IBNR RESERVES UNDER STOCHASTIC INTEREST RATES[*]

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Abstract: This paper intends to evaluate the present value of IBNR reserves, when future interest rates are unknown. We first derive a result for the Laplace transform of the present value, when it is assumed that the interest rates are stochastic and can be modelled by means of a stochastic process which is similar to the model of Cox, Ingersoll and Ross (1985). Starting from this Laplace transform, it is shown how the probability distribution for the quantity under investigation can be found. The results are illustrated numerically.

Keywords : IBNR reserves, discount factor, probability density, Laplace transform.

1. INTRODUCTION

The correct estimation of loss reserves for IBNR (= Incurred But Not Reported) claims represents an important topic for insurance companies. In the first place the balance sheet should give a correct picture of the company's liabilities, besides a systematic underestimation of the IBNR reserves will lead to premiums that are too low.
One of the sub-problems in this respect consists of the discounting of the future estimates in the IBNR run-off triangle, when interest rates and inflation are not known for certain.

Suppose that for a certain year of origin estimates are available for the claim amounts ω_j in the different years of development $j = 1, \ldots, n$.

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If these values are discounted in a deterministic way. the present value equals

$$Y_{\overline{n}|} = \sum_{j=1}^{n} \omega_j v^j = \sum_{j=1}^{n} \omega_j e^{-\delta j} \quad ; \tag{1.1}$$

if we want to make the discount factor stochastic. we can write

$$Y_{\overline{n}|R} = \sum_{j=1}^{n} \omega_j e^{-x_j} , \tag{1.2}$$

where x_j is a realisation of a certain stochastic process.

In stead of using a Brownian motion with drift (see e.g. De Schepper et al. (1992), Vanneste et al. (1994), De Schepper et al. (1997)) , defined by

$$dX(\tau) = \delta\, d\tau + \sigma^2\, dW(\tau) \tag{1.3}$$

where $W(\tau)$ is a standard Wiener process, we choose a differential equation with a variance depending on the process (cfr. Cox, Ingersoll and Ross (1985)) :

$$dI(\tau) = \left(\alpha + \beta\, e^{I(\tau)}\right) d\tau + 2\sigma\, e^{I(\tau)/2}\, dW(\tau). \tag{1.4}$$

A transformation of this equation to the process $J(\tau)$ defined by $I(\tau) = -2 \ln J(\tau)$ results in a new form for the present value (1.2). We get

$$Y_{\overline{n}|R} = \sum_{j=1}^{n} \omega_j x_j^2 \tag{1.5}$$

where now x_j is a realisation of the process $J(\tau)$ with differential equation (see Goovaerts & Dhaene (1997))

$$dJ(\tau) = \left(-\frac{\alpha}{2} J(\tau) + \frac{\sigma^2 - \beta}{2} \frac{1}{J(\tau)} \right) d\tau - \sigma\, dW(\tau) \quad . \tag{1.6}$$

There are good arguments for this choice of stochastic process. Indeed, when we consider the differential equation (1.6) in the deterministic case – which means that the stochastic term $\sigma\, dW(\tau)$ must be zero – the solution with boundary condition $J(0) = 1$ equals

$$J^2(\tau) = \left(1 - \frac{\sigma^2 - \beta}{\alpha} \right) e^{-\alpha\tau} + \frac{\sigma^2 - \beta}{\alpha} \quad , \tag{1.7}$$

such that for the present value we get with a choice of $z = (\sigma^2 - \beta)/\alpha$

$$Y_{\overline{n}|z} = z \sum_{j=1}^{n} \omega_j + (1 - z) \sum_{j=1}^{n} \omega_j e^{-\alpha j} \quad , \tag{1.8}$$

which is a credibility average between the nominal and discounted values of the estimated payments.

In this contribution we aim at calculating the probability density for the present value $Y_{\overline{n}|R}$ of (1.5), when the underlying process is defined by (1.6). This will provide us with a mean to make decisions about the amount of loss reserves when future interest rates are unknown.

Other interpretations of the application of these techniques in the framework of IBNR are possible of course. Indeed one could consider a run-off triangle with non cumulative loss figures. A linear model for the logarithms of these payments according to the three dimensions of the problem (development, year of origin and calendar year direction) could be considered, with a restricted number of parameters. One can imagine that in the development and the year of origin direction, some information is available from past years to "estimate" these type of evolutions for the following calendar year. However, less information is available for the trends and the evolution of e.g. inflation in claim figures or discount factors for the calendar year to come. This explaines e.g. the enormous differentiation in IBNR forecasts based on separation methods where several models are available for the extrapolation of future calendar years. To cope with the uncertainty in this direction, one could consider aggregated figures ω_j for the future calendar years j and evaluate the present value of the quantities as given in (1.2). This provides us with the distribution of the total IBNR provision rather than the distribution of the provision for each year of origin.

In the following section we discuss the transition probability for the stochastic process representing the uncertainty in the discount factor. Section 3 consists of the derivation of two expressions for the Laplace transform of the present value $Y_{\overline{n}|R}$. This Laplace transform can be used to find the successive moments of the disount factor, which is done in section 4. In section 5 we show how the Laplace transform can be used to find the probability density function for $Y_{\overline{n}|R}$. For an explicit calculation, we have to switch over to numerical methods. These results are presented in section 6.

2. Calculation of the Transition Probability

We start this section by means of a lemma, giving two integral expressions that are needed to prove some of the results in the rest of this paper. For a proof of this lemma, we rely upon some formulaes that can be found in Gradshteyn & Ryzhik (1980).

LEMMA 2.1: *The following integral equalities hold :*

$$\int_0^{+\infty} dx\, x\, e^{-\alpha x^2}\, I_\nu(\beta x)\, I_\nu(\gamma x) = \frac{1}{2\alpha}\, e^{\frac{\beta^2+\gamma^2}{4\alpha}}\, I_\nu\left(\frac{\beta\gamma}{2\alpha}\right) \tag{2.1}$$

and

$$\int_0^{+\infty} dx\, x^{\alpha-1}\, e^{-\beta x^2}\, I_\alpha(2\gamma x) = \frac{1}{2}\,\frac{\gamma^\alpha}{\beta^{\alpha+1}}\, e^{\frac{\gamma^2}{\beta}}. \tag{2.2}$$

We now return to the (stochastic) present value of the IBNR-payments for a certain year of origin

$$Y_{\bar{n}|R} = \sum_{j=1}^{n} \omega_j\, x_j^2 \tag{2.3}$$

where x_j is a realisation of the process $J(\tau)$ with differential equation

$$dJ(\tau) = \left(-\frac{\alpha}{2} J(\tau) + \frac{\sigma^2-\beta}{2}\frac{1}{J(\tau)} \right) d\tau - \sigma\, dW(\tau). \tag{2.4}$$

In order to find an expression for the transition probability of this stochastic process, we will make use of the following result :

THEOREM 2.1: *A stochastic process satisfying the stochastic differential equation*

$$dX(\tau) = A(X(\tau),\tau)\, d\tau + dW(\tau) \tag{2.5}$$

has a transition probability that can be calculated by means of a path integral as

$$p(s,y\,;t,x) = \frac{d}{dx} Prob\,(\,X(t) \le x \mid X(s) = y\,)$$

$$= e^{\int_y^x A(z,s)\,dz} \int_{(s,y)}^{(t,x)} Dx(\tau)\; e^{-\frac{1}{2}\int_s^t \dot{x}^2\, d\tau}\; e^{-\frac{1}{2}\int_s^t \left(\frac{\partial A}{\partial x} + A^2\right) d\tau} . \tag{2.6}$$

PROOF : A proof can be found in De Schepper & Goovaerts (1997). □

It is clear that the structure of the differential equation (2.4) allows us to implement the previous result to the stochastich process under investigation, or :

