

# IS THERE A RATIONAL EVIDENCE FOR AN INFINITE VARIANCE ASSET PRICING MODEL ?

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## Abstract.

Distributions defined over the whole real line arise when modelling the financial return generating process. In case a financial return is interpreted as a difference between assets and liabilities, we propose distributions from a  $\delta$ -family, in particular independent differences, for which both positive and negative tails are of  $\delta$ -exponential type. Under the assumption that the standardized price logarithm belongs to the corresponding one-parametric family of symmetric standardized independent difference distributions with  $\delta$ -exponential tails, one obtains explicit option pricing formulas. This new approach has the advantage to take into account higher values of the kurtosis parameter than is allowed by the classical model. If one requires a risk-neutral forward rate of return, then the Black-Scholes rate of return does not lead to a feasible option pricing model with  $\delta$ -exponential tails of return, which shows the existence of a distinct theoretical model. The economic problem of minimizing the expected future value of an asset return guarantee is considered. Under our  $\delta$ -exponential rate of return model, the minimum is attained for a distribution with infinite variance, which provides a new rational evidence for the existence of a financial asset pricing model with infinite variance.

Keywords : financial return, option pricing, return guarantee, infinite variance, independent difference, exponential tail, kurtosis

## **Introduction.**

Besides the actuarial interest put forward in Hürlimann(1991a), an important motivation for considering probabilistic models of *independent differences* is the following elementary setting in asset and liability risk analysis. Let  $L$  represent the accumulated random insurance liabilities during some fixed time period and let  $A$  be the accumulated random asset value of the premium income during the same period, which is supposed to be invested in a risky portfolio fund protected by portfolio insurance. Given that  $A$  and  $L$  are *independent*, a problem is to find a practical and approximate *distribution of the surplus*  $G=A-L$ . Up to a location parameter, it seems reasonable to approximate roughly the distribution of  $A$  by a  $\delta$ -exponential family as defined in Subsection 1.1 (see e.g. Clarke and Arnott(1988)). In case the distribution of  $L$  is continuous, for example a translated gamma distribution, as can be reasonably assumed in Risk Theory (see e.g. Dickson and Waters(1993)), our formulas of Subsection 1.1 can be used to evaluate the distribution of the surplus. For some specific problems, for example the determination of "financial risk premiums" associated to financial risk protection models (see e.g. Hürlimann(1991b/95a/96)), algorithms for the direct evaluation of stop-loss premiums  $E[(G-c)_+]$ ,  $c \in \mathbb{R}$ , might also be useful.

Distributions defined over the whole real line arise in particular when modelling the financial return generating process of some securities. In case a financial return is interpreted as a difference between assets and liabilities, the natural question is whether a financial return generating process can be modelled by independent differences. A definitive answer to such questions needs of course empirical work. However, from a theoretical point of view, an equally important issue might be addressed as follows. Given some model of financial return, what economic implications can be derived from it? Our main interest in this paper is devoted to the problem of minimizing the expected future value of an asset return guarantee. As a main result, we show in Section 3 that there exists a well-defined financial asset return generating process with *infinite variance* under which this minimum is attained.

A brief outline of the paper follows. In the preliminary Section 1, some specific models of independent differences are considered. First, the negative tail is supposed to belong to a  $\delta$ -family, which includes in particular compound Poisson distributions. Then, in Section 1.1, one supposes a  $\delta$ -exponential positive tail, and in Section 1.2 both positive and negative tails are supposed to be of  $\delta$ -exponential type. In Section 2, option pricing formulas are derived in case the standardized price logarithm belongs to the one-parametric family of symmetric standardized independent difference distributions with  $\delta$ -exponential tails considered in Section 1.2. This new approach has the advantage to take into account the kurtosis parameter, which is known to be far greater than for the standardized normal distribution. Moreover the mathematical analysis is very tractable and can be done in closed form. As an interesting feature it turns out that an "infinite variance" model might be feasible in this setting. Moreover, if one requires a risk-neutral valuation (through a risk-neutral forward rate of return), then the Black-Scholes rate of return does not lead to a feasible option pricing model of  $\delta$ -exponential type. This result implies in particular the existence of alternative risk-neutral valuation option pricing models. In Section 3, the economic problem of minimizing the expected future value of an asset return guarantee is investigated. It is shown that under a  $\delta$ -exponential rate of return model, the minimum is attained for the special "infinite variance" model. In particular, this solution provides new rational evidence for the existence of a financial asset pricing model with infinite variance. Knowing the instantaneous volatility and kurtosis parameter, the optimal time period which minimizes the expected future value of an asset return guarantee can be estimated. A numerical illustration based on published data is presented.

