Catastrophe Risk Bonds

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Abstract We examine the pricing of catastrophe risk bonds. Catastrophe risk cannot be hedged by traditional securities and thus the pricing of catastrophe risk bonds must be examined in an incomplete markets setting and therefore entails special difficulties in the pricing methodology. We present techniques for pricing these bonds and discuss this theory in the context of equilibrium pricing and its relationship to the standard arbitrage-free valuation framework. It is found that a general pricing approach may be developed based on the prescription of a term structure model and a probability structure for the catastrophe risk exposure. The pricing methodology can also be used to assess the relative default spread on catastrophe risk bonds compared with traditional defaultable securities.

"It is indeed most wonderful to witness such desolation produced in three minutes of time." - Charles Darwin commenting on the February 20, 1835 earthquake in Chile.

1. Introduction

Catastrophe risk bonds are an innovative technique for securitising reinsurance risk. The value of these bonds are linked to particular catastrophic events such as earthquakes, hurricanes, or floods. In the event of a catastrophe, a catastrophe risk bond behaves much like a defaultable corporate bond. The "default" of a catastrophe risk bond occurs when a catastrophe of some degree occurs. Unlike corporate bonds, the default of a catastrophe risk bond has no correlation with underlying financial market variables such as interest rate levels or aggregate consumption. Consequently, the payments from a catastrophe risk bond cannot be well approximated[1] by a portfolio of the ordinary types of securities that are traded in financial markets such as bonds or common stocks. Therefore, the pricing of catastrophe bonds must be examined in an incomplete markets framework as there do not exist primitive securities which can be used to study the catastrophe risk bond as a pure derivative security. Fortunately though, the fact that the occurrence of the catastrophe is essentially uncorrelated with movements in underlying economic variables renders the incomplete markets theory somewhat simpler than would be possible in the case where there was significant correlation between the states of the world which depend on the outcome of the catastrophe and the underlying financial market variables. This fact may be used to develop a simple approach for pricing catastrophe risk bonds.

The model we present for pricing catastrophe risk bonds is based on equilibrium pricing considerations. The model is practical in that the valuation may be done in a two stage procedure. Firstly, one must select or estimate the interest rate dynamics[2] in the states of the world which do not involve the occurrence or non-occurrence of the catastrophe. In this stage, one is essentially constructing a term structure model which is a relatively well understood and practiced procedure. Secondly, one must estimate the probability[3] of the catastrophe occurring. Valuation for the full model is then accomplished by combining the estimated probability of the catastrophe occurring and the interest rate dynamics from the term structure model. One may implement the valuation using the standard tool of a risk-neutral valuation measure. The full model is arbitrage-free.

The paper begins in section 2 by describing how catastrophe risk bonds arise from the securitisation of liabilities. Section 3 provides a quick overview and motivation of how pricing may be carried out for catastrophe risk bonds. Section 4 details the inherent pricing problems one faces with catastrophe risk bonds because of the incomplete markets setting. Section 5 describes our formal model and provides a numerical example and section 6 concludes the paper.

2. Catastrophe Reinsurance as a High-Yield Bond

Most investment banks, some insurance brokers and most large reinsurers developed OTC insurance derivatives by 1995. This is a form of liability securitisation, but instead of exchange traded contracts these securities are handled like private placements or customized forwards or options. Tilley (1995) describes securitised catastrophe reinsurance in terms of a high-yield bond. Froot et al. (1996) describe a similar one-period product. These products illustrate how catastrophe risk can be distributed through capital markets in a clever way. The following description is an abstraction and simplification but useful for illustrating the concepts.

Consider a one-period reinsurance contract under which the reinsurer agrees to pay a fixed amount L at the end of the period if a defined catastrophic event occurs. The reinsurer issues a one-period reinsurance contract that pays L at the end of the period, if there is a catastrophe. It pays nothing if no catastrophe occurs. L is known when the policy is issued. If q_{cat} denotes the probability of a catastrophic event and P the price of the reinsurance, then the fair value of the reinsurance is

$$P = \frac{1}{1+r} q_{cat} L$$

where r is the one period effective default-free interest rate. This defines a one-to-one correspondence between bond prices and probabilities of a catastrophe. Since the reinsurance market will determine the price P, it is natural to call q_{cat} the reinsurance market assessment of the probability of a catastrophe.

From where does the capital to support the reinsurer come? The reinsurer will have no customers unless it can convince them that it has capital at least equal to L. Suppose that before it sells the reinsurance, the reinsurer borrows capital by issuing a defaultable bond, i.e., a junk bond. Investors know when they buy a junk bond that it may default but they buy anyway because the bonds do not often default and they have higher returns than more reliable bonds. The reinsurer issues enough bonds to raise an amount of cash C determined so that

$$(P + C)(1 + r) = L.$$

This satisfies the reinsurer's customers. They see that the reinsurer has enough capital to pay for a catastrophe. The bondholders know that the bonds will be worthless if there is a catastrophe. In this case they get nothing. If there is no catastrophe, they get their cash back plus a coupon $R = L - C$. The bond market will determine the price per unit of face value. In terms of discounted expected cash flow, the price per unit can be written in the form

$$\frac{1}{1+r}(1+c)(1-q_B)$$

where $c = R/C$ is the coupon rate and q_B denoted the bondholders' assessment of the probability of default on the bonds. We can assume that the investment bank designing the bond contract sets c so that the bonds sell at face value. Thus, c is determined so that investors pay 1 in order to receive $1 + c$ one year later, if there is no catastrophe. This is expressed as

$$1 = \frac{1}{1+r}(1+c)(1-q_B).$$

Of course, default on the bonds and a catastrophe are equivalent events. The probabilities may differ because bond investors and reinsurance customers may have different information about catastrophes. The reinsurance company sells bonds once c is determined to raise the required capital C. The corresponding bond market probability is found by solving for q_B:

$$q_B = \frac{c-r}{1+c}$$

The implied price for reinsurance is

$$P_B = \frac{1}{1+r} \cdot \frac{c-r}{1+c} \cdot L$$

Provided the reinsurance market premium P (the fair price determined by the reinsurance market) is at least as large as P_B, the reinsurance company will function smoothly. It will collect C from the bond market and P from the reinsurance market at the beginning of the policy period. The sum invested for one period at the risk free rate will be at least L. This is easy to see mathematically using the relation $R = L - C$:

$$
\begin{aligned}
(P+C)(1+r) \ge (P_B+C)(1+r) \\
&= \frac{c-r}{1+c}\cdot L+(1+r)C \\
&= \frac{R-rC}{C+R}\cdot L+(1+r)C \\
&= \frac{R-rC}{L}\cdot L+(1+r)C = R+C = L
\end{aligned}
$$

So long as P_B does not exceed P, or equivalently, so long as $q_{cat} \ge \frac{c-r}{1+c}$, there will be a economically viable market for reinsurance capitalized by borrowing in the bond market. We note that borrowing (issuing bonds) to finance losses is not new. In the late 1980s, when US liability insurance prices were high and interest rates were moderate, some traditional insurance customers replaced insurance with self-insurance programs financed by bonds. The catastrophe property market in the 1990s, according to Froot *et al.* (1995), Lane (1995), and Tilley (1995), satisfies this condition -- insurance prices are high enough to attract investors.

The fund has adequate cash to pay the loss if a catastrophic event occurs. If no catastrophe occurs, the fund goes to the bond owners. From the bond owners' perspective, the bond contract is like lending money subject to credit risk, except the risk of "default" is really the risk of a catastrophic event. Note that the reinsurer has adequate cash at the beginning of the period to make the loss payment with probability one. Tilley (1995) describes this as a fully collateralised reinsurance contract. This scheme is a simple version of how a traditional reinsurer works with the following differences.

- The traditional reinsurance company owners buy shares of stock instead of bonds.
- Losses are based on a portfolio of risks rather than single exposure.
- Simplifying and specializing makes it possible to sell single exposures through the capital markets, in contrast to shares of stock of a reinsurer, which are claims on the aggregate of outcomes.

Tilley (1995) demonstrates this technique in a more general setting in which the reinsurance and bond are N period contracts. This one period model illustrates the key ideas.

3. Modelling Catastrophe Risk Bonds

In the previous section we discussed the securitisation underlying catastrophe risk bonds. In this section we shall adopt a standardised definition of a catastrophe risk bond for the purposes of analysing this security using financial economics. We shall be informal in this section, leaving the definition of some technical terms until section 5.

A *catastrophe risk bond* with face amount of 1 is an instrument that is scheduled to make a coupon payment of c at the end of each period and a final principal repayment of 1 at the end of the last period [labelled time T] so long as a catastrophe does not occur. An investor enters into a position in the catastrophe risk bond by making an initial principal investment of 1. However, if a catastrophe should occur the bond makes a fractional coupon payment and a fractional principal repayment that period and is then wound up. The fractional payment is assumed to be of the fraction f so that if a catastrophe occurs, the payment made at the end of the period in which the catastrophe occurs is equal to f(c + 1). At present we are not allowing for varying severity in the claims associated with the catastrophe. Varying severity would occur in practice and we mention this modelling issue later.

Financial economics theory tells us that when an investment market is *arbitrage-free*, there exists a probability measure, which we shall denote by Q, referred to as the *risk-neutral measure*, such that the price at time 0 of each uncertain cash flow stream $\{c(\omega, k) : k = 1, 2, \ldots, T\}$ is given by the following expectation under the probability measure Q,

$$E^Q\left[\sum_{k=1}^{T} \frac{1}{[1 + r(\omega, 0)] \cdots [1 + r(\omega, k-1)]} c(\omega, k)\right]. \tag{3.1}$$

The process $\{r(\omega, k)\}$ is the process of one-period interest rates. We shall denote the price at time 0 of a non-defaultable zero coupon bond maturing at time n by P(n). Therefore we have,

$$P(n) = E^Q\left[\frac{1}{[1 + r(\omega, 0)] \cdots [1 + r(\omega, n-1)]}\right]. \tag{3.2}$$

We shall let τ denote the time of the first occurrence of the catastrophe. A catastrophe may or may not occur prior to the scheduled maturity of the catastrophe risk bond at time T. If a catastrophe occurs then evidently $\tau \in \{1, 2, \ldots, T\}$. The cash flow stream received by the holder of the catastrophe risk bond may be described as

$$c(\omega, k) = \begin{cases} c\, 1_{\{\tau > k\}} + f(c+1)\, 1_{\{\tau = k\}} & k = 1, 2, \ldots, T-1 \\ (c+1)\, 1_{\{\tau > T\}} + f(c+1)\, 1_{\{\tau = T\}} & k = T. \end{cases} \tag{3.3}$$

Let us assume that we are trading catastrophe risk bonds in an investment market which is arbitrage-free with risk-neutral valuation measure Q and that the time of the catastrophe is independent of the term structure under the probability measure Q. We shall formalise these notions[4] in section 5. We may apply relation (3.1) to the cash flow stream in (3.3) and find that the price at time 0 of the cash flow stream provided by the catastrophe risk bond is given by the expression

$$c\sum_{k=1}^{T} P(k)\, Q(\tau > k) + P(T)\, Q(\tau > T) + f(c+1)\sum_{k=1}^{T} P(k)\, Q(\tau = k)\,. \qquad (3.4)$$

The term $Q(\tau > k)$ is the probability under the risk neutral valuation measure that the catastrophe does not occur within the first k periods. The other probabilistic terms may be verbalised similarly. No assumption has been made about the distribution of τ but the assumption that only one degree of severity can occur is clearly being used here. Of course, the distribution of τ will depend on the structure of the catastrophe risk exposure.