THEOREM 2.2: *The transition probability for the process appearing in the present value (2.3) can be written – with a suitable choice for g and γ – as*

$$p(s,y;t,x) = \left(\frac{x}{y}\right)^{a-1/2} \int_{(s,y)}^{(t,x)} D_{\cdot}x(\tau)\, e^{-\frac{1}{2\sigma^2}\int_s^t \dot{x}^2 d\tau - g\int_s^t \frac{d\tau}{x^2} - \gamma\int_s^t x^2 d\tau} \tag{2.7}$$

where $a = \sqrt{2g/\sigma^2 + 1/4}$ *; the remaining path integral can be calculated and results in*

$$p(s,y;t,x) = \frac{\sqrt{2\gamma/\sigma^2}}{\sinh[\sqrt{2\gamma\sigma^2}(t-s)]}\left(\frac{x}{y}\right)^a x\, I_a\left(\frac{\sqrt{2\gamma/\sigma^2}\;yx}{\sinh[\sqrt{2\gamma\sigma^2}(t-s)]}\right)$$

$$\times \exp\left\{-\frac{1}{2}\sqrt{2\gamma/\sigma^2}\coth[\sqrt{2\gamma\sigma^2}(t-s)](y^2+x^2)\right\} \quad . \tag{2.8}$$

PROOF : Expression (2.7) immediately follows from theorem 2.1, applied to the differential equation (2.4). The calculation of the path integral can be performed when use is made of a more general result of Vanneste et al. (1994). In that contribution the authors find a result for a path integral as the one given in (2.7), but with a general function $\gamma(\tau)$ in stead of the constant γ. □

An investigation of the specific form of the integrand in the path integral of (2.7) provides us with an extra argument for the appropriateness of the underlying stochastic process. In the first place, we can remark that for small values of g and γ, the transition probability equals the transition for a Brownian motion. Besides, both terms can be interpreted seperately. Indeed, the term e^{-g/x^2} causes a small density when x lies in the neighbourhood of zero, and can be needed due to the influence of control authorities and regulation measurements ; the term $e^{-\gamma x^2}$ gives a small density when x goes to infinity, and can be explained as the payment of a dividend for large values of x.

The transition probability of (2.8) satifies the important Kolmogorov property, which is useful for the calculations in the next sections :

LEMMA 2.2: *For the transition probability of (2.8), the Kolmogorov property is satisfied, or*

$$p(0,u;t,x) = \int_0^{\infty} p(0,u;s,y)\, p(s,y;t,x)\, dy \quad . \tag{2.9}$$

PROOF : A substitution of expression (2.8) for the transition probabilities in the right hand side of (2.9) gives

$$\int_0^{\infty} p(0,u;s,y)\, p(s,y;t,x)\, dy = \frac{\sqrt{2\gamma/\sigma^2}}{\sinh[\sqrt{2\gamma\sigma^2}s]} \frac{\sqrt{2\gamma/\sigma^2}}{\sinh[\sqrt{2\gamma\sigma^2}(t-s)]} \left(\frac{x}{u}\right)^a x$$

$$\times \exp\left\{-\frac{1}{2}\sqrt{2\gamma/\sigma^2}\coth[\sqrt{2\gamma\sigma^2}s]\ u^2 - \frac{1}{2}\sqrt{2\gamma/\sigma^2}\coth[\sqrt{2\gamma\sigma^2}(t-s)]\ x^2\right\}$$

$$\times \int_0^{\infty} dy\, y\ \exp\left\{-\frac{1}{2}\sqrt{2\gamma/\sigma^2}\left(\coth[\sqrt{2\gamma\sigma^2}s] + \coth[\sqrt{2\gamma\sigma^2}(t-s)]\right) y^2\right\}$$

$$\mathrm{I}_a\left(\frac{\sqrt{2\gamma/\sigma^2}\ uy}{\sinh[\sqrt{2\gamma\sigma^2}s]}\right) \mathrm{I}_a\left(\frac{\sqrt{2\gamma/\sigma^2}\ yx}{\sinh[\sqrt{2\gamma\sigma^2}(t-s)]}\right) \quad . \tag{2.10}$$

The integral can be performed when use is made of equation (2.1), together with the equality

$$\coth[\sqrt{2\gamma\sigma^2}s] + \coth[\sqrt{2\gamma\sigma^2}(t-s)] = \frac{\sinh[\sqrt{2\gamma\sigma^2}t]}{\sinh[\sqrt{2\gamma\sigma^2}s]\ \sinh[\sqrt{2\gamma\sigma^2}(t-s)]} \quad . \tag{2.11}$$

This results in

$$\int_0^{\infty} p(0,u;s,y)\, p(s,y;t,x)\, dy$$

$$= \frac{\sqrt{2\gamma/\sigma^2}}{\sinh[\sqrt{2\gamma\sigma^2}t]} \left(\frac{x}{u}\right)^a x\ \mathrm{I}_a\left(\frac{\sqrt{2\gamma/\sigma^2}\ ux}{\sinh[\sqrt{2\gamma\sigma^2}t]}\right) \tag{2.12}$$

$$\times\ \exp\left\{-\frac{1}{2}\sqrt{2\gamma/\sigma^2}\left(\coth[\sqrt{2\gamma\sigma^2}s] - \frac{\sinh[\sqrt{2\gamma\sigma^2}(t-s)]}{\sinh[\sqrt{2\gamma\sigma^2}s]\sinh[\sqrt{2\gamma\sigma^2}t]}\right)u^2\right\}$$

$$\times\ \exp\left\{-\frac{1}{2}\sqrt{2\gamma/\sigma^2}\left(\coth[\sqrt{2\gamma\sigma^2}(t-s)] - \frac{\sinh[\sqrt{2\gamma\sigma^2}s]}{\sinh[\sqrt{2\gamma\sigma^2}(t-s)]\sinh[\sqrt{2\gamma\sigma^2}t]}\right)y^2\right\} \quad ;$$

finally the lemma is proved when remarking that both hyperbolic expressions in the exponential factors can be reduced to $\coth[\sqrt{2\gamma\sigma^2}t]$. □

3. THE LAPLACE TRANSFORM OF THE PRESENT VALUE

A straightforward calculation of the probability density for the functional under investigation $Y_{\bar{n}|R}$ is impossible. Yet a transition to the Laplace transform of this functional enables us to get a result for the probability density. Indeed, if the Laplace transform

$$M(\kappa) = \mathrm{E}\left[\, e^{-\kappa Y_{\bar{n}|R}} \mid x(0) = u \,\right] \tag{3.1}$$

is known, an inversion with respect to κ leads to an expression for the density of $Y_{\bar{n}|R}$.

This Laplace transform now can be found by means of a recursion, which is stated in the following theorem. In order not to complicate the resulting expressions, we introduce the notations

$$s = \frac{\sinh\sqrt{2\gamma\sigma^2}}{\sqrt{2\gamma/\sigma^2}} \quad , \text{ and} \tag{3.2}$$

$$c = \sqrt{2\gamma/\sigma^2}\,\coth\sqrt{2\gamma\sigma^2} \quad . \tag{3.3}$$

THEOREM 3.1: *The Laplace transform of the present value $Y_{\bar{n}|R}$ as defined in (2.3) can be calculated as*

$$M(\kappa) = \frac{C_n}{s^n}\, e^{-\frac{c}{2}u^2}\, \frac{f_n}{2}\left(\frac{\alpha_n}{2}\right)^a \frac{1}{(p_n - c/2)^{a-1}} \cdot e^{-q_n u^2 + \frac{\alpha_n^2 u^2}{4(p_n - c/2)}} \quad , \tag{3.4}$$

where the coefficients f_n, α_n, p_n, q_n *are determined by the following recursion relations :*

$$\begin{cases} f_{j+1} = f_j \cdot \dfrac{1}{2p_j} \\ q_{j+1} = q_j - \dfrac{\alpha_j^2}{4p_j} \\ p_{j+1} = \beta_{j+1} - \dfrac{1}{4s^2 p_j} \\ \alpha_{j+1} = \dfrac{\alpha_j}{2sp_j} \end{cases} \qquad j = 1, \ldots, n-1 \tag{3.5}$$

with starting values

$$\begin{cases} f_1 = 1 \\ q_1 = 0 \\ p_1 = \beta_1 \\ \alpha_1 = 1/s \end{cases} \tag{3.6}$$

and with $\beta_j = c + \kappa\,\omega_j$ *. The constant* C_n *has to be determined by means of the constraint* $M(0) = 1$.