## 1. Preliminaries on modelling independent differences.

Let  $X$  be a random variable defined on  $\mathbb{R}$  and let us write it as difference  $X=X^+-X^-$  of "positive" random variables  $X^+$ ,  $X^-$  defined on  $\mathbb{R}^+$ . In the present paper let us assume that the *positive tail*  $X^+$  and the *negative tail*  $X^-$  are independent. Then the representation  $X=X^+-X^-$  is called an *independent difference*. Distributions are denoted by  $F(x)=F_X(x)=\Pr(X\leq x)$ ,  $F^+(x)=F_{X^+}(x)=\Pr(X^+\leq x)$ ,  $F^-(x)=F_{X^-}(x)=\Pr(X^-\leq x)$ , and it is assumed that the corresponding probability densities  $f(x)=f_X(x)$ ,  $f^+(x)=f_{X^+}(x)$ ,  $f^-(x)=f_{X^-}(x)$  exist. The distribution of an independent difference is given by the convolution

$$(1.1) \quad F(x) = (F_{X^+} * F_{X^-})(x) = \int_{-\infty}^{\min(0,x)} F_{X^+}(x-y)f_{X^-}(y)dy,$$

obtained by observing that  $F_{X^-}(-x)=F^-(x)$ ,  $f_{X^-}(-x)=f^-(x)$ . To get explicit results, it is necessary to specify further the probability structure of  $X^+$  and  $X^-$ .

Let  $X^-$  be distributed as a  $\delta$ -family of the type

$$(1.2) \quad \begin{aligned} f^-(x) &= \varepsilon^- \delta(x) + (1-\varepsilon^-)g^-(x), \quad x \geq 0, \quad 0 \leq \varepsilon^- < 1, \\ F^-(x) &= \varepsilon^- + (1-\varepsilon^-)G^-(x), \end{aligned}$$

where  $\delta(x)$  is Dirac's delta function and  $g^-(x)$  is continuous such that  $g^-(0)=0$ . Note that  $\delta$ -families include compound Poisson distributions, which usually are applied to model insurance risk liabilities. With the choice (1.2), formula (1.1) reads

$$(1.3) \quad F(x) = \begin{cases} (1-\varepsilon^-)(F_{X^+} * G_{X^-})(x), & x < 0, \\ \varepsilon^- F_{X^+}(x) + (1-\varepsilon^-)(F_{X^+} * G_{X^-})(x), & x \geq 0, \end{cases}$$

where for convention one sets  $G_{X^-}(-x)=G^-(x)$ ,  $g_{X^-}(-x)=g^-(x)$ . The formula (1.3) shows that the evaluation of distributions is reduced to the case  $\varepsilon^-=0$ .

### 1.1. $\delta$ -exponential positive tail and continuous negative tail.

For the reason mentioned after (1.3), assume that  $f^-(x)$  is continuous and  $f^-(0)=0$ . A simple but useful and very tractable two-parametric specification for the positive tail is a  $\delta$ -exponential family defined by

$$(1.4) \quad \begin{aligned} f^+(x) &= \varepsilon^+ \delta(x) + (1-\varepsilon^+)ae^{-ax}, \quad x \geq 0, \quad 0 \leq \varepsilon^+ < 1, \quad a > 0, \\ F^+(x) &= 1 - (1-\varepsilon^+)e^{-ax}, \quad x \geq 0. \end{aligned}$$

It has been shown in Hürliemann(1990), formula (4.15), that for  $\varepsilon^+>0$  the  $\delta$ -exponential density  $f^+(x)$  is compound Poisson with parameter  $\lambda=-\ln(\varepsilon^+)$  and "jump density"