Formula (3.4) expresses the price of the catastrophe risk bond in terms of known parameters, including the coupon rate c. As we described at the beginning of this section, the principal amount of the catastrophe risk bond is fixed at the time of issue and the coupon rate is varied to ensure that the price of the cash flows provided by the bond are equal to the principal amount. One may apply the valuation formula (3.4) obtain a formula for the coupon rate as

$$c = \frac{1 - P(T)\, Q(\tau > T) - f\sum_{k=1}^{T} P(k)\, Q(\tau = k)}{\sum_{k=1}^{T} P(k)\, Q(\tau > k) + f\sum_{k=1}^{T} P(k)\, Q(\tau = k)}\,. \qquad (3.5)$$

If there was a conditional severity distribution, denoted by μ, formula (3.4) would be modified to

$$c\sum_{k=1}^{T} P(k)\, Q(\tau > k) + P(T)\, Q(\tau > T) + \sum_{k=1}^{T} P(k)\, Q(\tau = k) \int_0^{\infty} x\, d\mu(x)\,. \qquad (3.6)$$

On comparing formula (3.4) and (3.6) we see that there is little difference between the two formulas. Generally, the conditional probability measure μ is embedded as part of the risk-neutral measure Q.

Let us suppose that the catastrophe risk structure is such that the conditional probabilities under the risk neutral measure of no catastrophe each period are equal to a constant θ_0. Furthermore, suppose that should a catastrophe occur there is a single severity level resulting in a payment equal to $f(c + 1)$ at the end of the period in which the catastrophe occurs. Let $\theta_1 = 1 - \theta_0$. In this case, formula (3.4) simplifies to the expression given in Tilley (1995) for the price at time 0 of the catastrophe risk bond, namely

$$c\sum_{k=1}^{T} P(k)\,(1-\theta_1)^k + P(T)\,(1-\theta_1)^T + f(c+1)\sum_{k=1}^{T} P(k)\,\theta_1\,(1-\theta_1)^{k-1}. \qquad (3.7)$$

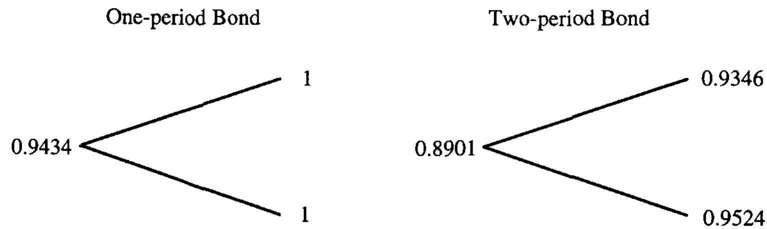
In order to apply Tilley's formula (3.7), one must know what the conditional risk-neutral probability θ_1 [or equivalently θ_0] is. At this point, θ_1 has not been related to the empirical conditional probability of a catastrophe occurring. Therefore, the formula (3.7) is not quite "closed". In order to close the model we need to link the valuation formula (3.7) with observable quantities that can be used to estimate the parameters needed to apply the valuation model. Although we began the discussion of the pricing model with an assumption about the existence of a valuation measure Q, it is possible to justify an interpretation of θ_1 as the empirical conditional probability of a catastrophe occurring. We shall address and clarify this point in section 5.

4. Incompleteness in the Presence of Catastrophe Risk

The introduction of catastrophe risk into a securities market model implies that the resulting model is incomplete. The pricing of uncertain cash flow streams in an incomplete model is substantially weaker in the interpretation of the pricing results that can be obtained than is the case for pricing in complete securities markets. In this section we discuss market completeness and explain the nature of the incompleteness problem for models with catastrophe risk exposures. For simplicity, we work with a one-period model although similar notions may be developed for multi-period models.

Let us consider a single-period model in which two bonds are available for trading, one of which is a one-period bond and the other a two period bond. For convenience we shall assume that both bonds are zero coupon bonds. We further assume that the financial markets will evolve to one of two states at the end of the period, "interest rates go up" or "interest rates go down" and that the price of each bond will assume to behave according to the binomial model depicted in the following figures.

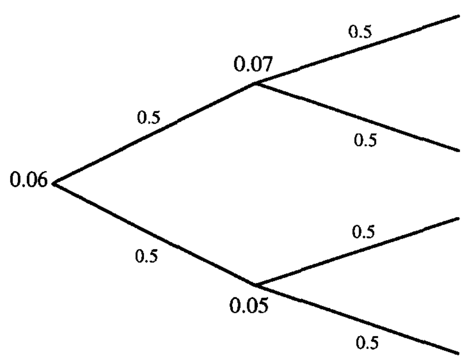
Figure 1



The bond prices for this model could be derived from the equivalent information in the following tree diagram for which the one-period model is embedded. We specified the bond prices directly to avoid bringing a two-period model into our discussion of the one-period case. The prices reported in figure 1 have been rounded from what one would compute from figure 2.