PROOF : The evaluation of the Laplace transform of (3.1) involves the integration of the transition probability of (2.8) over the successive realisations x_j, or

$$\begin{aligned} M(\kappa) &= C_n \int_0^{+\infty} dx_1 \int_0^{+\infty} dx_2 \ldots \int_0^{+\infty} dx_n\, e^{-\kappa \sum_{j=1}^n \omega_j x_j^2} \\ &\quad \times p(0,u;1,x_1)\, p(1,x_1;2,x_2) \ldots p(n-1,x_{n-1};n,x_n) \end{aligned} \tag{3.7}$$

where from (2.8) it follows that

$$p(j,x_j;j+1,x_{j+1}) = \frac{1}{s}\left(\frac{x_{j+1}}{x_j}\right)^a x_{j+1}\, e^{-\frac{c}{2}(x_j^2 + x_{j+1}^2)}\, \mathrm{I}_a\!\left(\frac{x_j x_{j+1}}{s}\right) \; . \tag{3.8}$$

If we use the notation $\beta_j = c + \kappa\,\omega_j$, equation (3.7) can be rewritten as

$$M(\kappa) = \frac{C_n}{s^n} \int_0^{+\infty} x_1\, dx_1 \int_0^{+\infty} x_2\, dx_2 \dots \int_0^{+\infty} x_n\, dx_n \left(\frac{x_n}{u} \right)^a e^{-\frac{c}{2}u^2 + \frac{c}{2}x_n^2}$$

$$\times\; e^{-\beta_1 x_1^2 - \beta_2 x_2^2 - \dots - \beta_n x_n^2}\, I_a\!\left(\frac{u x_1}{s} \right) I_a\!\left(\frac{x_1 x_2}{s} \right) \dots I_a\!\left(\frac{x_{n-1} x_n}{s} \right) . \qquad (3.9)$$

The integrations can now be performed by making use of formula (2.1).

We start with the integration over x_1

$$\int_0^{+\infty} dx_1\, x_1\, e^{-\beta_1 x_1^2}\, I_a\!\left(\frac{u x_1}{s} \right) I_a\!\left(\frac{x_1 x_2}{s} \right)$$

$$= \int_0^{+\infty} dx_1\, f_1\, x_1\, e^{-q_1 u^2 - p_1 x_1^2}\, I_a(\alpha_1 u x_1)\, I_a\!\left(\frac{x_1 x_2}{s} \right)$$

$$= f_1 \frac{1}{2p_1} e^{-(q_1 - \frac{\alpha_1^2}{4p_1}) u^2 + \frac{x_2^2}{4s^2 p_1}}\, I_a\!\left(\frac{\alpha_1}{2 s p_1} u x_2 \right) \qquad (3.10)$$

where the initial values of (3.6) are introduced, and go on with the integrations over x_j for $j = 2, \dots, n-1$

$$\int_0^{+\infty} dx_j\, x_j\, e^{-\beta_j x_j^2}\, I_a\!\left(\frac{x_j x_{j+1}}{s} \right) \times f_{j-1} \frac{1}{2p_{j-1}} e^{-(q_{j-1} - \frac{\alpha_{j-1}^2}{4p_{j-1}}) u^2 + \frac{x_j^2}{4s^2 p_{j-1}}}\, I_a\!\left(\frac{\alpha_{j-1}}{2 s p_{j-1}} u x_j \right)$$

$$= \int_0^{+\infty} dx_j\, f_j\, x_j\, e^{-q_j u^2 - p_j x_j^2}\, I_a(\alpha_j u x_j)\, I_a\!\left(\frac{x_j x_{j+1}}{s} \right)$$

$$= f_j \frac{1}{2p_j} e^{-(q_j - \frac{\alpha_j^2}{4p_j}) u^2 + \frac{x_{j+1}^2}{4s^2 p_j}}\, I_a\!\left(\frac{\alpha_j}{2 s p_j} u x_{j-1} \right)$$

$$= f_{j+1}\, e^{-q_{j+1} u^2 - p_{j+1} x_{j+1}^2}\, I_a(\alpha_{j+1} u x_{j+1}) \qquad (3.11)$$

constructing the recursion as given in (3.5).

The last integral but one results in

$$\mathrm{M}(\kappa) = \frac{C_n}{s^n}\, e^{-\frac{c}{2}u^2} \int_0^{+\infty} dx_n\, f_n\, x_n \left(\frac{x_n}{u}\right)^a e^{-q_n u^2 - p_n x_n^2 + \frac{c}{2}x_n^2}\, \mathrm{I}_a\left(\alpha_n u x_n\right) \;; \tag{3.12}$$

the integration over x_n can be evaluated by means of formula (2.2), after having carried through a substitution $y_n = x_n^2$ .We then find the expression mentioned in equation (3.4). □

In order to solve the recursion relations of (3.4) and (3.5), we rewrite the coefficients by means of the solutions of a difference equation. The following lemma provides us with the technical requisites.

LEMMA 3.1: *Let* $\Delta v_i = v_{i+1} - v_i$ *and* $\Delta^2 v_i = \Delta(\Delta v_i)$.
If $\Delta^2 v_i = \gamma_i\, v_{i+1}$ *then*

$$r_i = v_i \sum_{j=2}^{i} \frac{1}{v_j v_{j-1}} \tag{3.13}$$

is a solution of the same difference equation. In case $v_0 = 0$ *and* $v_1 = 1$ *, the starting values for the solution* r_i *are* $r_0 = -1$ *and* $r_1 = 0$.
It immediately follows that

$$\frac{r_i}{v_i} = \sum_{j=2}^{i} \frac{1}{v_j v_{j-1}} \,. \tag{3.14}$$

PROOF : Define r_i by means of (3.13). Then it follows that

$$\Delta r_i = \Delta v_i \sum_{j=2}^{i} \frac{1}{v_j v_{j-1}} + v_{i-1} \frac{1}{v_{i+1} v_i} \quad \text{, and} \tag{3.15}$$

$$\Delta^2 r_i = \Delta^2 v_i \sum_{j=2}^{i+1} \frac{1}{v_j v_{j-1}} + \Delta v_i \frac{1}{v_{i+1} v_i} - \Delta v_i \frac{1}{v_{i+1} v_i} \;, \tag{3.16}$$

having taken in account that $\Delta x_i y_i = y_{i-1}\, \Delta x_i + x_i\, \Delta y_i$.

Hence,

$$\Delta^2 r_i = \gamma_i \, v_{i+1} \sum_{j=2}^{i-1} \frac{1}{v_j v_{j-1}} = \gamma_i \, r_{i-1} \quad . \tag{3.17}$$

With $v_0 = 0$, $v_1 = 1$ and $r_0 = -1$, $r_1 = 0$ we get the same values for r_i by making use of either of the two equations (3.13) or (3.17). □

We will now show how the previous lemma can be used for the calculation of the Laplace transform of $Y_{\overline{n}|R}$. We therefore examine for $j = 1, \ldots, n$

$$2p_j s = \frac{v_{j+1}}{v_j} \tag{3.18}$$

which satisfies the difference equation

$$\Delta^2 v_j = (2s\beta_{j+1} - 2)\, v_{j-1} \tag{3.19}$$

with $v_0 = 0$ and $v_1 = 1$.

