A comparison of optimal and dynamic control strategies for continuous-time pension fund models

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Abstract

This paper considers continuous-time stochastic pension fund models in which there are two risky assets as well as randomness in the level of benefit outgo. We consider Markov control strategies which optimise over the contribution rate and over the range of possible asset-allocation strategies.

A quadratic loss function is used which extends that of Boulier *et al.* (1995). Using this loss function it is shown that both the optimal contribution rates and the asset-allocation strategies are linear functions of the fund size.

Boulier *et al.* (1995) proved the perhaps surprising result that under the optimal asset-allocation strategy as the level of surplus in the pension fund increases the proportion of the fund invested in high-return, high-risk assets, such as equities decreases. This paper demonstrates that this result applies to a much larger class of models and loss functions.

Dynamic optimal solutions are compared with their static counterparts.

It is also found that the optimal strategies do not depend upon the amount of uncertainty in the level of benefit outgo. This means that, for quadratic loss functions, small pension funds should operate in precisely the same way as large pension funds.

Keywords: continuous time; stochastic differential equation; asset-allocation; contribution strategy; optimal control.

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1 Introduction

In this paper we consider continuous time stochastic models for pension fund dynamics which allow for two risky assets and for noise in the level of benefit outgo. The general form of this simple model is:

$$
\begin{aligned}
dX(t) &= X(t).d\delta(t, X(t)) + C(t).dt - B.dt - \sigma_b.dZ_b(t) \\
\text{where } X(t) &= \text{fund size at } t \\
d\delta(t, X(t)) &= \text{return on assets between } t \text{ and } t + dt \\
C(t) &= C(t, X(t)) \\
&= \text{contribution rate} \\
B &= \text{expected rate of benefit outgo} \\
\text{and } \sigma_b &= \text{volatility in benefit outgo}
\end{aligned}
$$

Discrete-time models have been considered by Cairns (1995), Dufresne (1988, 1989, 1990) and Haberman & Sung (1994). Analytical results for these models harder to come by. Continuous-time models, which are, in some ways, more idealised, yield more analytical results (for example, see Dufresne, 1990, Boulier *et al.*, 1995, and Cairns, 1996). Similar results can then be sought empirically in discrete time models.

For simplicity in this development we will consider the rate of benefit outgo as fixed in monetary terms with noise added to model demographic variation in the membership. However, in Cairns (1996) it was discussed how X_t could be measured in terms of the salary roll of the pension fund. This adds a more realistic dimension to the problem by allowing liabilities, benefit outgo and contributions to be taken as stationary relative to the salary roll rather than in absolute monetary terms. When amounts are considered in terms of the salary roll the final noise term $\sigma_b dZ_b(t)$ can be considered to include an allowance for salary increases which are not predictable.

The contribution rate, $C(t)$, is a predictable process and provides us with one of the means of controlling the dynamics of the pension fund. Dufresne (1990) and Cairns (1996) considered continuous-time models in which the contribution rate was a linear function of the current fund size, $X(t)$. Boulier *et al.* (1995) considered more general forms for $C(t)$ but found that the optimal solution to a simple control problem was that the contribution rate should indeed be linear in $X(t)$.

The other means of control is through the asset-allocation strategy. There are

two assets, the prices of which follow correlated geometric brownian motion: that is,

$$\frac{dS_1(t)}{S_1} = d\delta_1(t) = \delta_1.dt + \sigma_{11}.dZ_1(t) + \sigma_{12}.dZ_2(t)$$
$$\frac{dS_2(t)}{S_2} = d\delta_2(t) = \delta_2.dt + \sigma_{21}.dZ_1(t) + \sigma_{22}.dZ_2(t) \ .$$

Let us assume that $\delta_1 > \delta_2$ and that $\sigma_{11}^2 + \sigma_{12}^2 > \sigma_{21}^2 + \sigma_{22}^2$, so that asset 1 provides a higher return than asset 2 in return for a higher level of risk. No assumption is made about the level of correlation between the returns on the two stocks including, for example, the benefits (or otherwise) of diversification. The proportion of the assets invested in asset 1 is denoted by $p(t, X(t))$. Cairns (1996) considered the stationary distribution of the fund when $\sigma_b^2 = 0$ and $p(t, x) = (p_0 + p_1 x)/x$.

We use two risky assets rather than one risky and one risk-free because the experience of the UK pension funding scene is that pension funds only use cash for short-term liquidity rather than as a serious asset. Instead funds use government bonds (fixed interest and index linked) as low risk (but non-zero-risk) assets. However, it can readily be assumed that $\sigma_{12} = \sigma_{21} = \sigma_{22} = 0$ if one wishes.