Figure 2

One-Period Rates and Risk-Neutral Probabilities



Suppose that we select a portfolio of the one-period and two-period bonds. Let us denote the number of one-period bonds held in this portfolio by ϑ_1 and the number of two-period bonds held in this portfolio by ϑ_2. This portfolio will have a value in each of the two states at time 1. Let us represent the state dependent price of each bond at time 1 using a column vector. Then we may represent the value of our portfolio at time 1 by the following matrix equation.

$$\begin{bmatrix} 1 & 0.9346 \\ 1 & 0.9524 \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} \tag{4.1}$$

The cost of this portfolio is given by

$$0.9434\ \vartheta_1 + 0.8901\ \vartheta_2. \tag{4.2}$$

The 2×2 matrix of bond prices at time 1 appearing in equation (4.1) is nonsingular. Therefore, any vector of cash flows at time 1 may be generated by forming the appropriate portfolio of these two bonds. For instance, if we want the vector of cash flows at time 1 given by the column vector,

$$\begin{bmatrix} c^u \\ c^d \end{bmatrix} \tag{4.3}$$

then we form the portfolio

$$\begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0.9346 \\ 1 & 0.9524 \end{bmatrix}^{-1} \begin{bmatrix} c^u \\ c^d \end{bmatrix}$$

at a cost of $0.9434\ \vartheta_1 + 0.8901\ \vartheta_2$. Carrying out the arithmetic, one finds that the price of each cash flow of the form (4.3) is given by the expression

$$0.4716\ c^u + 0.4718\ c^d. \tag{4.4}$$

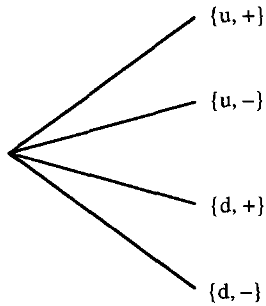
Since every such set of cash flows at time 1 can be obtained and priced in the model we say that the one-period model is *complete*. The notion of pricing in this complete model is justified by the fact that the price we assign to each uncertain cash flows stream is exactly equal to the price of the portfolio of one-period and two-period bonds that generates the value of the cash flow stream at time 1.

Let us see how the model is changed when catastrophe risk exposure is incorporated as part of the information structure. Suppose that we have the framework of the previous model with the addition of catastrophe risk. Furthermore, let us suppose that the catastrophic event occurs independently of the underlying financial market variables. Therefore, there will be four states in the model which we may identify as follows.

$$\begin{array}{c}\{\text{interest rates go up, catastrophe occurs}\} \equiv \{u, +\} \\ \{\text{interest rates go up, catastrophe does not occur}\} \equiv \{u, -\} \\ \{\text{interest rates go down, catastrophe occurs}\} \equiv \{d, +\} \\ \{\text{interest rates go down, catastrophe does not occur}\} \equiv \{d, -\}\end{array} \tag{4.5}$$

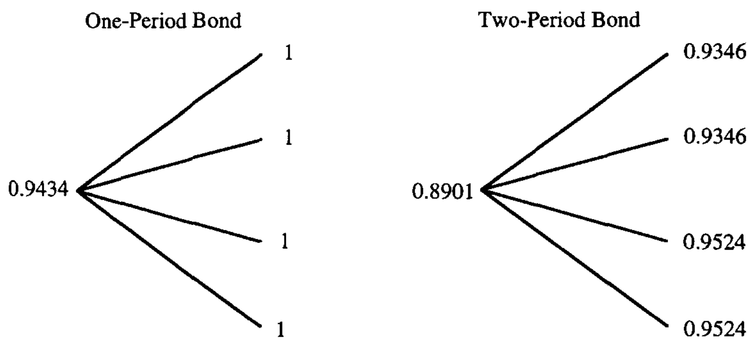
The reader will note that the symbol {u, +} is shorthand for "interest rates go up" and "catastrophe occurs" and so forth. This information structure could be represented on a single-period tree with four branches such as is shown in figure 3.

Figure 3



The values at time 1 of the one-period bond and the two-period bond are not linked to the occurrence or nonoccurrence of the catastrophic event and therefore do not depend on the catastrophic risk variable. We may represent the prices of the one-period and two-period bond in the extended model as shown in figure 4.

Figure 4



In contrast to equation (4.1), the value at time 1 of a portfolio of the one-period and two-period bonds is now given by the following matrix equation.

$$\begin{bmatrix} 1 & 0.9346 \\ 1 & 0.9346 \\ 1 & 0.9524 \\ 1 & 0.9524 \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} \tag{4.6}$$

The cost of this portfolio is still given by $0.9434\ \vartheta_1 + 0.8901\ \vartheta_2$. The most general vector of cash flows at time 1 in this model is of the following form.

$$\begin{bmatrix} c^{u,+} \\ c^{u,-} \\ c^{d,+} \\ c^{d,-} \end{bmatrix} \tag{4.7}$$

On reviewing equation (4.6) we see that the span of the assets available for trading in the model [i.e. the one-period and two-period bonds] are not sufficient to span all cash flows of the form (4.7). Consequently, we cannot derive a pricing relation such as (4.4) that is valid for all cash flow vectors of the form (4.7). The best we can do is to obtain bounds on the price of a general cash flow vector so that its price is consistent with the absence of arbitrage. To do this we recall that this one-period securities market model is arbitrage-free if and only if there exists a vector

$$\psi \equiv [\psi^{u,+}, \psi^{u,-}, \psi^{d,+}, \psi^{d,-}], \tag{4.8}$$

each component of which is positive, such that

$$[\psi^{u,+}, \psi^{u,-}, \psi^{d,+}, \psi^{d,-}] \begin{bmatrix} 1 & 0.9346 \\ 1 & 0.9346 \\ 1 & 0.9524 \\ 1 & 0.9524 \end{bmatrix} = \begin{bmatrix} 0.9434 \\ 0.8901 \end{bmatrix}. \tag{4.9}$$

Such a vector is called a state price vector[5]. One may solve (4.9) for all such vectors to find that the class of all state price vectors for this model is of the form

$$\{[0.4716 - s, s, 0.4718 - t, t] \ : \ 0 < s < 0.4716,\ 0 < t < 0.4718\}. \tag{4.10}$$

For each cash flow of the form (4.7), there is a range of prices that are consistent with the absence of arbitrage. This is given by the expression

$$0.4716\, c^{u,+} + 0.4718\, c^{d,+} + s\,(c^{u,-} - c^{u,+}) + t\,(c^{d,-} - c^{d,+})\,, \tag{4.11}$$

where s and t range through all feasible values of the parameters s and t. For instance, the price of the cash flow stream which pays 1 if no catastrophe occurs and 0.5 if a catastrophe occurs has the price range given by the expression

$$0.4716\,(0.5) + 0.4718\,(0.5) + s\,(1 - 0.5) + t\,(1 - 0.5)\,, \tag{4.12}$$

where s and t range through all feasible values of the parameters s and t. On completing the algebra, one finds that the range of prices for this cash flow stream are the open interval (0.4717, 0.9434). These price bounds are not very tight. However, they are all that can be said working solely from the absence of arbitrage.