All of the coefficients f_n, α_n, p_n, q_n can be expressed in terms of the special solutions v_j and r_j of the difference equation (3.19). Consequently, the recursive calculation of the Laplace transform as stated in theorem 3.1 can be transformed into a calculation by means of the solutions of equation (3.19). The result can be written down as in the following theorem.

THEOREM 3.2: *The Laplace transform of the present value* $Y_{\overline{n}|R}$ *as defined in (2.3) can be calculated as*

$$M(\kappa) = C_n^{*} \frac{1}{(v_{n+1} - c s v_n)^{a+1}} \cdot e^{\frac{1}{2 s v_n}\left(r_n + \frac{1}{v_{n-1} - c s v_n}\right) u^2} \quad , \tag{3.20}$$

where the quantities v_n, v_{n+1}, r_n *are determined as the solutions of*

$$v_{j+2} = -v_j + 2s\,\beta_{j+1}\, v_{j+1} \qquad \textit{with} \qquad v_0 = 0 \, , \; v_1 = 1 \tag{3.21}$$

and

$$r_{j+2} = -r_j + 2s\,\beta_{j+1}\, r_{j+1} \qquad \textit{with} \qquad r_0 = -1 \, , \; r_1 = 0 \, , \tag{3.22}$$

and where $\beta_j = c + \kappa\,\omega_j$.

The constant C_n^* is determined by means of the constraint $M(0) = 1$.

PROOF : We start with equation (3.4), from which it is clear that alternative expressions are needed for f_n, α_n, p_n, q_n .
Since from (3.5) and (3.6) we know that

$$2sp_{j+1} + \frac{1}{2sp_j} = 2s\,\beta_{j-1} \qquad \text{with} \qquad p_1 = \beta_1 , \tag{3.23}$$

it can easily be shown that with $2sp_j = v_{j-1}/v_j$, v_j is a solution of (3.21), and thus

$$p_n = \frac{v_{n+1}}{2s\,v_n} . \tag{3.24}$$

It also follows from (3.5) and (3.6) that

$$\frac{\alpha_{j+1}}{\alpha_j} = \frac{1}{2sp_j} = \frac{v_j}{v_{j+1}} \qquad \text{with} \qquad \alpha_1 = \frac{1}{s} , \tag{3.25}$$

and therefore

$$\alpha_n = \frac{1}{s\,v_n} . \tag{3.26}$$

For the quantity f_n we start from

$$\frac{f_{j+1}}{f_j} = \frac{1}{2p_j} = s\,\frac{v_j}{v_{j+1}} \qquad \text{with} \qquad f_1 = 1 , \tag{3.27}$$

or

$$f_n = \frac{s^n}{s\,v_n} . \tag{3.28}$$

Finally q_n can be found since

$$q_{j+1} = q_j - \frac{\alpha_j^2}{4p_j} = q_j - \frac{1}{2s}\frac{1}{v_j v_{j+1}} \qquad \text{with} \qquad q_1 = 0 , \tag{3.29}$$

such that

$$q_n = -\frac{1}{2s}\frac{r_n}{v_n} . \tag{3.30}$$

A substitution of (3.24), (3.26), (3.28) and (3.30) into (3.4) immediately results in expression (3.20).

The constant C_n^* can be calculated as

$$C_n^* = (v_{n+1}^* - csv_n^*)^{a+1} \cdot e^{-\frac{1}{2sv_n^*}\left(r_n^* + \frac{1}{v_{n+1}^* - csv_n^*}\right) u^2} , \tag{3.31}$$

where v_n^* and r_n^* are the solutions of (3.21) and (3.22), but with c in stead of β_{j+1} . □

4. Moments of the Discount Factor

In this section, we want to evaluate the moments of the discount factor x_τ^2 where x_τ is a realisation of the stochastic differential equation (2.4). Just as for the derivation of the probability density, we start by calculating the Laplace transform of this discount factor, since then use can be made of the well known formula

$$\mathrm{E}\left[Z^m \right] = (-1)^m \frac{d^m}{d\lambda^m} \mathrm{E}\left[e^{-\lambda Z} \right] . \tag{4.1}$$

The result for the calculation of this Laplace transform can be formulated as follows :

THEOREM 4.1: *The Laplace transform of x_τ^2 where the transition probability of x_τ is given by (2.8), can be calculated as*

$$\mathrm{E}\left[e^{-\lambda x_\tau^2} \mid x(0) = u \right] = B^{a+1} e^{-\frac{G^2 u^2}{B}} \frac{1}{(\lambda + B)^{a+1}} e^{\frac{G^2 u^2}{\lambda + B}} , \tag{4.2}$$

where

$$B = \frac{1}{2}\sqrt{2\gamma/\sigma^2} \frac{\cosh[\sqrt{2\gamma\sigma^2}\, t]}{\sinh[\sqrt{2\gamma\sigma^2}\,\tau]\, \cosh[\sqrt{2\gamma\sigma^2}\,(t-\tau)]} \tag{4.3}$$

and

$$G = \frac{1}{2}\sqrt{2\gamma/\sigma^2}\;\frac{1}{\sinh[\sqrt{2\gamma\sigma^2}\,\tau]} \;. \tag{4.4}$$

PROOF : The evaluation of the Laplace transform involves an integration of the transition probability of (2.8), or

$$M(\lambda) = \mathrm{E}\left[\, e^{-\lambda x_\tau^2} \mid x(0)=u \,\right] = \int_0^{+\infty} dx \int_0^{+\infty} dy\; e^{-\lambda x^2}\, p(0,u;\tau,x)\; p(\tau,x;t,y)$$

$$= C\,\frac{\sqrt{2\gamma/\sigma^2}}{\sinh[\sqrt{2\gamma\sigma^2}\,\tau]}\;\frac{\sqrt{2\gamma\sigma^2}}{\sinh[\sqrt{2\gamma\sigma^2}\,(t-\tau)]}\left(\frac{1}{u}\right)^{a} e^{-\frac{1}{2}\sqrt{2\gamma/\sigma^2}\coth[\sqrt{2\gamma\sigma^2}\,\tau]\,u^2}$$

$$\times \int_0^{+\infty} dx\; x\; e^{-\lambda x^2}\; e^{-\frac{1}{2}\sqrt{2\gamma/\sigma^2}\,\frac{\sinh[\sqrt{2\gamma\sigma^2}\,t]}{\sinh[\sqrt{2\gamma\sigma^2}\,\tau]\,\sinh[\sqrt{2\gamma\sigma^2}\,(t-\tau)]}\,x^2}\; \mathrm{I}_a\!\left(\frac{\sqrt{2\gamma/\sigma^2}\;u\,x}{\sinh[\sqrt{2\gamma\sigma^2}\,\tau]}\right)$$

$$\times \int_0^{+\infty} dy\; y^{a+1}\; e^{-\frac{1}{2}\sqrt{2\gamma/\sigma^2}\coth[\sqrt{2\gamma\sigma^2}\,(t-\tau)]\,y^2}\; \mathrm{I}_a\!\left(\frac{\sqrt{2\gamma/\sigma^2}\;x\,y}{\sinh[\sqrt{2\gamma\sigma^2}\,(t-\tau)]}\right) \;. \tag{4.5}$$