In this paper we will consider two alternative approaches: dynamic control and static control.

In Sections 2 and 3 we investigate the dynamic control-theoretical framework first applied by Boulier et $al.$ (1995) to a pension funding problem and apply it to the more general model described above. Boulier et $al.$ (1995) made use of the loss function $L(t, C(t)) = C(t)^2$. This analysis considers the more general loss function $L(t, X(t), C(t)) = (C(t) - c_m)^2 + k(X(t) - x_p)^2$. It is found that the optimal solution (C^*, p^*) to this problem is of the form

$$C^*(t, x) = c_0 - c_1 X(t)$$
$$p^*(t, x) = p_1(\tilde{x}_m - x)$$

These optimal solutions depend upon the δ_i, the σ_{ij}, c_m, x_p and k. They do not depend upon σ_b, the volatility of the benefit outgo.

This solution is of the same form as Boulier et $al.$ (1995) but, of course, applies to a much more general problem. It was commented in Cairns (1996)

that, while $C^*(t,x)$ is very much a conventional formula for the contribution rate (that is, linear and decreasing), the form for $p^*(t,x).x$ (also linear and decreasing) goes very much against conventional actuarial wisdom which says that surplus should be invested in more risky assets rather than low-risk assets. The rationale behind this solution is that the optimal strategy aims to get as quickly as possible to the ideal contribution rate and fund size and then to stay as close as possible to that ideal value by investing in low risk assets.

The linearity of the optimal solutions leads us into Section 4 which considers the stationary distribution of the fund size. This is shown to be a Pearson Type IV distribution and was first described in the context of pension funding by Cairns (1996). In the stationary state we also minimize the expected value of the loss function and compare the solution with that obtained in Section 3.

1.1 Motivation

The model used here is relatively simple. Simple models have the advantage of allowing the derivation of analytical results. It is rarely possible, on-the-other-hand, to find analytical results using complex models. Often with such models it is very difficult to explain the cause of certain effects because there may be many interacting components at work. The existence of analytical results for simple models provides a focus for investigations of more complex models. Very often the same results can be observed empirically through simulation of a complex model.

This paper considers the important question of the optimal control of a pension fund. Given the results in this paper and in that of Boulier *et al.* (1995) attention needs to be drawn to the following issues:

- what objective functions (if any) are used by pension funds?

- what constraints (if any) on contributions and investments are appropriate?

- is too much emphasis placed on the calculation of the so-called actuarial liability when this may have no relationship to the target funding level under the optimised objective function.

2 The control theoretic approach

Following Boulier *et al.* (1995) we define the value function for a general controlled pension fund process

$$W(t,x)(C,p) = E\left[\int_t^\infty \exp(-\beta s)L\left(s,C(s,X(s)),X(s)\right)ds \mid X(t) = x\right]$$

Let $V(t,x) = \inf_{(C,p)} W(t,x)(C,p) = W(t,x)(C^*,p^*)$ assuming that such optimal control strategies C^* and p^* exist. Then $V(t,x)$ satisfies the Hamilton-Jacobi-Bellman equation (for example, see Merton, 1990, or Øksendal, 1992):

$$0 = \inf_{C,p}\left(e^{-\beta t}L(t,C,x) + V_t + (\delta_2 x + \lambda px + C - B)V_x\right.$$
$$\left. + \frac{1}{2}V_{xx}[x^2(\epsilon_2 p^2 + \epsilon_1 p + \epsilon_0) + \sigma_b^2]\right)$$

where
$$\lambda = \delta_1 - \delta_2$$
$$\epsilon_2 = (\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2$$
$$\epsilon_1 = 2[(\sigma_{11} - \sigma_{21})\sigma_{21} + (\sigma_{12} - \sigma_{22})\sigma_{22}]$$
$$\epsilon_0 = \sigma_{21}^2 + \sigma_{22}^2$$
$$V_t \equiv \partial V/\partial t$$
$$V_x \equiv \partial V/\partial x$$
$$V_{xx} \equiv \partial^2 V/\partial x^2$$

Taking partial derivatives with respect to C and p, it follows (generalising Boulier *et al.*, 1995) that the optimal policy (C^*, p^*) satisfies:

$$e^{-\beta t}L_c(t,C,x) + V_x = 0 \tag{1}$$

$$\lambda x V_x + \frac{1}{2}V_{xx}x^2(2\epsilon_2 p + \epsilon_1) = 0 \tag{2}$$

$$\Rightarrow C^*(t,x) = L_c^{-1}\left(e^{\beta t}V_x\right) \tag{3}$$

$$p^*(t,x) = -\frac{\lambda V_x + \frac{1}{2}V_{xx}\epsilon_1 x}{V_{xx}x\epsilon_2} \tag{4}$$

The precise form for $V(t,x)$ is, of course, still not yet known: we only have expressions for C^* and p^* involving $V(t,x)$.