Let us consider the case of a one-period catastrophe risk bond with $f = 0.3$. Therefore, in return for a principal deposit of $1 at time 0, the investor will receive the uncertain cash flow stream at time 1 of the form

$$\begin{bmatrix} 0.3\,(1 + c) \\ (1 + c) \\ 0.3\,(1 + c) \\ (1 + c) \end{bmatrix}. \tag{4.13}$$

We may apply the relation (4.11) to find that the range of values on the coupon that must be paid to the investor have the range in the open interval (0.06, 2.5333). The coupon rate of the catastrophe risk bond is not uniquely defined. There is but a range of values for the coupon that are consistent with the absence of arbitrage. Although this is a very wide range of coupon rates, this is the strongest statement about how the coupon values can be set subject only to the criterion that the resulting securities market is arbitrage-free. Evidently, we need to bring in some additional theory if we are to obtain useful, if benchmark, pricing formulas for catastrophe risk bonds. In fact, we shall see that we can tighten these bounds, even to the point of generating an explicit price, by embedding in the model the probabilities of the catastrophe occurring. The difficulty with this approach is that the price can no longer be justified by arbitrage considerations alone [i.e. the cost of a portfolio of existing assets that gives the appropriate payoffs - since there is no such portfolio] and thus we lose the uniqueness of prices. Such is the nature of incomplete markets.

In the following section we shall describe a method of obtaining explicit prices for catastrophe risk bonds. In the case of the example we have been studying, the coupon rate may be expressed as a function of the probability of a catastrophe occurring. Let us denote this probability by q. Then the coupon rate for the catastrophe risk bond is given the following explicit formula in terms of the probability q.

$$c = \frac{0.0566 + 0.66038\, q}{0.9434\ - 0.66038\, q}. \tag{4.14}$$

Formula (4.14) gives a "tight" number for the coupon rate but it requires the pegging of the probability of a catastrophe occurring.

5. A Formal Model

In section 3 we gave a preliminary presentation of the basic formulation of a valuation model for catastrophe risk bonds and discussed the type of valuation formulas described in Tilley (1995). The discussion offered in section 3 should be considered as motivation for the formal model that we now develop. The formal model we describe is designed to combine primary financial market variables with catastrophe risk variables to yield a theoretical valuation model for catastrophe risk bonds. Of course, the mathematics of the model may be used in other contexts regardless of the interpretation we give to the components of the model.

The financial market variables are assumed to be modelled on the filtered probability space $(\Omega^{(1)}, \mathcal{F}^{(1)}, P_1)$. The filtration $\mathcal{F}^{(1)}$ represents how information evolves in the financial market and in the simplest models may be thought of as an information tree. The catastrophe risk variables are assumed to be modelled on the filtered probability space $(\Omega^{(2)}, \mathcal{F}^{(2)}, P_2)$. P_2 is interpreted as the probability measure governing the catastrophe structure. The probability space for our full model is taken to be the product space $\Omega := \Omega^{(1)} \times \Omega^{(2)}$. Ω is also referred to as the sample space. Therefore, a typical element of the probability space for the full model is of the form $\omega = (\omega^{(1)}, \omega^{(2)})$: $\omega^{(1)} \in \Omega^{(1)}, \omega^{(2)} \in \Omega^{(2)}$ and such an element describes the state of the financial market variables and the catastrophe risk variables. It should again be emphasised that under this construction, the embedded sample space $\Omega^{(1)}$ represents the primary financial market variables, which for the purposes of valuing catastrophe risk bonds is essentially the term structure of interest rates, while the embedded sample space $\Omega^{(2)}$ represents risk variables related to the catastrophe exposure. The probability measure on the sample space Ω is given the product measure structure $P(\omega) := P_1(\omega^{(1)})\, P_2(\omega^{(2)})$. This assumption ensures the independence of the economic and catastrophe risk variables. It is easily checked that the partitions $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ are independent under the probability measure P.

The benchmark financial economics technique used to price uncertain cash flow streams in an incomplete markets setting is the representative agent. We now describe this technique in the context of the probability structure we have just defined. The representative agent technique consists of an assumed representative utility function and an aggregate consumption process. The aggregate consumption process may be thought of as the total consumption available in the economy at each point in time and in each state. We shall denote the aggregate consumption process by $\{C^*(\omega, k) \mid k = 0, 1, \ldots, T\}$. We shall assume that the representative agent's utility is time-additive and separable as well as differentiable. Time-additive and separable means that there are utility functions $u_0, u_1, \ldots, u_T$ such that the utility the agent gets from a generic consumption stream $\{c(\omega, k) \mid k = 0, 1, \ldots, T\}$ is given by

$$E^P\left[\sum_{k=0}^{T} u_k(c(\omega, k))\right]. \tag{5.1}$$

It follows from the theory of the representative agent[6] that the price of a generic cash flow stream $\{c(\omega, k) \mid k = 0, 1, \ldots, T\}$ at time 0 is given by the expectation

$$E^P\left[\sum_{k=0}^{T} \frac{u'_k(C^*(\omega, k))}{u'_0(C^*(\omega, 0))}\, c(\omega, k)\right]. \tag{5.2}$$

Note that the aggregate consumption process plays a role in the pricing relation. In many implementations of this pricing relation the aggregate consumption process is assumed to evolve according to an exogenous process. This will not be an issue for us. Both the form of the utility function and the aggregate endowment process will be removed from the pricing analysis by relating the pricing relation to the valuation measure approach to arbitrage-free pricing.