The integration over y can be performed when use is made of formula (2.2). Rearranging the hyperbolic terms results in

$$M(\lambda) = C\,\frac{\sqrt{2\gamma/\sigma^2}}{\sinh[\sqrt{2\gamma\sigma^2}\,\tau]}\left(\frac{1}{\cosh[\sqrt{2\gamma\sigma^2}\,(t-\tau)]}\right)^{a+1}\left(\frac{1}{u}\right)^{a} e^{-\frac{1}{2}\sqrt{2\gamma/\sigma^2}\coth[\sqrt{2\gamma\sigma^2}\,\tau]\,u^2}$$

$$\times \int_0^{+\infty} dx\; x^{a+1}\; e^{-\lambda x^2}\; e^{-\frac{1}{2}\sqrt{2\gamma/\sigma^2}\,\frac{\cosh[\sqrt{2\gamma\sigma^2}\,t]}{\sinh[\sqrt{2\gamma\sigma^2}\,\tau]\,\cosh[\sqrt{2\gamma\sigma^2}\,(t-\tau)]}\,x^2}\; \mathrm{I}_a\!\left(\frac{\sqrt{2\gamma/\sigma^2}\;u\,x}{\sinh[\sqrt{2\gamma\sigma^2}\,\tau]}\right) . \tag{4.6}$$

Again, formula (2.2) enables us to perform the remaining integration, or

$$M(\lambda) = C\left(\frac{\sqrt{2\gamma/\sigma^2}}{2\sinh[\sqrt{2\gamma\sigma^2}\,\tau]\;\cosh[\sqrt{2\gamma\sigma^2}\,(t-\tau)]}\right)^{a+1} e^{-\frac{1}{2}\sqrt{2\gamma/\sigma^2}\coth[\sqrt{2\gamma\sigma^2}\,\tau]\,u^2}$$

$$\times \left(\frac{1}{\lambda + \frac{1}{2}\sqrt{2\gamma/\sigma^2}\,\frac{\cosh[\sqrt{2\gamma\sigma^2}\,t]}{\sinh[\sqrt{2\gamma\sigma^2}\,\tau]\,\cosh[\sqrt{2\gamma\sigma^2}\,(t-\tau)]}} \right)^{a+1} \tag{4.7}$$

$$\times \exp\left\{ \frac{\gamma/\sigma^2}{2\sinh^2[\sqrt{2\gamma\sigma^2}\,\tau]}\,u^2 \cdot \left(\lambda + \frac{1}{2}\sqrt{2\gamma/\sigma^2}\,\frac{\cosh[\sqrt{2\gamma\sigma^2}\,t]}{\sinh[\sqrt{2\gamma\sigma^2}\,\tau]\,\cosh[\sqrt{2\gamma\sigma^2}\,(t-\tau)]} \right)^{-1} \right\}.$$

The constant C can be determined by the constraint $M(0) = 1$, or

$$C = \left(\cosh[\sqrt{2\gamma\sigma^2}\,t] \right)^{a+1} e^{\frac{1}{2}\sqrt{2\gamma/\sigma^2}\tanh[\sqrt{2\gamma\sigma^2}\,t]\,u^2} . \tag{4.8}$$

A substitution of (4.8) in (4.7) results in the expression as given in (4.2). □

Formula (4.1) together with the result for the Laplace transform (4.2) then provides us with expressions for the moments of the discount factor, as presented in the following theorem.

THEOREM 4.2: *The moments of x_τ^2 where the transition probability of x_τ is given by (2.8), can be calculated as*

$$E\left[x_\tau^2 \mid x(0) = u \right] \tag{4.9}$$

$$= \frac{a+1}{\sqrt{\gamma/2\sigma^2}}\,\frac{\sinh[\sqrt{2\gamma\sigma^2}\,\tau]\,\cosh[\sqrt{2\gamma\sigma^2}\,(t-\tau)]}{\cosh[\sqrt{2\gamma\sigma^2}\,t]} + \left(\frac{\cosh[\sqrt{2\gamma\sigma^2}\,(t-\tau)]}{\cosh[\sqrt{2\gamma\sigma^2}\,t]} \right)^2 u^2$$

and

$$\mathrm{Var}\left[x_\tau^2 \mid x(0) = u \right] = \frac{1}{\sqrt{\gamma/2\sigma^2}}\sinh[\sqrt{2\gamma\sigma^2}\,\tau]\,\frac{\cosh[\sqrt{2\gamma\sigma^2}\,(t-\tau)]}{\cosh[\sqrt{2\gamma\sigma^2}\,t]} \tag{4.10}$$

$$\times \left\{ \frac{a+1}{\sqrt{\gamma/2\sigma^2}} \; \frac{\sinh[\sqrt{2\gamma\sigma^2}\,\tau]\;\cosh[\sqrt{2\gamma\sigma^2}\,(t-\tau)]}{\cosh[\sqrt{2\gamma\sigma^2}\,t]} + 2\left(\frac{\cosh[\sqrt{2\gamma\sigma^2}\,(t-\tau)]}{\cosh[\sqrt{2\gamma\sigma^2}\,t]} \right)^2 u^2 \right\}.$$

PROOF : For the first moment we have

$$\mathrm{E}\left[x_\tau^2 \mid x(0)=u \right] = -\frac{dM(\lambda)}{d\lambda}\Bigg]_{\lambda=0}$$

$$= \left(\frac{B}{\lambda+B} \right)^{a+1} e^{-\frac{G^2u^2}{B}+\frac{G^2u^2}{\lambda+B}} \left\{ \frac{a+1}{\lambda+B} + \frac{G^2u^2}{(\lambda+B)^2} \right\} \Bigg]_{\lambda=0} \tag{4.11}$$

$$= \frac{a+1}{B} + \frac{G^2u^2}{B^2} \;,$$

with B and G as defined in (4.3) and (4.4).

The variance can be calculated as

$$\mathrm{Var}\left[x_\tau^2 \mid x(0)=u \right] = \frac{d^2M(\lambda)}{d\lambda^2}\Bigg]_{\lambda=0} - \left(\mathrm{E}\left[x_\tau^2 \mid x(0)=u \right] \right)^2$$

$$= \left(\frac{B}{\lambda+B} \right)^{a+1} e^{-\frac{G^2u^2}{B}+\frac{G^2u^2}{\lambda+B}} \left\{ \left[\frac{a+1}{\lambda+B} + \frac{G^2u^2}{(\lambda+B)^2} \right]^2 + \left[\frac{a+1}{(\lambda+B)^2} + \frac{2G^2u^2}{(\lambda+B)^3} \right] \right\} \Bigg]_{\lambda=0} - \left(\frac{a+1}{B} + \frac{G^2u^2}{B^2} \right)^2 \tag{4.12}$$

$$= \frac{a+1}{B^2} + \frac{2G^2u^2}{B^3} \;.$$

A substitution of (4.3) and (4.4) into this expression results in (4.10). □

In case the term t is large ($t \gg 1$), an interpretation of the result for the expected value is obvious. Indeed, the right hand side of (4.9) can be simplified to

$$\mathrm{E}\left[\, x_\tau^2 \mid x(0)=u \,,\, t \gg 1 \,\right] = \frac{a+1}{\sqrt{\gamma/2\sigma^2}} \sinh[\sqrt{2\gamma\sigma^2}\,\tau]\; e^{-\sqrt{2\gamma\sigma^2}\,\tau} + e^{-2\sqrt{2\gamma\sigma^2}\,\tau}\, u^2$$

$$= \frac{a+1}{\sqrt{2\gamma\sigma^2}} \left(1 - e^{-2\sqrt{2\gamma\sigma^2}\,\tau} \right) + e^{-2\sqrt{2\gamma\sigma^2}\,\tau}\, u^2 \;. \tag{4.13}$$

For a choice of $\tau=0$, (4.13) reduces to the starting value u^2 ; a large value of τ results in the asymptotic value $\dfrac{a+1}{\sqrt{2\gamma/\sigma^2}}$.