(Note it is necessary that the loss function is a strictly convex function of C. This ensures that the inverse of L_c exists. This requirement excludes, for example, downside loss functions which are convex but not strictly convex.)

Here we restrict ourselves to the case

$$
\begin{aligned}
L(t, C, x) &= (C - c_m)^2 + k(x - x_p)^2 \\
\Rightarrow L_c(t, C, x) &= 2(C - c_m) \\
\Rightarrow L_c^{-1}\left(e^{\beta t} V_x\right) &= c_m - \frac{1}{2} e^{\beta t} V_x
\end{aligned}
$$

This loss function is of the same form as that suggested by Haberman and Sung (1994).

2.1 A special case

Boulier *et al.* (1995) considered the case $\sigma_{11} = \sigma$, $\sigma_{12} = \sigma_{21} = \sigma_{22} = 0$, $\sigma_b = 0$ (that is, asset 2 is risk-free and there is no volatility in the level of benefit outgo), $c_m = 0$ and $k = 0$.

They found that the optimal solution was of the following form:

$$
\begin{aligned}
C^*(t, x) &= \left(2\delta_2 - \beta - \frac{\lambda^2}{\sigma^2}\right)(x_m - x) \\
p^*(t, x) &= \frac{\lambda}{\sigma^2} \frac{(x_m - x)}{x} \\
\text{where } x_m &= B/\delta_2
\end{aligned}
$$

Here we make some further observations:

- The value of $x_m = B/\delta_2$ is the self-financing, risk-free steady state: that is, once $X(t)$ reaches x_m it will stay there and the fund will provide precisely enough interest to pay all future benefits as they arise.

- If $X(0) > x_m$ then the fund will have more than enough interest to pay off the benefits and the fund size will drift off to plus infinity unless refunds to the sponsoring employer are allowable.

- If $X(0) < x_m$ then the fund size will remain below x_m with probability 1. The process may fall below 0, but typically with a very low probability during any specified finite period of time.

- It is simple to show that under the optimal strategy

$$
x_m - X(t) = [x_m - X(0)].\exp\left[-\left(\delta_2 - \beta + \frac{1}{2}\frac{\lambda^2}{\sigma^2}\right)t + \frac{\lambda}{\sigma}Z(t)\right]
$$

where $Z(t)$ is a standard Brownian Motion.

Thus $x_m - X(t)$ is a Geometric Brownian Motion and we are able to say much more about the dynamics of $X(t)$ than is normally possible.

- If the term $\delta_2 - \beta + \frac{1}{2}\lambda^2/\sigma^2$ is negative then $X(t)$ will tend to minus infinity almost surely.

- We require $\delta_2 - \beta + \frac{1}{2}\lambda^2/\sigma^2 > 0$ (that is $\beta < \delta_2 + \frac{1}{2}\lambda^2/\sigma^2$) to ensure that $X(t) \to x_m$ almost surely.

As can be seen from these remarks the process $X(t)$ under the optimal strategy has a trivial stationary state (that is, the constant x_m with probability 1).

Such a solution is unlikely to occur in practice because of:

- the reluctance of the fund trustees to invest in low-return, risk-free assets (preferring low-risk assets to risk-free);

- unexpected variation in the level of benefit outgo (for example, due to demographic variation);

- random variation in the rate of salary growth relative to the risk-free rate of interest (not considered here).

These possibilities plus the more general loss function are considered in the next section.