$h^+(x) = \frac{\exp(-ax) - \exp(-\frac{ax}{\varepsilon^+})}{x \cdot \ln(\varepsilon^+)}$  (see also Panjer and Willmot(1992), p. 13-14). It follows

that a  $\delta$ -exponential density is infinitely divisible. Introducing (1.4) into (1.1) one gets

$$\begin{aligned} F(x) &= \int_{-\min(0,x)}^{\infty} f^-(y)dy - (1-\varepsilon^+)e^{-ax} \cdot \int_{-\min(0,x)}^{\infty} e^{-ay} f^-(y)dy \\ &= 1 - F^-(-\min(0,x)) - (1-\varepsilon^+)e^{-ax} \cdot \int_{-\min(0,x)}^{\infty} e^{-ay} f^-(y)dy \\ &= 1 - F^-(-\min(0,x)) - (1-\varepsilon^+)e^{-ax} \cdot ((Lf^-)(a) - \int_0^{-\min(0,x)} e^{-ay} f^-(y)dy), \end{aligned}$$

where  $(Lf^-)(a) = \int_0^{\infty} e^{-ax} f^-(x)dx$  denotes the Laplace transform of the negative tail  $f^-(x)$ .

Simplifying further, one obtains

$$(1.5) \quad F(x) = \begin{cases} 1 - (1-\varepsilon^+)e^{-ax} (Lf^-)(a), & x \geq 0, \\ 1 - F^-(-x) - (1-\varepsilon^+)e^{-ax} ((Lf^-)(a) - \int_0^{-x} e^{-ay} f^-(y)dy), & x < 0. \end{cases}$$

In case  $x < 0$  an alternative formula, obtained through partial integration, is given by

$$(1.6) \quad F(x) = 1 - \varepsilon^+ F^-(-x) - (1-\varepsilon^+)e^{-ax} (Lf^-)(a) + (1-\varepsilon^+)ae^{-ax} \int_0^{-x} e^{-ay} F^-(y)dy, \quad x < 0.$$

Since  $F(0^-) = F(0^+)$  one sees that  $F(x)$  is everywhere continuous (because  $f^-(x)$  is continuous). In case  $f^-(x)$  is infinitely divisible  $F(x) = (F_{x^+} * F_{-x^-})(x)$  is also infinitely divisible (because  $F^+(x)$  is infinitely divisible). The probability density function is given by

$$(1.7) \quad f(x) = \begin{cases} (1-\varepsilon^+)ae^{-ax} (Lf^-)(a), & x \geq 0, \\ \varepsilon^+ f^-(-x) + (1-\varepsilon^+)ae^{-ax} ((Lf^-)(a) - \int_0^{-x} e^{-ay} f^-(y)dy), & x < 0, \end{cases}$$

or alternatively

$$(1.8) \quad \begin{aligned} f(x) &= \varepsilon^+ f^-(-x) - (1-\varepsilon^+)aF^-(-x) \\ &+ (1-\varepsilon^+)ae^{-ax} ((Lf^-)(a) - a \int_0^{-x} e^{-ay} F^-(y)dy), \quad x < 0. \end{aligned}$$

Since  $f(0^-) = f(0^+)$  the probability density  $f(x)$  is everywhere continuous.

### 1.2. $\delta$ -exponential positive and negative tails.

Let us assume that both positive and negative tails of an independent difference follow  $\delta$ -exponential densities specified by the four-parametric family

$$(1.9) \quad \begin{aligned} f^+(x) &= \varepsilon^+ \delta(x) + (1 - \varepsilon^+) a e^{-ax}, \quad x \geq 0, \quad a > 0, \quad 0 \leq \varepsilon^+ < 1, \\ f^-(x) &= \varepsilon^- \delta(x) + (1 - \varepsilon^-) b e^{-bx}, \quad x \geq 0, \quad b > 0, \quad 0 \leq \varepsilon^- < 1. \end{aligned}$$

Applying formulas (1.3) and (1.5) one shows that

$$(1.10) \quad F(x) = \begin{cases} F^- e^{bx}, & x < 0, \\ 1 - (1 - F^+) e^{-ax}, & x \geq 0, \end{cases}$$

with

$$(1.11) \quad \begin{aligned} F^- &:= F(0^-) = (1 - \varepsilon^-) \left( \frac{a + \varepsilon^+ b}{a + b} \right), \\ F^+ &:= F(0^+) = 1 - (1 - \varepsilon^+) \left( \frac{b + \varepsilon^- a}{a + b} \right). \end{aligned}$$

The corresponding probability density is equal to

$$(1.12) \quad f(x) = \begin{cases} F^- b e^{bx}, & x < 0, \\ (F^+ - F^-) \delta(x) + (1 - F^+) a e^{-ax}, & x \geq 0. \end{cases}$$