Remark that in the limit for γ going to zero, these moments simplify to

$$\lim_{\gamma\to 0} \mathrm{E}\left[\, x_\tau^2 \mid x(0)=u \,\right] = 2\,\sigma^2 (a+1)\,\tau + u^2 \tag{4.14}$$

and

$$\lim_{\gamma\to 0} \mathrm{Var}\left[\, x_\tau^2 \mid x(0)=u \,\right] = 4\,\sigma^4 (a+1)\,\tau^2 + 4\,\sigma^2\,\tau\, u^2 \;, \tag{4.15}$$

since

$$\lim_{\gamma\to 0} \frac{\sinh[\sqrt{2\gamma\sigma^2}\,\tau]}{\sqrt{\gamma/2\sigma^2}} = 2\,\sigma^2\,\tau \;. \tag{4.16}$$

Next, if we go further and also consider the limit for g going to zero, a reaches 1/2, such that

$$\lim_{\gamma\to 0}\lim_{g\to 0} \mathrm{E}\left[\, x_\tau^2 \mid x(0)=u \,\right] = 3\,\sigma^2\,\tau + u^2 \tag{4.17}$$

and

$$\lim_{\gamma\to 0}\lim_{g\to 0} \mathrm{Var}\left[\, x_\tau^2 \mid x(0)=u \,\right] = 6\,\sigma^4\,\tau^2 + 4\,\sigma^2\,\tau\, u^2 \;. \tag{4.18}$$

5. A PROBABILITY DENSITY FOR THE PRESENT VALUE

As pointed out in section 3, the knowledge of the Laplace transform enables us to find a result for the probability density of the investigated quantity. So, in order to derive the density for the present value $Y_{\overline{n}|R}$, we look for an expression of the Laplace transform $M(\kappa)$ for which the inversion with respect to κ can be performed.

We therefore return to the result of theorem 3.2. which states that

$$M(\kappa) = C_n^* \frac{1}{(v_{n+1} - csv_n)^{a-1}} \cdot e^{\frac{1}{2sv_n}\left(r_n + \frac{1}{v_{n+1} - csv_n}\right)u^2} , \tag{5.1}$$

with v_n and r_n special solutions of the difference equations

$$v_{j-2} = -v_j + 2s(c + \kappa\omega_j)v_{j-1} \qquad \text{with} \qquad v_0 = 0 \text{ , } v_1 = 1 \tag{5.2}$$

and

$$r_{j+2} = -r_j + 2s(c + \kappa\omega_j)r_{j+1} \qquad \text{with} \qquad r_0 = -1 \text{ , } r_1 = 0 \text{ ,} \tag{5.3}$$

and with C_n^* determined by means of the constraint $M(0) = 1$.

In order to be able to invert (5.1) with respect to κ, we remark that v_n and r_n are polynomials in κ due to the structure of the difference equations (5.2) and (5.3). The following lemma provides us with recursion relations for the coefficients in the polynomial expressions mentioned above.

LEMMA 5.1: *The special solution* v_n *of the difference equation (5.2) is a polynomial of degree* $n-1$ *(if* $n>0$*)*

$$v_n = \vartheta_n^0 + \vartheta_n^1 \kappa + \ldots + \vartheta_n^{n-1} \kappa^{n-1} , \tag{5.4}$$

where the coefficients ϑ_n^i *(* $i = 0, \ldots, n-1$ *) can be determined by means of the recursion (* $j>2$ *):*

$$\begin{cases} \vartheta_j^0 = 2sc\vartheta_{j-1}^0 - \vartheta_{j-2}^0 \\ \vartheta_j^i = 2s\left(c\vartheta_{j-1}^i + \omega_{j-1}\vartheta_{j-1}^{i-1}\right) - \vartheta_{j-2}^i \qquad (\ i = 1, \ldots, j-3\ ) \\ \vartheta_j^{j-2} = 2s\left(c\vartheta_{j-1}^{j-2} + \omega_{j-1}\vartheta_{j-1}^{j-3}\right) \\ \vartheta_j^{j-1} = 2s\omega_{j-1}\vartheta_{j-1}^{j-2} \end{cases} \tag{5.5}$$

with starting values

$$\left\{ \vartheta_1^0 = 1 \right. \qquad \text{and} \qquad \left\{ \begin{aligned} \vartheta_2^0 &= 2sc \\ \vartheta_2^1 &= 2s\omega_1 \quad . \end{aligned} \right. \tag{5.6}$$

The special solution r_n *of the difference equation (5.3) is a polynomial of degree* $n-2$ *(if* $n>1$*)*

$$r_n = \delta_n^0 + \delta_n^1 \kappa + \ldots + \delta_n^{n-2} \kappa^{n-2} \quad , \tag{5.7}$$

where the coefficients δ_n^i *(* $i=0,\ldots,n-2$ *) can be determined by means of the recursion (* $j>3$ *):*

$$\left\{ \begin{aligned} \delta_j^0 &= 2sc\delta_{j-1}^0 - \delta_{j-2}^0 \\ \delta_j^i &= 2s\left(c\delta_{j-1}^i + \omega_{j-1}\delta_{j-1}^{i-1}\right) - \delta_{j-2}^i \qquad (i=1,\ldots,j-4) \\ \delta_j^{j-3} &= 2s\left(c\delta_{j-1}^{j-3} + \omega_{j-1}\delta_{j-1}^{j-4}\right) \\ \delta_j^{j-2} &= 2s\omega_{j-1}\delta_{j-1}^{j-3} \end{aligned} \right. \tag{5.8}$$

with starting values

$$\left\{ \delta_2^0 = 1 \right. \qquad \text{and} \qquad \left\{ \begin{aligned} \delta_3^0 &= 2sc \\ \delta_3^1 &= 2s\omega_2 \quad . \end{aligned} \right. \tag{5.9}$$

As a consequence the difference $v_{n+1} - cs\,v_n$ *is a polynomial of degree n (* $n>0$ *)*

$$v_{n+1} - cs\,v_n = \gamma_n^0 + \gamma_n^1 \kappa + \ldots + \gamma_n^n \kappa^n \tag{5.10}$$

where the coefficients γ_n^i *(* $i=0,\ldots,n$ *) can be determined as*

$$\left\{ \begin{aligned} \gamma_n^i &= \vartheta_{n+1}^i - cs\,\vartheta_n^i \qquad i=0,\ldots,n-1 \\ \gamma_n^n &= \vartheta_{n+1}^n \quad . \end{aligned} \right. \tag{5.11}$$

PROOF : The coefficients can be found straightforwardly if the difference equations (5.2) and (5.3) are solved recursively. ■

The previous lemma enables us to rewrite the Laplace transform of (5.1) by means of polynomials in κ.

THEOREM 5.1: *The Laplace transform of the present value $Y_{\overline{n}|R}$ as defined in (2.3) can be calculated as*

$$M(\kappa) = \mathrm{E}\left[e^{-\kappa Y_{\overline{n}|R}} \mid x(0)=u \right]$$

$$= C_n^* \left(\frac{1}{\gamma_n^0 + \gamma_n^1 \kappa + \ldots + \gamma_n^n \kappa^n} \right)^{a+1} \exp\left\{ \frac{u^2}{2s} \frac{1}{\vartheta_n^0 + \vartheta_n^1 \kappa + \ldots + \vartheta_n^{n-1} \kappa^{n-1}} \right.$$

$$\left. \cdot \left(\delta_n^0 + \delta_n^1 \kappa + \ldots + \delta_n^{n-2} \kappa^{n-2} + \frac{1}{\gamma_n^0 + \gamma_n^1 \kappa + \ldots + \gamma_n^n \kappa^n} \right) \right\} , \qquad (5.12)$$

where C_n^* follows from $M(0) = 1$, and where the coefficients γ_n^i, ϑ_n^i and δ_n^i are determined in lemma 5.1.

PROOF : A substitution of the results of lemma 5.1 into (5.1) immediately gives the desired expression. □

Finally, in order to be able to perform the inversion with respect to κ, we switch over to the decompositions of the different polynomials in (5.12). This leads us to the following result, for which the proof is trivial.