3 Solution for the general model

Here we consider the case $L(t, C, x) = (C - c_m)^2 + k(x - x_p)^2$. We have from Equations (3) and (4)

$$C^*(t,x) = L_c^{-1}\left(e^{\beta t}V_x\right)$$

$$= c_m - \frac{1}{2}e^{\beta t}V_x$$

$$p^*(t,x) = -\frac{\lambda V_x + \frac{1}{2}V_{xx}\epsilon_1 x}{V_{xx}x\epsilon_2}$$

We put these back into the Hamilton-Jacobi-Bellman equation to get

$$
\begin{aligned}
0 = {} & e^{-\beta t}\left[\left(-\frac{1}{2}e^{\beta t}V_x\right)^2 + k(x - x_p)^2\right] + V_t + (\delta_2 x - B)V_x \\
& + \lambda x\left[-\frac{\lambda V_x + \frac{1}{2}V_{xx}\epsilon_1 x}{V_{xx}x\epsilon_2}\right]V_x + \left(c_m - \frac{1}{2}e^{\beta t}V_x\right)V_x \\
& + \frac{1}{2}\left[x^2\left\{\epsilon_2\left(\frac{\lambda V_x + \frac{1}{2}V_{xx}\epsilon_1 x}{V_{xx}x\epsilon_2}\right)^2 - \epsilon_1\left(\frac{\lambda V_x + \frac{1}{2}V_{xx}\epsilon_1 x}{V_{xx}x\epsilon_2}\right) + \epsilon_0\right\} + \sigma_b^2\right]
\end{aligned}
$$

Now, the process is Markov and time-homogeneous and the discount factor is exponential in t. It follows that $V(t,x)$ must be of the form $\exp(-\beta t)F(x)$. If we substitute in $V(t,x) = \exp(-\beta t)F(x)$ we find after some straightforward simplifications:

$$
\begin{aligned}
0 = {} & -\frac{1}{4}(F_x)^2 + k(x - x_p)^2 - \beta F + (\delta_2 x - B)F_x - \frac{1}{2}\lambda^2\frac{(F_x)^2}{F_{xx}\epsilon_2} - \frac{1}{2}\frac{\lambda F_x\epsilon_1 x}{\epsilon_2} \\
& + c_m F_x - x^2\frac{\epsilon_1^2 F_{xx}}{8\epsilon_2} + \frac{1}{2}\epsilon_0 x^2 F_{xx} + \frac{1}{2}\sigma_b^2 F_{xx} \qquad (5)
\end{aligned}
$$

We try (as in Boulier et al., 1995) for a solution of the form $F(x) = px^2 + qx + r$. We apply this to Equation (5) and, after a bit of rearranging we get:

$$
\begin{aligned}
0 = {} & x^2\left(-p^2 + k - \beta p + 2\delta_2 p - \frac{\lambda^2}{\epsilon_2}p - \frac{\lambda \epsilon_1}{\epsilon_2}p - \frac{1}{4}\frac{\epsilon_1^2}{\epsilon_2}p + \epsilon_0 p\right) \\
& + x\left(-pq - 2kx_p - \beta q - 2Bp + \delta_2 q - \frac{\lambda^2}{\epsilon_2}q - \frac{\lambda \epsilon_1}{2\epsilon_2}q + 2c_m p\right) \\
& + \left(-\frac{1}{4}q^2 + kx_p^2 - \beta r - Bq - \frac{\lambda^2}{4\epsilon_2}\frac{q^2}{p} + c_m q + \sigma_b^2 p\right)
\end{aligned}
$$

The constant and the coefficients of x and x^2 must all be equal to zero. First consider the coefficient of x^2. If $k = 0$ solutions are $p = 0$ (which is inappropriate for this problem) and

$$p = 2\delta_2 - \beta - \frac{\lambda^2}{\epsilon_2} - \frac{\lambda\epsilon_1}{\epsilon_2} - \frac{1}{4}\frac{\epsilon_1^2}{\epsilon_2} + \epsilon_0 = \hat{p}$$

say.

As k increases above 0 we have

$$\hat{p}(k) = \frac{\hat{p} \pm \sqrt{\hat{p}^2 + 4k}}{2}$$

Note that $\hat{p}(k)$ does not depend on c_m, x_p or σ_b^2.

One solution is negative and not relevant to this problem. The other solution is equal to $\hat{p}(k) = \hat{p} + k/\hat{p} + o(k)$. For numerical work the precise value of $\hat{p}(k)$ should be used.

Setting the coefficient of x equal to zero gives

$$\hat{q}(k) = \frac{2\left((B - c_m)\hat{p}(k) + kx_p\right)}{\left(-\hat{p}(k) - \beta + \delta_2 - \frac{\lambda^2}{\epsilon_2} - \frac{\lambda\epsilon_1}{2\epsilon_2}\right)}$$

Finally setting the constant equal to zero we get

$$\hat{r}(k) = \frac{1}{\beta}\left(-\frac{1}{4}\hat{q}(k)^2 + kx_p^2 - B\hat{q}(k) - \frac{\lambda^2}{4\epsilon_2}\frac{\hat{q}(k)^2}{\hat{p}(k)} + c_m\hat{q}(k) + \sigma_b^2\hat{p}(k)\right)$$