It is clear that  $f(x)$  is infinitely divisible because  $f^+(x)$ ,  $f^-(x)$  are. The moments of  $X$  are obtained from the moments of  $X^+$ ,  $X^-$  and are given by

$$(1.13) \quad E[X^k] = k! \left\{ \frac{F^-}{b^k} + (-1)^k \frac{1 - F^+}{a^k} \right\}, \quad k = 1, 2, \dots$$

The four-parametric family of independent difference distributions with  $\delta$ -exponential positive and negative tails contains an interesting subfamily, useful for modelling financial return generating processes (to be done in Section 2), namely the subclass of *symmetric standardized* distributions defined by the moment conditions

$$(1.14) \quad \begin{aligned} E[X] &= \frac{F^-}{b} - \frac{1 - F^+}{a} = 0, \\ E[X^2] &= 2 \left( \frac{F^-}{b^2} + \frac{1 - F^+}{a^2} \right) = 1, \\ E[X^3] &= 6 \left( \frac{F^-}{b^3} - \frac{1 - F^+}{a^3} \right) = 0. \end{aligned}$$

Comparing the first and third condition one gets  $b=a$ . The second condition implies now  $F^- = 1 - F^+ = \frac{a^2}{4}$ . The obtained family of distributions is summarized by the following result.

**Proposition 1.1.** The class of symmetric standardized independent difference distributions with  $\delta$ -exponential positive and negative tails is described by the one-parametric infinitely divisible distribution

$$(1.15) \quad F(x) = \begin{cases} \frac{1}{4}\alpha^2 e^{\alpha x}, & x < 0, \\ 1 - \frac{1}{4}\alpha^2 e^{-\alpha x}, & x \geq 0, \end{cases}$$

with corresponding probability density

$$(1.16) \quad f(x) = (1 - \frac{\alpha^2}{2})\delta(x) + \frac{1}{4}\alpha^3 e^{-\alpha|x|}, \quad 0 < \alpha \leq \sqrt{2}.$$

The kurtosis of a member of this class is equal to

$$(1.17) \quad E[X^4] = \frac{12}{\alpha^2} \geq 6, \quad 0 < \alpha \leq \sqrt{2}.$$

Furthermore the only continuous member of this class is the Laplace distribution characterized by  $\alpha = \sqrt{2}$ .

**Remark 1.1.** For the purpose of statistical inference, the maximum likelihood estimator  $\hat{\alpha}$  of the parameter  $\alpha$  is easily derived. If  $X_1, \dots, X_n$  is a sample of size  $n$ , then  $\hat{\alpha}$  is solution of the non-linear equation

$$(1.18) \quad \frac{3}{\alpha} \cdot n + \frac{2(\alpha^2 - 6)}{\alpha(\alpha^3 - 2\alpha^2 + 4)} \cdot n_0 = \sum_{i=1}^n |X_i|,$$

where  $n_0$  is the number of zero observations.

## 2. Option pricing for $\delta$ -exponential tails of return.

Suppose the price of a unit of a risky asset at time  $t$  is described by the random variable  $S_t$  with  $S_0=1$ . Let  $X_t=\ln(S_t)$  be the price logarithm. The classical model of Black and Scholes(1973) for the valuation of call-options in the stock market assumes that the standardized price logarithm  $Z_t = \frac{X_t - \mu t}{\sigma \sqrt{t}}$  follows a standard normal variable, where  $\mu, \sigma^2$  are constants representing the expected value and variance of return per unit of time. Furthermore, if  $r$  is the instantaneous risk-free rate of return, then in order to satisfy the arbitrage-free condition one has necessarily  $\mu = r - \frac{1}{2}\sigma^2$ . Let  $k$  be the relative exercise price, that is the ratio of exercise price to the stock's present price. Then, in a risk-neutral world, Black-Scholes call-option price is equal to the discounted expected value at time  $t$  of the claim  $(S_t - k)_+$  given by

$$(2.1) \quad C(d) = e^{-r} E[(S_t - k)_+] = N(d) - ke^{-r} N(d - \sigma \sqrt{t}), \quad \text{with}$$

$$(2.2) \quad d = \frac{rt - \ln(k)}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}.$$

Usually, the Black-Scholes assumptions are considered as acceptable provided the remaining life of an option is less than six months. Over longer periods of time, it is known that daily returns on stocks are not independent observations from the same normal distribution. In particular the *kurtosis* parameter of the standardized daily return generating process in most cases exceeds 6 (see Taylor(1992), p. 44), which is far greater than the kurtosis of a standard normal distribution, which is equal to 3. For this reason option valuation for alternative return generating processes must be analyzed.