THEOREM 5.2: *The Laplace transform of the present value $Y_{\overline{n}|R}$ as defined in (2.3) can be calculated as*

$$M(\kappa) = \mathrm{E}\left[e^{-\kappa Y_{\overline{n}|R}} \mid x(0)=u \right]$$

$$= C_n^* \left(\frac{1}{\gamma_n^n} \right)^{a+1} \left(\frac{1}{(\kappa+\beta_1)(\kappa+\beta_2)\dots(\kappa+\beta_n)} \right)^{a+1}$$

$$\times \exp \left\{ \frac{u^2}{2s} \frac{1}{\vartheta_n^{n-1}} \frac{1}{(\kappa+\alpha_1)(\kappa+\alpha_2)\dots(\kappa+\alpha_{n-1})} \right. \tag{5.13}$$

$$\left. \cdot \left(\delta_n^0 + \delta_n^1 \kappa + \dots + \delta_n^{n-2} \kappa^{n-2} + \frac{1}{\gamma_n^n} \frac{1}{(\kappa+\beta_1)(\kappa+\beta_2)\dots(\kappa+\beta_n)} \right) \right\} ,$$

where $-\alpha_i$ $(i=1,\dots,n-1)$ *are the roots of the equation*

$$\vartheta_n^0 + \vartheta_n^1 \kappa + \dots + \vartheta_n^{n-1} \kappa^{n-1} = 0 , \tag{5.14}$$

where $-\beta_i$ $(i=1,\dots,n)$ *are the roots of the equation*

$$\gamma_n^0 + \gamma_n^1 \kappa + \dots + \gamma_n^n \kappa^n = 0 , \tag{5.15}$$

and with C_n^* *chosen such that* $M(0) = 1$.

In this form, the inversion needed to get the probability density can be performed, leading to a convolution with a compound Poisson distribution.

THEOREM 5.3: *The probability density for the present value* $Y_{\bar{n}|R}$ *as defined in (2.3) can be calculated as*

$$f_{Y_{\bar{n}|R}}(y) = \frac{d}{dy} Prob\left(Y_{\bar{n}|R} \le y \mid x(0) = u \right) \tag{5.16}$$

$$= \frac{(\beta_{\max})^{n(a+1)}}{\Gamma[n(a+1)]} \int_0^y dz \, (y-z)^{n(a+1)-1} \, e^{-\beta_{\max}(y-z)} \, f_2(z)$$

where $\beta_{\max} = \max(\beta_1,\dots,\beta_n)$ *(see theorem 5.2), and where* $f_2(z)$ *is defined in formula (5.29).*

PROOF : From (5.13) it follows that

$$M(\kappa) = C_n^* \left(\frac{1}{\gamma_n^n} \right)^{a+1} \left(\frac{1}{\kappa + \beta_{\max}} \right)^{n(a+1)} \prod_{j=1}^{n} \left(\frac{\kappa + \beta_{\max}}{\kappa + \beta_j} \right)^{a+1}$$

$$\times \exp \left\{ \frac{u^2}{2s} \frac{1}{\vartheta_n^{n-1}} \frac{\delta_n^0 + \ldots + \delta_n^{n-2} \kappa^{n-2}}{(\kappa + \alpha_1) \ldots (\kappa + \alpha_{n-1})} \right. \tag{5.17}$$

$$\left. + \frac{u^2}{2s} \frac{1}{\vartheta_n^{n-1} \gamma_n^n} \frac{1}{(\kappa + \alpha_1) \ldots (\kappa + \alpha_{n-1})(\kappa + \beta_1) \ldots (\kappa + \beta_n)} \right\} .$$

In order to write the Laplace inversion of $M(\kappa)$ – which is the density of $Y_{\bar{n}:R}$ –in a suitable form, we rewrite $M(\kappa)$ by means of the following auxiliary functions :

$$M(\kappa) = C_n^* \left(\frac{1}{\gamma_n^n} \right)^{a+1} e^{(a+1)\xi} \, . \, g_1(\kappa) \, e^{(a+1)\xi \left\{ \frac{g_2^*(\kappa)}{\xi} - 1 \right\}} , \tag{5.18}$$

where $$g_1(\kappa) = \left(\frac{1}{\kappa + \beta_{\max}} \right)^{n(a+1)} \tag{5.19}$$

and where we split $g_2^*(\kappa)$ as the sum of $g_{2,1}^*(\kappa)$, $g_{2,2}^*(\kappa)$, $g_{2,3}^*(\kappa)$,

with $$g_{2,1}^*(\kappa) = \sum_{j=1}^{n} \ln \left(\frac{\kappa + \beta_{\max}}{\kappa + \beta_j} \right)$$

$$= \int_0^{+\infty} e^{-\kappa y} \left[\sum_{j=1}^{n} \frac{e^{-\beta_j y} - e^{-\beta_{\max} y}}{y} \right] dy \tag{5.20}$$

$$g_{2,2}^*(\kappa) = \frac{u^2}{2s(a+1)} \frac{1}{\vartheta_n^{n-1}} \frac{\delta_n^0 + \ldots + \delta_n^{n-2} \kappa^{n-2}}{(\kappa + \alpha_1) \ldots (\kappa + \alpha_{n-1})}$$

$$= \frac{u^2}{2s(a+1)} \frac{1}{\vartheta_n^{n-1}} \sum_{j=1}^{n-1} \frac{\lambda_j}{\kappa + \alpha_j} \tag{5.21}$$

$$\left.g_{2,3}^{*}(\kappa) = \frac{u^2}{2s(a+1)} \frac{1}{\vartheta_n^{n-1}\gamma_n^n} \frac{1}{(\kappa+\alpha_1)\dots(\kappa+\alpha_{n-1})(\kappa+\beta_1)\dots(\kappa+\beta_n)} \right\} . \tag{5.22}$$

The constant ξ is defined by $\xi = g_2^*(0)$ in order to normalize the function $g_2^*(\kappa)$.
Since the constant C_n^* is determined by means of the constraint $M(0) = 1$. a choice of $\kappa = 0$ in equation (5.18) immediately results in

$$C_n^* = (\gamma_n^n)^{a+1} (\beta_{\max})^{n(a+1)} e^{-(a+1)\xi} . \tag{5.23}$$

Each of these auxiliary functions can now be inverted, relying upon some formulae that can be found in Gradshteyn & Ryzhik (1980) and Oberhettinger & Badii (1973). If we denote the Laplace inversion with respect to κ of $g(\kappa)$ with $f(y)$, we then have

$$f_1(y) = \frac{1}{\Gamma[n(a+1)]} y^{n(a+1)-1} e^{-\beta_{\max} y} \tag{5.24}$$

$$f_{2,1}^*(y) = \sum_{j=1}^{n} \frac{e^{-\beta_j y} - e^{-\beta_{\max} y}}{y} \tag{5.25}$$

$$f_{2,2}^*(y) = \frac{u^2}{2s(a+1)} \frac{1}{\vartheta_n^{n-1}} \sum_{j=1}^{n-1} \lambda_j e^{-\alpha_j y} \tag{5.26}$$

$$\begin{aligned} f_{2,3}^*(y) = \frac{u^2}{2s(a+1)} \frac{1}{\vartheta_n^{n-1}\gamma_n^n} \Bigg\{ & \sum_{j=1}^{n-1} e^{-\alpha_j y} \left(\prod_{\substack{k=1 \\ k\neq j}}^{n-1} \frac{1}{\alpha_k - \alpha_j} \right) \left(\prod_{i=1}^{n} \frac{1}{\beta_i - \alpha_j} \right) \\ & + \sum_{j=1}^{n} e^{-\beta_j y} \left(\prod_{k=1}^{n-1} \frac{1}{\alpha_k - \beta_j} \right) \left(\prod_{\substack{i=1 \\ i\neq j}}^{n} \frac{1}{\beta_i - \beta_j} \right) \Bigg\} , \end{aligned} \tag{5.27}$$

as far as all the roots α_j and β_j are different. If not. the inversion can be made with one or more convolutions.