If $\epsilon_1 = \epsilon_0 = c_m = k = \sigma_b^2 = 0$ then we can show that $\hat{r}(k) = \hat{q}(k)^2/4\hat{p}(k)$ and $\hat{p}(k) = 2\delta_2 - \beta - \lambda^2/\epsilon_2$ which is the solution derived by Boulier et al. (1995). The general solution is of the form:

$$
\begin{aligned}
F(x) &= \hat{p}(k)(x - x_m)^2 + \hat{r}'(k) \\
\text{where } x_m &= -\frac{\hat{q}(k)}{2\hat{p}(k)} \\
\hat{r}'(k) &= \hat{r}(k) - \hat{p}(k)x_m^2 \\
\Rightarrow p^*(t, x) &= p^*(x)
\end{aligned}
$$

$$= \frac{\left(\lambda + \frac{1}{2}\epsilon_1\right)}{\epsilon_2} \frac{(\tilde{x}_m - x)}{x}$$

$$\text{where} \quad \tilde{x}_m = \frac{\lambda}{\lambda + \frac{1}{2}\epsilon_1} x_m$$

$$\text{or} \quad p^*(t, x) = \frac{p_0^* + p_1^* x}{x}$$

$$\text{where} \quad p_0^* = \frac{\lambda x_m}{\epsilon_2}$$

$$p_1^* = -\frac{\lambda + \frac{1}{2}\epsilon_1}{\epsilon_2}$$

$$\text{Also} \quad C^*(t, x) = C^*(x)$$

$$= (c_m + \hat{p}(k)x_m) - \hat{p}(k)x$$

Under normal circumstances p_1^* will be negative. p_1^* can only be positive if ϵ_1 is sufficiently negative.

It can be noted that the amortization rate depends on k, δ_2, λ, ϵ_0, ϵ_1 and ϵ_2 but not on c_m, x_p or σ_b^2. c_0 on-the-other-hand also depends on x_p and c_m but again not on σ_b^2.

It will be shown in the next section that a condition for stationarity of the fund size under the optimal strategy is

$$\hat{p}(k) > \delta_2 + p_1^* \lambda - \frac{1}{2}\gamma$$

$$\text{where} \quad p_1^* = -\frac{\lambda + \frac{1}{2}\epsilon_1}{\epsilon_2}$$

$$\text{and} \quad \gamma = ((1 + p_1^*)\sigma_{11} - p_1^*\sigma_{21})^2 + ((1 + p_1^*)\sigma_{12} - p_1^*\sigma_{22})^2$$

($\sqrt{\gamma}$ is the asymptotic volatility of the fund size as $|x|$ gets very large.)

Thus it has been possible to generalise significantly the results of Boulier *et al.* (1995) while still retaining the simple linear forms of the control functions $C^*(x)$ and $p^*(x).x$.

4 The stationary distribution of $X(t)$

Assume now that

$$\begin{aligned} C(t,x) &= c_0 - c_1 x \\ p(t,x) &= \frac{p_0 + p_1 x}{x} \end{aligned}$$

The dynamics of the fund size, $X(t)$, are then

$$\begin{aligned} dX &= (p_1 X + p_0)d\delta_1 + ((1-p_1)X - p_0)d\delta_2 + (c_0 - c_1 X - B)dt + \sigma_b dZ_b \\ &= (c_0 - B + p_0\lambda)dt - (c_1 - \delta_2 - p_1\lambda)X dt \\ &\quad + [(p_1 X + p_0)\sigma_{11} + ((1-p_1)X - p_0)\sigma_{21}]\,dZ_1 \\ &\quad\quad + [(p_1 X + p_0)\sigma_{12} + ((1-p_1)X - p_0)\sigma_{22}]\,dZ_2 + \sigma_b dZ_b \\ &\overset{d}{=} \mu.dt - \nu X.dt + (\alpha + \beta X + \gamma X^2)^{1/2}dZ \end{aligned}$$

where $Z(t)$ is a standard Brownian Motion and

$$\begin{aligned} \lambda &= \delta_1 - \delta_2 \\ \mu &= c_0 - B + p_0\lambda \\ \nu &= c_1 - \delta_2 - p_1\lambda \\ \alpha &= p_0^2\left[(\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2\right] + \sigma_b^2 \\ &= \epsilon_2 p_0^2 + \sigma_b^2 \\ \beta &= 2p_0\left[(\sigma_{11} - \sigma_{21})(p_1\sigma_{11} + (1-p_1)\sigma_{21}) + (\sigma_{12} - \sigma_{22})(p_1\sigma_{12} + (1-p_1)\sigma_{22})\right] \\ &= p_0(2\epsilon_2 p_1 + \epsilon_1) \\ \gamma &= (p_1\sigma_{11} + (1-p_1)\sigma_{21})^2 + (p_1\sigma_{12} + (1-p_1)\sigma_{22})^2 \\ &= \epsilon_2 p_1^2 + \epsilon_1 p_1 + \epsilon_0 \end{aligned}$$