In order to proceed further from relation (5.2), we assume that aggregate consumption [or equivalently, the aggregate endowment since we are in equilibrium] does not depend on the catastrophe risk variables. Mathematically, this assumption is the condition that

$$C^*(\omega, k) \equiv C^*(\omega^{(1)}, k). \tag{5.3}$$

The hypothesis that aggregate consumption does not depend on catastrophe risk variables is a reasonable approximation since the overall economy is only marginally influenced by localised catastrophes such as earthquakes or hurricanes.

In order to relate the representative agent valuation formula to the usual valuation measure approach in arbitrage-free pricing we need to define the one-period interest rates implicit in the representative agent pricing model. We define the one-period interest rates, $\{r(\omega, k) \mid k = 0, 1, \ldots, T-1, \omega \in \Omega\}$, through the conditional expectations

$$\frac{1}{1 + r(\omega^{(1)}, k)} := E^P\left[\frac{u'_{k+1}(C^*(\omega, k+1))}{u'_k C^*(\omega, k))} \mid \mathcal{F}_k\right]. \tag{5.4}$$

The reader may check that the one-period interest rate process is independent of the catastrophe risk exposure[7]. We may define the Radon-Nikodym derivative

$$\frac{dQ}{dP} := [1 + r(\omega, 0)]\,[1 + r(\omega, 1)] \cdots [1 + r(\omega, T-1)]\,\frac{u'_T(C^*(\omega, T))}{u'_0(C^*(\omega, 0))}. \tag{5.5}$$

We can then rewrite the valuation formula (5.2) as

$$E^Q\left[\sum_{k=1}^{T} \frac{1}{[1 + r(\omega, 0)]\,[1 + r(\omega, 1)] \cdots [1 + r(\omega, k-1)]}\, c(\omega, k)\right]. \tag{5.6}$$

Equation (5.6) recasts the equilibrium valuation formula as a standard risk-neutral expectation.

Also, we note that under the valuation measure [which we are denoting by Q] we have

$$Q(\omega) = \left[P_1(\omega^{(1)})\,[1 + r(\omega^{(1)}, 0)] \cdots [1 + r(\omega^{(1)}, T-1)]\,\frac{u'_T(C^*(\omega^{(1)}, T))}{u'_0(C^*(\omega^{(1)}, 0))}\right] P_2(\omega^{(2)}). \tag{5.7}$$

Let us define the notation

$$Q_1(\omega^{(1)}) := P_1(\omega^{(1)})\,[1 + r(\omega^{(1)}, 0)] \cdots [1 + r(\omega^{(1)}, T-1)]\,\frac{u'_T(C^*(\omega^{(1)}, T))}{u'_0(C^*(\omega^{(1)}, 0))}. \tag{5.8}$$

Therefore, we see that $Q(\omega) = Q_1(\omega^{(1)})\, P_2(\omega^{(2)})$ for all sample points $\omega \in \Omega$. We know that $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ are independent under the probability measure P. It is also true that $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ are independent under the probability measure Q. Indeed, suppose that $A_1 \in \mathcal{F}^{(1)}$ and $A_2 \in \mathcal{F}^{(2)}$. Then we find by direct calculation that,

$$Q(A_1 \cap A_2) = \sum_{\omega \in \Omega} 1_{A_1}(\omega)\, 1_{A_2}(\omega)\, Q(\omega)$$

$$= \sum_{\omega^{(1)} \in \Omega} \sum_{\omega^{(2)} \in \Omega} 1_{A_1}(\omega^{(1)})\, 1_{A_2}(\omega^{(2)})\, Q_1(\omega^{(1)})\, P_2(\omega^{(2)}) \tag{5.9}$$

$$= Q_1(A_1)\, P_2(A_2) = Q(A_1)\, Q(A_2).$$

The independence of $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ under the probability measure Q simplifies the valuation problem for catastrophe risk bonds as we now show.

For simplicity, we shall suppose that the catastrophe risk variables have a stationary and finite tree structure of the following nature. Let $S \subset Z$ denote a subset of the integers that contains 0 and assume that there are n elements other than 0. Define the embedded probability space $\Omega^{(2)} := S^T$. We charge $\Omega^{(2)}$ using one-step probabilities which we will denote by $\theta_0, \theta_1, \ldots, \theta_n$. The one-step probability θ_0 will be assumed to correspond to $\omega_k^{(2)} = 0$ and is interpreted as the event that no catastrophe occurs in period k. $\Omega^{(2)}$ is charged as

$$P_2(\omega^{(2)}) := \prod_{k=1}^{T} \theta_{\omega_k^{(2)}}. \tag{5.10}$$

We define the time of the first of the catastrophe as

$$\tau(\omega) \equiv \tau(\omega^{(2)}) := \inf\{k \mid \omega_k^{(2)} \neq 0\}, \tag{5.11}$$

where $\tau := \infty$ if $\omega^{(2)} = (0, 0, \ldots, 0)$. We assume there is a mapping $X\colon S \to R_+$ representing the severity random variable. The range of the severity random variable will be labelled as $\{0, x_1, \ldots, x_n\}$. The conditional severity distribution is seen to be,

$$P(X = x_j \mid \tau = k) = \frac{\theta_j}{1 - \theta_0}. \tag{5.12}$$

The cash flows from the catastrophe risk bond are

$$c(\omega, k) = \begin{cases} c\, 1_{\{\tau(\omega^{(2)}) > k\}} + X(\omega_k^{(2)})\, 1_{\{\tau(\omega^{(2)}) = k\}}, & k = 1, 2, \ldots, T-1 \\ (c+1)\, 1_{\{\tau(\omega^{(2)}) > T\}} + X(\omega_T^{(2)})\, 1_{\{\tau(\omega^{(2)}) = T\}}, & k = T. \end{cases} \tag{5.13}$$