Returning to equation (5.18), it is clear that furthermore we need the inversion of

$$g_2(\kappa) = e^{(a+1)\xi\left\{\frac{g_2^*(\kappa)}{\xi} - 1\right\}} . \tag{5.28}$$

Since we know the inversion $f_2^*(y)$ of $g_2^*(\kappa)$, which is the sum of $f_{2,1}^*(y)$, $f_{2,2}^*(y)$, and $f_{2,3}^*(y)$, the inversion $f_2(y)$ of (5.27) can be seen as the density of a compound Poisson distributed variable, with parameter $(a+1)\xi$ and with claim size density $f_2^*(y)/\xi$.

The calculation can be performed by means of the recursion formulae of Panjer, or

$$\begin{cases} f_2(0) = e^{-(a+1)\xi} \\ f_2(y) = (a+1)\, e^{-(a+1)\xi} f_2^*(y) + \dfrac{a+1}{y}\displaystyle\int_0^y t\, f_2^*(t) f_2(y-t)\, dt , \qquad y>0 . \end{cases} \tag{5.29}$$

Finally from (5.18) it follows that the probability density of the present value $Y_{\bar{n}|R}$ can be written as

$$f_{Y_{\bar{n}|R}}(y) = \frac{d}{dy} Prob\left(Y_{\bar{n}|R} \le y \mid x(0) = u \right)$$

$$= (\beta_{\max})^{n(a+1)} \cdot f_1(y) * f_2(y) , \tag{5.30}$$

resulting in the expression given in (5.16). □

6. Numerical Illustration

In this last section we will show the definite results for the density function of the present value

$$Y_{\bar{n};R} = \sum_{j=1}^{n} \omega_j\, x_j^2 . \tag{6.1}$$

All of the calculations are based on the following run-off triangle $(n=6)$ consisting of non cumulative loss figures, completed by means of estimates for the future values.

Tabel 6.1 : Run-off triangle

4 186	3 786	1 413	989	762	660	452
4 664	4 689	1 507	1 118	917	746	583
5 759	5 312	1 786	1 254	1 081	894	680
6 267	5 694	1 903	1 398	1 068	1 032	694
6 724	6 108	1 986	1 527	1 235	1 025	796
7 080	6 290	2 181	1 668	1 381	1 111	814
7 560	6 295	2 241	1 584	1 265	992	871

We will explicitly calculate the density for the present value of the aggregated values, and we therefore make use of the quantities for the calendar year to come, marked in the table. More specifically, the previous run-off table provides us with the following values for ω_j :

Table 6.2: Discounted values

ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
12 548	6·253	4 072	2 545	1 477	583

Next to these values, we need four exogeneous parameters from which the three endogeneous parameters can be determined –see equations (3.2), (3.3) and the asymptotic value of (4.13). In a first illustration –for which the intermediate results will be developed completely– we make the following choice :

Table 6.3: Exogeneous parameters

u^2	γ	σ^2	δ
1	1	1	0.01

Table 6.4: Endogeneous parameters

s	c	a
1.368299	1.591892	0.331856

We will refer to this choice as the standard situation.

In order to find a result for the density function of the present value $Y_{\overline{n}|R}$, we start by solving the recursions for the moment generating function as mentioned in lemma 5.1. For a choice of the parameters as in the standard situation mentioned above, this results in the following coefficients :

Table 6.5: Recursion Coefficients

ϑ_6^0	1.2514×10^3
ϑ_6^1	2.3036×10^7
ϑ_6^2	1.4432×10^{11}
ϑ_6^3	3.9000×10^{14}
ϑ_6^4	4.5634×10^{17}
ϑ_6^5	1.8433×10^{20}

δ_6^0	3.0423×10^2
δ_6^1	2.9828×10^6
δ_6^2	9.7671×10^9
δ_6^3	1.2608×10^{13}
δ_6^4	5.3679×10^{15}

γ_6^0	2.4215×10^3
γ_6^1	4.6905×10^7
γ_6^2	3.2170×10^{11}
γ_6^3	1.0160×10^{15}
γ_6^4	1.5706×10^{18}
γ_6^5	1.1296×10^{21}
γ_6^6	2.9409×10^{23}

The knowledge of these data leads to the values for the roots α_i and β_i (see equation (5.14) & (5.15)) and for the fractions λ_j (see formula (5.21)), as summarized in the following table.

Table 6.6: Roots of the polynomials

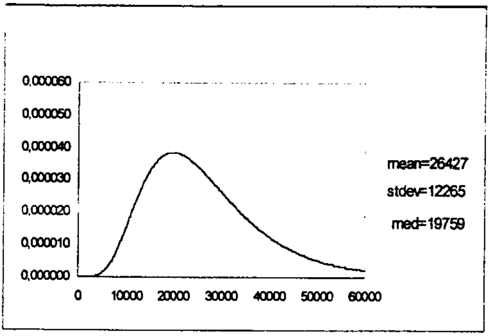
α_1	0.0001128
α_2	0.0002293
α_3	0.0003737
α_4	0.0006118
α_5	0.0011481

β_1	0.0001128
β_2	0.0002291
β_3	0.0003710
β_4	0.0005809
β_5	0.0008945
β_6	0.0016525

λ_1	4.7644×10^{15}
λ_2	5.6790×10^{14}
λ_3	3.5314×10^{13}
λ_4	3.3395×10^{11}
λ_5	1.6053×10^8

The numerical results of table 6.5 and 6.6 provide us with all the requisites needed to calculate the density of the present value under consideration. A graph of this density function is presented in figure 6.1.

Figure 6.1: Density for the present value in the standard situation



In order to visualize the influence of the exogeneous parameters, four more graphs are given below. Each graph corresponds with a situation in which one of the parameters given in table 6.3 is changed.

Figure 6.2 : Density in case σ^2=0.1

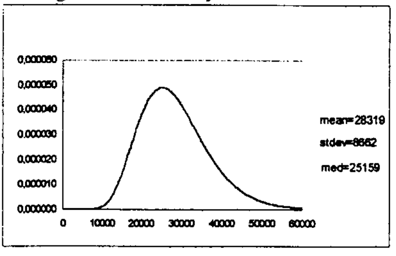


Figure 6.3 : Density in case γ=0.5

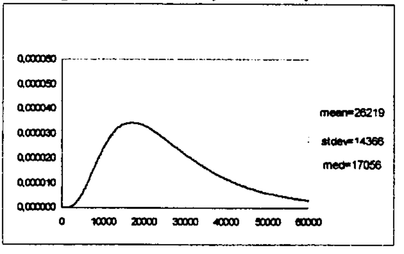


Figure 6.4 : Density in case u^2=3

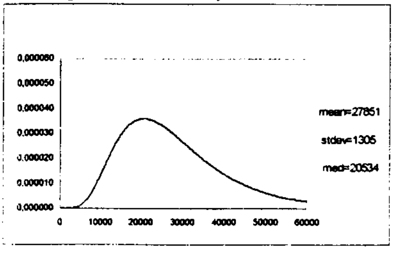
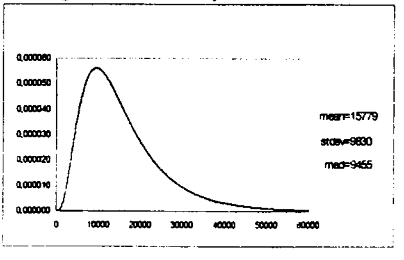


Figure 6.5 : Density in case δ=0.1



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