In the present paper, let us assume that the standardized price logarithm  $Z_t = \frac{X_t - \mu t}{\sigma\sqrt{t}}$  belongs to the one-parametric family of symmetric standardized independent difference distributions with  $\delta$ -exponential tails, as described in Proposition 1.1. Members of this family are infinitely divisible, and are discontinuous at  $x_t = \mu t$  with the exception of the Laplace distribution. This family can be characterized by its kurtosis parameter. If  $\kappa$  is the kurtosis per unit of time, then one has the relationship

$$(2.3) \quad \kappa t = \frac{12}{\alpha^2}, \quad 0 < \alpha^2 \leq 2.$$

Under this assumption, the distribution  $F_t(s)$  of the price  $S_t = \exp(X_t) = \exp(\mu t + \sigma\sqrt{t} \cdot Z_t)$  is given by

$$(2.4) \quad F_t(s) = \begin{cases} \frac{1}{4}\alpha^2 \exp\left\{\alpha\left(\frac{\ln s - \mu t}{\sigma\sqrt{t}}\right)\right\}, & s < e^{\mu t}, \\ 1 - \frac{1}{4}\alpha^2 \exp\left\{-\alpha\left(\frac{\ln s - \mu t}{\sigma\sqrt{t}}\right)\right\}, & s \geq e^{\mu t}. \end{cases}$$

In this return generating model, the call-option price is given by

$$(2.5) \quad C(d) = e^{-rt}\pi(d), \quad \text{with}$$

$$(2.6) \quad \pi(d) = E[(S_t - d)_+] = \int_d^{\infty} (s - d) dF_t(s),$$

where the integral has to be taken in the sense of Riemann-Stieltjes. For a risk-neutral valuation, one requires the following relationship

$$(2.7) \quad \pi(0) = E[S_t] = e^{rt},$$

which restricts the choice of the parameters  $\mu$ ,  $\sigma$ ,  $\kappa$ .

To evaluate the Stieltjes integral (2.6), distinguish between the two cases  $d \leq \exp(\mu t)$  and  $d > \exp(\mu t)$ , use the well-known formula

$$\int_d^{\infty} (s - d) dF_t(s) = \int_d^{\infty} (1 - F_t(s)) ds$$

obtained through partial integration, and add a correction term for the jump in  $s = \exp(\mu t)$  in case  $d < \exp(\mu t)$ . The integral exists under the *assumption*  $0 < \sigma\sqrt{t} < \alpha \leq \sqrt{2}$ , and one obtains the formulas :

$$(2.8) \quad \pi(d) = (2 - \frac{1}{2}\alpha^2)(\exp(\mu t) - d) + \frac{1}{2} \frac{\alpha^2 \sigma^2 t}{\alpha^2 - \sigma^2 t} \exp(\mu t) \\ + \frac{1}{4} \alpha^2 \frac{\sigma\sqrt{t}}{\alpha + \sigma\sqrt{t}} \exp\left\{ \mu t + \left( \frac{\alpha + \sigma\sqrt{t}}{\sigma\sqrt{t}} \right) (\ln(d) - \mu t) \right\}, \quad d \leq \exp(\mu t),$$

$$(2.9) \quad \pi(d) = \frac{1}{4} \alpha^2 \frac{\sigma\sqrt{t}}{\alpha - \sigma\sqrt{t}} \exp\left\{ \mu t - \left( \frac{\alpha - \sigma\sqrt{t}}{\sigma\sqrt{t}} \right) (\ln(d) - \mu t) \right\}, \quad d \geq \exp(\mu t).$$

In particular, the *mean* is equal to

$$(2.10) \quad \pi(0) = E[S_t] = \exp(\mu t) \cdot \left\{ 2 - \frac{1}{2}\alpha^2 + \frac{1}{2} \frac{\alpha^2 \sigma^2 t}{\alpha^2 - \sigma^2 t} \right\}.$$