It was shown in Cairns (1996) that stochastic differential equations of this type have a stationary distribution with the density function

$$\begin{aligned} f(x) &= K \exp\left[2a \tan^{-1}\frac{x+b}{c}\right](\alpha + \beta x + \gamma x^2)^{-1-\nu/\gamma} \\ &\qquad\qquad\qquad \text{for } -\infty < x < \infty \\ \text{where} \quad a &= \frac{1}{\sqrt{4\alpha\gamma - \beta^2}}\left(\frac{\nu\beta}{\gamma} + 2\mu\right) \end{aligned}$$

$$b = \frac{\beta}{2\gamma}$$

$$c = \frac{\sqrt{4\alpha\gamma - \beta^2}}{2\gamma}$$

K is a normalising constant which ensures that $f(x)$ integrates over the whole of the real line to 1.

We note the following points:

- This is known as the Pearson Type IV distribution. $f(x)$ is the solution to ordinary differential equation

$$\frac{f'(x)}{f(x)} = -\frac{k_3 + x}{k_0 + k_1 x + k_2 x^2}$$

$$\text{where} \quad k_0 = \frac{\alpha}{2(\gamma + \nu)}$$

$$k_1 = \frac{\beta}{2(\gamma + \nu)}$$

$$k_2 = \frac{\gamma}{2(\gamma + \nu)}$$

$$k_3 = \frac{\beta - 2\mu}{2(\gamma + \nu)}$$

This distribution has four degrees of freedom. The fifth degree of freedom used in the dynamics of the fund size determines the speed of the process.

- Following the notation of Johnson *et al.* (1994) we define $\mu'_r = E[X^r]$. We define $\mu'_{-1} = 0$ and have $\mu'_0 = 1$. It is easy to show that the μ'_r satisfy the following recursive relationship:

$$-k_0 r \mu'_{r-1} + (k_3 - (r+1)k_1)\mu'_r + (1 - (r+2)k_2)\mu'_{r+1} = 0$$

Hence we have

$$E[X] = \mu'_1$$

$$
\begin{aligned}
&= \frac{k_1 - k_3}{1 - 2k_2} \\
&= \frac{\mu}{\nu} \\
E[X^2] &= \mu_2' \\
&= \frac{k_0 + (2k_1 - k_3)\mu/\nu}{1 - 3k_2} \\
&= \frac{\alpha\nu + \beta\mu + 2\mu^2}{\nu(2\nu - \gamma)} \\
\Rightarrow Var[X] &= \frac{\alpha\nu + \beta\mu + 2\mu^2}{\nu(2\nu - \gamma)} - \frac{\mu^2}{\nu^2} \\
&= \frac{\nu^2\left(\alpha + \beta\frac{\mu}{\nu} + \gamma\frac{\mu^2}{\nu^2}\right)}{\nu^2(2\nu - \gamma)} \\
&= \frac{\alpha + \beta E[X] + \gamma E[X]^2}{2\nu - \gamma} \\
E[C(X)] &= c_0 - c_1 E[X] \\
&= c_0 - c_1\frac{\mu}{\nu} \\
Var[C(X)] &= c_1^2 Var[X] \\
&= c_1^2 \frac{\alpha + \beta E[X] + \gamma E[X]^2}{2\nu - \gamma}
\end{aligned}
$$

- The stationary distribution exists if and only if $2(1 + \nu/\gamma) > 1$: that is $c_1 > \delta_2 + p_1\lambda - \frac{1}{2}\gamma$ where $\gamma^{1/2}$ is the asymptotic volatility as $|x| \to \infty$.

  Similarly $E[X]$ exists if and only if $c_1 > \delta_2 + p_1\lambda$ and $Var[X]$ exists if and only if $c_1 > \delta_2 + p_1\lambda + \frac{1}{2}\gamma$.

  These results hold independent of the sign of p_1. For example, the optimal solution in Section 3 typically gives a negative value for p_1.

