Section A 14, 53-77.
*Hoeffding, W.* (1940). Massstabinvariante Korrelationstheorie. Schriften des Math. Instituts und des Instituts für Angewandte Mathematik der Universität Berlin 5, 179-233.
*Hürlimann, W.* (1993). Bivariate distributions with diatomic conditionals and stop-loss transform of random sums. Statistics and Probability Letters 24, 329-35.
*Hürlimann, W.* (1995). A stop-loss ordered extremal distribution and some of its applications. XXVI. ASTIN Colloquium, Leuven.
*Hürlimann, W.* (1997a). A unified approach to some loss variance bounds. Appears in Bulletin of the Swiss Association of Actuaries.
*Hürlimann, W.* (1997b). Bounds for expected positive differences by means of Hoeffding Fréchet extremal distributions. XXVIII. ASTIN Colloquium, Cairns, Australia.
*Kaas, R., Heerwaarden, Van A.E. and Goovaerts, M.J.* (1994). Ordering of Actuarial Risks. CAIRE Education Series 1, Brussels.
*Krengel, U. and L. Sucheston* (1987). Prophet compared to gambler : an inequality for transforms of processes. The Annals of Probability 15(4), 1593-99.
*Lorentz, G.C.* (1953). An inequality for rearrangements. American Mathematical Monthly 60, 176-79.
*Luskin, D.L.* (1988) (Editor). Portfolio Insurance : a guide to Dynamic Hedging. John Wiley.
*Maitra, A.P. and W.D. Sudderth* (1996). Discrete Gambling and Stochastic Games. Applications of Mathematics, Stochastic Modelling and Applied Probability, vol. 32. Springer.
*Shaked, M. and J.G. Shanthikumar* (1994). Stochastic Orders and Their Applications. Academic Press, New York.
*Tchen, A.H.* (1980). Inequalities for distributions with given marginals. Annals of Probability 8, 814-27.
*Vasicek, O.A.* (1977). An equilibrium characterization of the term structure. Journal of Financial Economics 5, 177-88.
*Whitt, W.* (1976). Bivariate distributions with given marginals. Annals of Statistics 4, 1280-89.