To calculate the *variance*, one uses the relationship

$$(2.11) \quad E[S_t^2] = \int_0^{\infty} 2\pi(s) ds,$$

which yields the formula

$$(2.12) \quad E[S_t^2] = 2 \exp(2\mu t) \cdot \left\{ \begin{aligned} & 1 - \frac{1}{4}\alpha^2 + \frac{1}{2} \frac{\alpha^2 \sigma^2 t}{\alpha^2 - \sigma^2 t} + \frac{1}{4} \frac{\alpha^2 \sigma^2 t}{(\alpha + \sigma\sqrt{t})(\alpha + 2\sigma\sqrt{t})} \\ & + \frac{1}{4} \frac{\alpha^2 \sigma^2 t}{(\alpha - \sigma\sqrt{t})(\alpha - 2\sigma\sqrt{t})} \end{aligned} \right\}.$$

In particular, an *infinite variance* is possible if  $\alpha = 2\sigma\sqrt{t}$  or, using (2.3), if

$$(2.13) \quad t = \frac{1}{\sigma} \sqrt{\frac{3}{\kappa}}, \quad \text{hence} \quad \alpha^2 = 4\sigma \sqrt{\frac{3}{\kappa}},$$

which is a feasible model provided  $\alpha^2 \leq 2$ , that is

$$(2.14) \quad \sigma \leq \frac{1}{2} \sqrt{\frac{\kappa}{3}}.$$

The risk-neutral condition (2.7) is with (2.10) equivalent to the relationship

$$(2.15) \quad \mu = r - \frac{1}{t} \ln \left\{ 2 - \frac{1}{2}\alpha^2 + \frac{1}{2} \frac{\alpha^2 \sigma^2 t}{\alpha^2 - \sigma^2 t} \right\}.$$

Under which condition does this constraint leads to a feasible model such that the assumption  $0 < \sigma^2 t < \alpha^2 \leq 2$  is fulfilled? Setting

$$(2.16) \quad \varepsilon = \exp\{(r - \mu)t\} - 1 > 0,$$

one sees that (2.15) is equivalent with the equation

$$(2.17) \quad \alpha^4 - 2(\sigma^2 t + 1 - \varepsilon)\alpha^2 + 2(1 - \varepsilon)\sigma^2 t = 0,$$

whose solution  $\alpha^2 > \sigma^2 t$  is given by

$$(2.18) \quad \alpha^2 = \sigma^2 t + (1 - \varepsilon) + \sqrt{\sigma^2 t + (1 - \varepsilon)^2}.$$

Now  $\alpha^2 \leq 2$  provided the following condition holds :

$$(2.19) \quad \varepsilon = \exp\{(r - \mu)t\} - 1 \geq \frac{1}{2}\sigma^2 t \cdot \left(\frac{3 - \sigma^2 t}{2 - \sigma^2 t}\right).$$

To summarize a *feasible option pricing model with  $\delta$ -exponential tails of return* is prescribed by a quadruple of parameters  $(r, \mu, \sigma, \kappa)$  such that the relations (2.3), (2.15) and the inequality (2.19) are fulfilled. The corresponding option prices are calculated from the formulas (2.5), (2.8) and (2.9).

**Proposition 2.1.** The Black-Scholes rate of return relationship  $\mu = r - \frac{1}{2}\sigma^2$  does not lead to a feasible option pricing model with  $\delta$ -exponential tails of return.

**Proof.** To be feasible  $\varepsilon = \exp\{(r - \mu)t\} - 1 = \exp\{\frac{1}{2}\sigma^2 t\} - 1$  should satisfy the inequality (2.19). However, the reverse inequality holds. Indeed, setting  $x = \frac{1}{2}\sigma^2 t$ , one gets

$$2(e^x - 1 - x) = x^2 + 2\left(\frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) < x(1 + x^2 + x^3 + \dots) = \frac{x}{1 - x},$$

hence

$$e^x - 1 < x\left(1 + \frac{1}{2(1 - x)}\right) = x\left(\frac{3 - 2x}{2 - 2x}\right),$$

which is reverse to the inequality (2.19).