Direct calculation shows that

$$E^Q[X(\omega_k^{(2)})\, 1_{\{\tau(\omega^{(2)}) = k\}}]$$

$$= \theta_0^{k-1}\,[x_1\theta_1 + x_2\theta_2 + \cdots + x_n\theta_n] \tag{5.14}$$

$$= \theta_0^{k-1}\,(1-\theta_0)\,[x_1\frac{\theta_1}{(1-\theta_0)} + x_2\frac{\theta_2}{(1-\theta_0)} + \cdots + x_n\frac{\theta_n}{(1-\theta_0)}].$$

This may be summarised as

$$E^Q[X(\omega_k^{(2)})\, 1_{\{\tau(\omega^{(2)}) = k\}}] = Q(\tau = k)\, E^Q[X \mid \tau = k]. \tag{5.15}$$

Let us employ the notation $\mu := \left[x_1\frac{\theta_1}{1-\theta_0} + \cdots + x_n\frac{\theta_n}{1-\theta_0}\right]$. Then, relying on the independence relation (5.9) to simplify the expectation (5.6) for the cash flow stream (5.13) shows the price of the catastrophe risk bond is given by the expression

$$c\sum_{k=1}^{T} P(k)\, Q(\tau > k) + P(T)\, Q(\tau > T) + \sum_{k=1}^{T} P(k)\, Q(\tau = k)\, \mu. \tag{5.16}$$

Examining relation (5.16) allows us to draw the following conclusion. We have established that valuation by a representative agent is equivalent to selecting a term structure model which is independent of the catastrophe risk structure and combining this term structure model with the probabilities of a catastrophe occurring to price the catastrophe risk bond.

The evolving catastrophe risk bond prices [i.e. prices at times other than time 0] may be obtained from computing conditional expectations such as

$$E^Q[X(\omega_k^{(2)})\, 1_{\{\tau(\omega^{(2)}) = k\}} \mid \mathcal{F}_j] = \theta_0^{k-j-1}\,[x_1\theta_1 + x_2\theta_2 + \cdots + x_n\theta_n]. \tag{5.17}$$

The general intertemporal valuation formula for the price of the catastrophe risk bond at time n when the financial market is in state $\omega^{(1)}$ and no catastrophe has occurred as of time n is given by

$$c\sum_{k=n+1}^{T} P(n, k, \omega^{(1)})\, Q(\tau > k \mid \tau > n) + P(n, T, \omega^{(1)})\, Q(\tau > T \mid \tau > n)$$
$$+ \sum_{k=n+1}^{T} P(n, k, \omega^{(1)})\, Q(\tau = k \mid \tau > n)\, \mu \tag{5.18}$$

where $P(n, k, \omega^{(1)})$ denotes the price at time n when the financial market is in state $\omega^{(1)}$ of a zero coupon bond maturing for 1 at time k. For our stationary model, the conditional probabilities are easy to compute. Indeed, for $n < k$

$$Q(\tau > k \mid \tau > n) = \theta_0^{k-n}, \quad Q(\tau = k \mid \tau > n) = \theta_0^{k-n-1}(1-\theta_0). \tag{5.19}$$

A special case of this model is the binomial model for which

$$\Omega^{(2)} = \{0, 1\}^T \quad \text{and} \quad P(\omega^{(2)}) = (1-\theta_0)^{\sigma(\omega^{(2)}, T)}\, \theta_0^{\,T-\sigma(\omega^{(2)}, T)} \tag{5.20}$$

where $\sigma(\omega^{(2)}, j) := \omega_1^{(2)} + \cdots + \omega_j^{(2)}$. This results in the following pricing formula for the catastrophe risk bond.

$$c\sum_{k=1}^{T} P(k)\,\theta_0^k + P(T)\,\theta_0^T + \sum_{k=1}^{T} P(k)\,(1-\theta_0)\,\theta_0^{k-1}\,\mu. \tag{5.21}$$

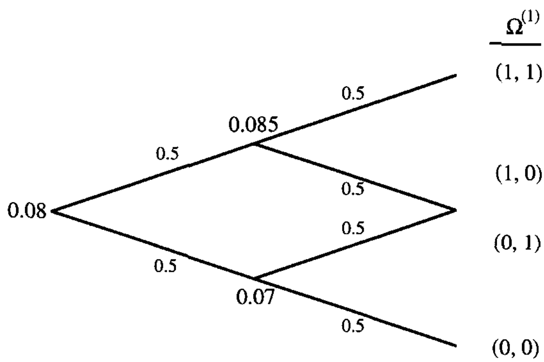
This formula has already made an appearance in section 3 [equation (3.7)] with θ_0 replaced by $1-\theta_1$. The model developed in Tilley (1995) may be thought of as the selection of a short-rate process $\{r(\omega^{(1)}, k) : k = 0, 1, \ldots, T-1\}$ on the filtered space $(\Omega^{(1)}, \mathcal{F}^{(1)})$ and a risk-neutral probability measure $Q^{(1)}$ on the probability space $\Omega^{(1)}$ [i.e. a term structure model defined by $\{r(\omega^{(1)}, k)\}$ and $Q^{(1)}$] crossed with a conditional binomial catastrophe structure of the form (5.20). The independence of the financial market risk from the catastrophe risk has permitted us to easily fit together these two probability structures to obtain a practical and economically meaningful model. The binomial formula (5.21) is easy to apply as all that is needed for pricing the catastrophe risk bond is an estimate of the probability of a catastrophe occurring within one-period and a knowledge of the current yield curve. The expression (5.21) is theoretically equivalent to Tilley's formula except that we are able to interpret the parameter θ_0 in a traditional actuarial fashion because we closed our model using the theory of a representative agent which naturally involves the empirical probabilities of the various risks in the model.