5 Numerical examples

We consider here an example in which the following parameters are fixed:

$$\delta_1 = 0.06, \delta_2 = 0.04, \quad \sigma_{11} = 0.2, \sigma_{12} = \sigma_{21} = 0.01, \sigma_{22} = 0.02, \quad B = 1$$

In Tables 1 (Dynamic optimisation) and 2 (static optimisation) below we give the values of the input parameters (c_m, k, x_p and β), the optimal values of p_0,

p_1, c_0 and c_1, and the mean and standard deviation of the stationary fund size and the contribution rate.

Table 1
Dynamic Optimisation

Ex.	c_m	k	x_p	β	σ_b	p_0	p_1	c_0	c_1	$E[X]$	$S.D.[X]$	$E[C]$	$S.D.[C]$
1	0.3	0	~	0.03	0.1	9.8	-0.60	0.96	0.037	16.9	4.3	0.329	0.161
2	0.3	0	~	0.04	0.1	9.8	-0.60	0.79	0.027	∞	∞	∞	∞
3 (*)	0.3	0.001	25	0.04	0.1	11.2	-0.60	1.28	0.048	24.7	4.1	0.085	0.198
4	0.3	0.01	25	0.04	0.1	12.6	-0.60	2.91	0.115	24.9	1.4	0.051	0.162
5	0.2	0.001	25	0.04	0.1	12.1	-0.60	1.26	0.048	24.6	3.5	0.068	0.167
6	0.3	0.001	20	0.04	0.1	10.2	-0.60	1.19	0.048	19.7	2.5	0.245	0.122
7	0.3	0.001	25	0.04	0.2	11.2	-0.60	1.28	0.048	24.7	4.2	0.085	0.204

Table 1
Static Optimisation

Ex.	c_m	k	x_p	β	σ_b	p_0	p_1	c_0	c_1	$E[X]$	$S.D.[X]$	$E[C]$	$S.D.[C]$
1	0.3	0	~	0.03	0.1	9.8	-0.60	1.50	0.067	17.5	1.4	0.313	0.097
2	0.3	0	~	0.04	0.1	9.8	-0.60	1.50	0.067	17.5	1.4	0.313	0.097
3 (*)	0.3	0.001	25	0.04	0.1	10.8	-0.60	1.86	0.080	20.6	1.5	0.206	0.119
4	0.3	0.01	25	0.04	0.1	12.4	-0.60	3.42	0.139	24.0	1.2	0.080	0.160
5	0.2	0.001	25	0.04	0.1	11.8	-0.60	1.91	0.080	22.1	1.5	0.145	0.123
6	0.3	0.001	20	0.04	0.1	10.1	-0.60	1.76	0.080	18.5	1.3	0.279	0.104
7	0.3	0.001	25	0.04	0.2	10.8	-0.60	1.86	0.080	20.6	1.6	0.206	0.127

5.1 Notes on the numerical examples

- Examples 1 and 2 show the effect of changing the risk-discount rate β. Note how the mean and variance in the dynamic case become infinite as β increases. However, when $\beta = 0.04$ the dynamic optimum still has a stationary distribution, albeit with infinite moments. There is no effect, obviously, on the static case.

  When $k = 0$ we can also see that the optimal asset-allocation strategies for the dynamic and static cases are the same and do not depend upon β. We also see that $c_1^D = c_1^S - \beta$ if $k = 0$.

  As β tends to 0 the optimal dynamic solutions converge to the same values as the optimal static solution. The effect of β is therefore to suppress variance in the short term through a lower value of c_1. A low value of c_1 may reduce variance in the short term but it increases it in the long run by allowing fluctuations in the fund size to persist.

In example 1we also see that $E[C] > c_m$. This reflects that fact that the minimum variance of C falls as $E[C]$ increases (and $E[X]$ falls).

- In example 1, the fund is invested 100% in asset 1 when X equals about 6 below this the fund goes long in asset 1 and short in asset 2. Conversely, when X reaches just above 16 the fund has 100% in asset 2, with a long position in asset 2 and a short position in asset 1 when X goes above this.

  Similar ranges apply for each of the other examples (most are shifted upwards slightly).

- Examples 2, 3 and 4 show the effect of increasing k. This shifts the emphasis onto reducing the variance of the fund size rather than of the contribution rate. The principle effect is that c_1 increases with k: that is, surplus or deficit is amortised more quickly. The changes in p_0 and c_0 are primarily a knock on effect.

- Examples 3 and 5 demonstrate the consequences of changing c_m. p_1 remains unchanged as it does throughout. The changes in the remaining control parameters have the effect of shifting the mean values principally but also affect the variances.