**Remark 2.1.** For modelling purposes, this result implies in particular that the  $\delta$ -exponential tails of return option-pricing model and Black-Scholes model cannot both be true theoretical models, and a comparison of these models is logically impossible

### 3. Minimizing the expected future value of an asset return guarantee.

A possible (forward) *measure of the asset return* over a time period  $t$  is the expected value of the price logarithm of a unit of investment. It is given by  $E[X_t] = E[\ln(S_t)] = \mu t$ . To guarantee at time  $t$  this asset return, an investor can buy a put-option with relative strike price  $k = \exp(\mu t)$  such that at time  $t$  one has the perfect hedge relation of portfolio insurance

$$(3.1) \quad S_t + (e^{\mu t} - S_t)_+ = e^{\mu t} + (S_t - e^{\mu t})_+.$$

The expected future value of this hedging strategy is equal to

$$(3.2) \quad V_t = e^{\mu t} + E[(S_t - e^{\mu t})_+].$$

Clearly, this value can be estimated only if the distribution of  $S_t$  is specified. Under the usual lognormal assumption, that is  $X_t = \ln(S_t)$  is normally distributed with mean  $\mu t$  and variance  $\sigma^2 t$ , one has

$$(3.3) \quad V_t = \frac{1}{2} \left\{ 1 + 2 \exp\left(\frac{1}{2} \sigma^2 t\right) N(\sigma \sqrt{t}) \right\} \cdot \exp(\mu t),$$

a value which is independent of any further characteristic of the return generating process, in particular the kurtosis parameter  $E[Z_t^4] = E\left[\left(\frac{X_t - \mu t}{\sigma \sqrt{t}}\right)^2\right]$ , which is always 3 under the lognormal assumption. By fixed  $\mu, \sigma$ , the value  $V_t$  depends on  $t$ , but there cannot exist within this model a rule, which will prefer one time period over another.

Suppose an investor has to decide, which time period  $t$  will provide the minimum value of  $V_t$  by fixed  $\mu, \sigma$ . Then, he must specify a return generating process  $Z_t = \frac{X_t - \mu t}{\sigma \sqrt{t}}$  with a varying characteristic. Since it seems reasonable to assume a symmetric  $Z_t$ , it is appropriate to specify a  $Z_t$  with varying kurtosis parameter. If the time unit is one day, the kurtosis is in most situations greater than 6, and a possible mathematically tractable distribution, which is able to model this feature, is a symmetric standardized independent difference distribution with  $\delta$ -exponential tails, as described in Proposition 1.1. Under such a distribution, the expected future value of the chosen asset return guarantee is equal to

$$(3.4) \quad V_t = \left\{ 1 + \frac{1}{4} \alpha^2 \left( \frac{\sigma \sqrt{t}}{\alpha - \sigma \sqrt{t}} \right) \right\} \cdot \exp(\mu t).$$

A straightforward calculation shows that  $V_t$  is *minimum* with respect to  $\alpha$  in case  $\alpha = 2\sigma\sqrt{t}$ , and then

$$(3.5) \quad V_t(\alpha = 2\sigma\sqrt{t}) = (1 + \sigma^2 t) \cdot \exp(\mu t),$$

a value which again depends only on  $\mu, \sigma, t$ . As follows from equation (2.12), this choice implies an *infinite variance*  $E[S_t] = \infty$ . Given this infinite variance model and the characteristic values  $(\mu, \sigma, \kappa)$  of the return process  $X_t$ , which time period should be chosen ?

A feasible infinite variance model necessarily satisfies the conditions (2.13) and (2.14), namely

$$(3.6) \quad \alpha^2 = 4\sigma\sqrt{\frac{3}{\kappa}}, \quad t = \frac{1}{\sigma}\sqrt{\frac{3}{\kappa}}, \quad \sigma \leq \frac{1}{2}\sqrt{\frac{\kappa}{3}}.$$

Risk-neutral option pricing with this model requires additionally the conditions (2.15) and (2.19), that is

$$(3.7) \quad (r - \mu)t = \ln\left\{2 - \frac{4}{3}\sigma^2 t\right\},$$

$$(3.8) \quad \varepsilon = \exp\{(r - \mu)t\} - 1 = 1 - \frac{4}{3}\sigma^2 t \geq \frac{1}{2}\sigma^2 t \cdot \left(\frac{3 - \sigma^2 t}{2 - \sigma^2 t}\right).$$

The latter inequality implies that  $11\sigma^4 t^2 - 31\sigma^2 t + 12 \geq 0$ , hence

$$(3.9) \quad \sigma^2 t \leq \frac{1}{22}(31 - \sqrt{433}) = 0.46324.$$

Leaving aside the option pricing valuation problem, let us concentrate on the conditions (3.6).

A solution to our minimization problem exists always provided  $\sigma \leq \frac{1}{2}\sqrt{\frac{\kappa}{3}}$ , and in this case

the optimal time period is equal to  $t = \frac{1}{\sigma}\sqrt{\frac{3}{\kappa}}$ .