The fact that a catastrophe risk model is necessarily incomplete means that there is no unique interpretation of the prices that we assign to the catastrophe risk bonds. This problem is inherent in any model that is used to attach a price to catastrophe risk bonds. The utility function of the representative agent, which we could loosely refer to as the risk aversion of the market or the market's attitude towards risk, is part of the assumed structure of the pricing rule. In our equivalent formulation of the pricing problem in terms of risk-neutral valuation, the incompleteness is embedded in the selection of the term structure model rather than as part of the catastrophe probabilities. In other words, for varying economic and catastrophe variables, the effect on the price dynamics of the catastrophe risk bond [equation (5.18)] appears through the implicit selection of the embedded term structure model. Although the bond pricing formula (5.16) seems to not depend on the embedded risk aversion, the dynamics of the catastrophe risk bond prices as shown in formula (5.18) depend on the full term structure model and thus on the embedded risk aversion. The fact that it is natural to select a term structure model for actuarial valuation problems hides the inherent difficulty associated with the fact that the catastrophe risk market is incomplete.

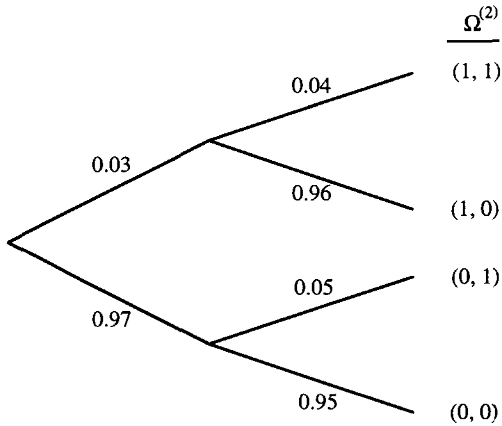
Example We illustrate the preceding pricing model for a two-period case combining a binomial term structure model and a binomial catastrophe risk structure. Assume that the severity is of the form $f(1 + c)$ where c is the coupon rate and $f = 0.3$. We wish to solve for the coupon rate for the catastrophe risk bond as described in equation (3.5). Here we have $\Omega^{(1)} = \{0, 1\}^2$ and $\Omega^{(2)} = \{0, 1\}^2$.

Embedded Term Structure Model - Figure 5

[one-period rates and risk neutral probabilities]



Catastrophe Risk Structure - Figure 6



One may easily verify that $P(1) = 0.9259$ and $P(2) = 0.8594$. One may also verify that $Q(\tau = 1) = 0.03$, $Q(\tau = 2) = 0.0485$, $Q(\tau > 1) = 0.97$, and $Q(\tau > 2) = 0.9215$. The coupon rate is computed to be $c = 0.1094$.

6. Concluding Remarks

We have discussed the financial economics involved in the pricing of catastrophe risk bonds. Furthermore, we have demonstrated how this theory may be utilised to construct a practical valuation model which can be justified within the framework of a representative agent equilibrium. A full implementation of the representative agent model could have been made but there is little point since in practice one is more likely to choose to work with the non-defaultable term structure model backing the valuation procedure. It is quite natural that the inputs to a valuation procedure for catastrophe risk bonds should be a an assumption about the term structure dynamics and the probability structure governing the occurrence of a catastrophe. As a first approximation to the pricing of catastrophe risk bonds, such a valuation framework seems to hold reasonable intuition and is theoretically sound.

A catastrophe risk bond cannot be fully hedged because of the lack of traditional securities that can be used to closely approximate the payoffs from the catastrophe risk bond [i.e. inherent market incompleteness]. Consequently, implicit in the coupon rate [or equivalently the price] of a catastrophe risk bond is the investor's attitude towards risk. Although we have provided a framework in which to attach a specific price to a catastrophe risk bond, the fact that the catastrophe risk bond cannot be perfectly hedged necessarily implies that there is a range of prices at which the catastrophe risk bond could sell without the existence of arbitrage in the market. The inability of investor's to efficiently hedge the risk in catastrophe risk bonds also suggests that were Charles Darwin to observe a catastrophe bond market during a major catastrophe he might comment "[i]t is indeed most wonderful to witness such financial desolation produced in three minutes of time." At such a time, catastrophe risk bondholders would generally find that the "high yields" they were receiving were insufficient to protect them from the bare risk that is inherent in such an unhedgeable security.

There is a substantial literature dealing with the problem of incomplete markets. In the end however, no matter how one chooses to look at the valuation problem in incomplete markets there is simply no way to assign exact prices to securities. Readers interested in these pricing issues may consult Chan and van der Hoek (1996) as an introduction to several techniques for pricing cash flows in incomplete markets. In the end, one is hard pressed to come up with completely convincing pricing theories for catastrophe risk bonds.

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[1]Financial economists would say that the payments from a catastrophe risk bond cannot be *spanned* by a portfolio of primitive assets [stocks and bonds].

[2]For those familiar with state prices, this is equivalent to estimating local state prices for the states of the world which do not depend on the catastrophe.

[3]More generally, one estimates the probability distribution for the varying degrees of severity of the catastrophe risk.

[4]These are the assumptions made in Tilley (1995) although they are not stated in quite this terminology.

[5]On reviewing section 3, the reader may easily check that the components of the state price vector are precisely the risk neutral probability of each state discounted by the short rate. In other words, the components of the state price vector are $Q(\omega_j) / [1 + r(\omega, 0)]$.

[6]See Karatzas (1997) for the a rigorous discussion of the theory of the representative agent. Embrechts and Meister (1996) discuss a related method from an alternative viewpoint.

[7]Indeed, the random variable under the expectation operator depends only on the financial market information in $\mathcal{F}^{(2)}$.