- Examples 3 and 6 consider the effect of changing the target fund size x_p. There is no change in c_1 or p_1. p_0 and c_0 change in order to shift the mean fund size. The variance falls because the target fund size is being moved towards the more natural mean observed in Example 1. This also relaxes considerably the tension on the mean contribution rate since a target fund size of 25 is not consistent with a target contribution rate of 0.3.

- Examples 3 and 7 show the influence of the uncertainty in the level of benefit outgo. The increases in the variances are small indicating that at this level ($\sigma_b = 0.1$ or 0.2) the main source of variability in the contribution rate is due to investment risk.

- The stationary distributions for the fund size for the dynamic and the static optima in example 3 are plotted in Figure 1. The equivalent distributions for the contribution rate are plotted in Figure 2. The differences in the mean values are quite clear as is the fact that the dynamic solution gives rise to a much higher long-term variance. In other examples
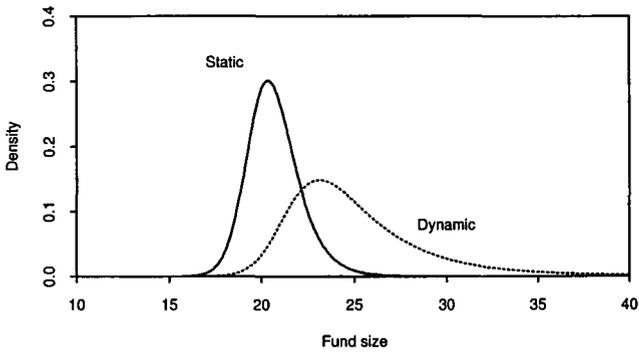
Figure 1: Example 3: Comparison of the stationary distribution of the fund sizes for the dynamic and static optimal solutions.
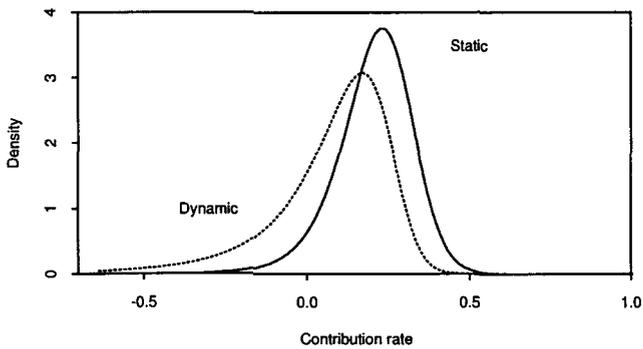

Figure 2: Example 3: Comparison of the stationary distribution of the contribution rates for the dynamic and static optimal solutions.

(4 and 6) the differences between the dynamic and static solutions are less marked.

6 Conclusions

This paper has considered the optimal control of a pension fund using the asset-allocation strategy and the contribution strategy.

Using a quadratic loss function optimal solutions for both the dynamic and static cases have been considered. In most cases the contribution strategy appears to be sensible and conforms with current practice. In most cases also, the optimal asset-allocation strategy derived is very counter-intuitive: moving, say, out of equities into bonds when the level of surplus is growing. A typical fund in the UK, for example, will have between 60% and 80% of the fund invested in risky assets such as equities.

This has one of two explanations. Funds may be operating in a very non-optimal way. Alternatively, they may be operating optimally but with different objectives. For example, in the UK, the government has recently introduced minimum funding legislation. This should lead to loss functions which heavily penalise events when the fund size falls below the legal minimum. Boulier *et al.* (1996) considered a related problem in which the contribution rate was subject to an upper constraint (say, twice the target rate). However, in the present framework (in which all assets are risky and where there is volatility in the benefit outgo) it is not possible to constrain the contribution rate in this way, for otherwise the fund size would ultimately drift off to minus infinity.

There is, however, some sense in a shift out of equities if the fund size is well above its target level. First if there is too much surplus then there will be pressure on the sponsoring employer to use this surplus to pay for discretionary pension increases which, perhaps, had not been promised. In any event the members would be benefitting from good investment returns while the employer has to pay when things go badly. Second if the employer is able to take a refund, the refund may be liable to tax (for example, in the UK this is 40% with the aim of inhibiting exploitation of the tax advantages enjoyed by a pension fund). Third, too much surplus may lead to the removal of part or all of the fund's special tax status (again this is the case in the UK). All of these reasons mean that it should be advantageous to put a bigger proportion of the fund into low-risk assets when the fund has a large surplus. The results described in this paper back up this viewpoint.

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