Based on the empirical data given in Taylor(1992), Table 2.3 and 2.4, p. 33-34, a numerical illustration follows. The average optimal time period is 45.2 days for stocks, 29.3 for metals and 48.8 for futures. Among stocks the variability of the optimal time period is less than among futures. For stocks the minimum optimal time period is 22 and the maximum is 68 while for futures it is 15 respectively 129. Details are reported in Table 3.1 below.

### **Remarks 3.1.**

(i) The considered *financial optimization problem* and its solution provides *rational evidence* (in form of an objective mathematical proof) for the existence of a financial asset pricing model with infinite variance. One should note that up to now only *subjective evidence* (including more or less objective empirical evidence) seems to have been put forward in favour of an infinite variance model (e.g. Granger and Orr(1972), p. 277, and subsequent work up to Peters(1994)).

(ii) In the field of actuarial science, another kind of *rational evidence* in favour of an infinite variance Pareto model, of use in the reinsurance of large claims, has been put forward by Aebi et al.(1992).

(iii) A further probabilistic model with infinite variance, which has been introduced and applied in Hürlimann(1993/94/95b), is a stop-loss ordered extremal distribution with an infinite range.

**Table 3.1** : time period minimizing the expected future value of an asset return guarantee

Series (months)	years	100 $\sigma$	$\kappa$	$t = \frac{1}{\sigma} \sqrt{\frac{3}{\kappa}}$ (days)
<b>Stocks</b>				
Allied	1966-76	1.85	6.05	38
Alcoa		1.76	7.40	36
Am. Can		1.29	5.70	56
AT and T		1.01	6.44	68
Am. Brands		1.34	6.17	52
Anaconda		2.35	11.25	22
Bethlehem		1.71	5.47	43
Chrysler		2.44	6.96	27
Dupont		1.34	4.76	59
Kodak		1.57	6.35	44
G. Electric		1.54	5.25	49
G. Foods		1.52	5.67	48
G. Motors		1.39	7.36	46
G. Telephone		1.70	4.97	46
Harvester		1.65	5.81	44
<b>Stock index</b>				
FT 30	1975-82	1.29	3.51	72
<b>Metals</b>				
Gold	1975-82	1.85	11.42	28
Silver		1.89	11.50	27
Copper		1.85	10.09	29
Lead		2.02	9.76	27
Tin		1.28	10.13	43
Zinc		2.04	15.15	22
<b>Currency</b>				
Pound/Dollar	1974-82	0.59	8.36	102
<b>Futures</b>				
Corn(12)	1963-76	1.30	8.29	46
Corn(6)		1.22	6.65	55
Corn(3)		1.22	6.57	55
Cocoa(12)	1971-81	1.89	10.09	16
Cocoa(6)	1971-80	1.95	9.09	29
Cocoa(3)		2.03	9.19	28
Coffee(12)	1971-81	2.26	26.99	15
Coffee(6)	1970-80	2.13	17.90	19
Coffee(3)		2.18	22.02	17
Sugar(12)	1961-81	2.34	8.34	26
Sugar(6)	1961-79	2.24	8.61	26
Sugar(3)		2.29	9.10	25
Wool(12)	1966-78	1.32	23.25	27
Sterling(6)	1974-81	0.60	5.99	118
Deutschmark(6)		0.60	5.03	129
Swissfranc(6)		0.73	4.38	113
T-bond	1981-83	1.01	3.96	86

**Remarks 3.1.** (continued)

(iv) With the present paper, we hope to stimulate further research along the line of the famous empirical and theoretical work started by Mandelbrot(1963), Fama(1965) and others on stable distribution models. Recent research in this area include Rachev and coworkers (e.g. Mittnik and Rachev(1993)) and the two books by Peters(1991/94). A chaotic analysis of "Treasury" interest rates using the fractional Brownian motion of Mandelbrot and Van Ness(1968) is presented in Craighead(1994).

(v) Besides the stable Paretian with infinite variance, the most popular alternatives are a Student t-distribution (e.g. Praetz(1972), Blattberg and Gonedes(1974), Kon(1984)) and mixtures of normals (e.g. Praetz(1972), Clark(1973)). To validate the practical use of a  $\delta$ -exponential tail model requires further research, which is beyond the scope our paper.

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