PRICING CONTINGENT-CLAIMS CONCERNING THE ITALIAN PENSION PLAN*

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Abstract

The object of this paper is to present a valuation model suitable for pricing contingent-claims intervening in pension schemes. To this end we adopt the multifactor model of Cox, Ingersoll and Ross for describing the stochastic evolution of real and nominal interest rates, price level, expected inflation rate and, in a nominal setting, we introduce two further state-variables: the individual salary and the value of an index such as, for instance, the Gross Domestic Product, whose performance is taken into account for determining the interest rate credited to the accumulated contributions. We apply then this framework to the valuation of particular contingent-claims recently introduced in Italy by a new legislation reforming the National Pension Plan.

Keywords: pension plans, contingent-claims, greater of benefits

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1. Introduction

A new legislation reforming the National Pension Plan came in force in Italy in August 1995. This reform sanctions the transition from a defined-benefit to a defined-contribution scheme and, as is the custom, states a set of transitional provisions in order to safeguard vested rights. In particular, only people entering the pension scheme after the 31st of December 1995 are entirely subject to the new system and, more precisely, they are entitled to an initial annual pension proportional to the value of contributions paid during their whole working life. The value of these contributions, that are proportional to salaries, is obtained by accruing interests at a rate exclusively depending on the performance of the nominal Gross Domestic Product (GDP). The initial pension instalment obtained in this way is subsequently adjusted according to the price level.

People that at 31-12-1995 have at least 18 years of service are entirely subject to the old system, i.e. they are entitled to a defined benefit. In this case the initial annual pension, subsequently adjusted according to the price level, depends on the length of service and on an average of a certain number of last salaries adjusted according to the price level as well. Finally, employees already working at 31-12-1995 but with less than 18 years of service are subject to a mixed system; their pension is constituted by two portions: the first one, referred to the years of service until 31-12-1995, is computed according to the old system, while the second one is computed on a contributory basis.

However, these last two classes of transient persons can switch to the new system, i.e. they can opt for a totally contributory pension, provided that their length of service is at least 15 years, 5 of which (at least) after 31-12-1995. Since the rate of contribution is anyway the same in both systems, this option leads a rational and non-satiated employee to choice the scheme which grants the greater value of the benefits promised.

A similar "Greater Of Benefits" feature (GOB henceforth), characterizing particular pension plans in force in some foreign countries such as, for instance, Australia, Canada and USA, was also studied by Bell and Sherris (1991), Britt (1991), Sherris (1993, 1995), Cohen and Bilodeau (1996). All these authors, under different assumptions and by various methodologies, approached the problem of "valuing" the financial option embedded in such provision. In particular Sherris, applying the contingent claims valuation approach, obtains a partial differential equation for the GOB value, and then solves it by numerical techniques.

The object of this paper is to price the GOB option which an individual belonging to the first class of transient persons (i.e. with more than 18 years of service at 31-12-1995) is entitled to. We are particularly interested in determining the price of such an option because it measures the additional cost that the Italian Government has to bear for allowing a given individual of this transient class to choice the new system instead of forcing him/her to remain in the old one. This additional cost could indeed make vain, at least in a transition's period of above twenty years, one of the purposes of the reform, that is to reduce the Social Security deficit.

Coming now to our specific problem and using the same words of Margrabe (1978), it is just a matter of valuing an option to exchange one asset (the defined-benefit pension) for another (the defined-contribution pension). To this end we adopt the multifactor model of Cox, Ingersoll and Ross (1985; see in particular Moriconi, 1995) for describing the stochastic evolution of real and nominal interest rates, price level, expected inflation rate and, in a nominal setting, we introduce two further state-variables: the individual salary and the GDP. Assuming neutrality with respect to the demographic risk (i.e. with respect to mortality, disability, withdrawal, survivors, and so on) and independence of all demographic variables from the financial and economic ones we obtain, by exploiting the martingale approach, a valuation formula for the GOB option that can be estimated by means of Monte Carlo simulation.

A very crucial problem concerns the particular moment(s) at which the option can be exercised. If indeed the option is of European style with maturity the retirement date, the assets to be exchanged consist exclusively of the retirement pension (reversible to survivors). If instead the option, although of European style, matures before the retirement date, the assets are more complex since they include disability, death and withdrawal benefits; the same happens if the option is of American style, i.e. if it can be exercised in any moment on or before its maturity, whatever this maturity is. Unfortunately on this subject nothing is clear at the moment (January 1997), since the Italian Government has still to issue a decree for regulating the matter. For this reason we consider, in the paper, two cases, in both of which we assume that the option is of European style, i.e. that it can be exercised only at maturity. In the first case we assume that it matures at retirement, in the second one that it matures before retirement and after the moment in which the employee acquires the option right, i.e. after 31-12-2000.

The paper is structered as follows. In Section 2 we introduce our notation and assumptions concerning the economic framework. Section 3, where we describe the complete model and derive our valution formula, is divided into two subsections: in Section 3.1 we treat the case in which the GOB option matures at retirement, while in Section 3.2 we deal with the more complex case in which its maturity precedes retirement. Section 4 concludes the paper.

2. The economic framework

In this section we introduce our notation and assumptions concerning the economic framework. Since we assume that all demographic variables are independent from the economic ones, we delay instead until the next sections the definitions and notation concerning life contingencies.

We consider the following state-variables:

$e(t)$ real instantaneous interest rate
$r(t)$ nominal instantaneous interest rate
$i(t)$ expected instantaneous inflation rate
$p(t)$ price level
$s(t)$ nominal instantaneous individual salary
$g(t)$ nominal GDP.

We assume that markets are populated by rational and non-satiated agents, all sharing the same information, and that all traded securities are perfectly divisible and default-risk-free. Moreover, let markets be perfectly competitive, frictionless and free of arbitrage opportunities. We assume the existence of index-linked and nominal bonds that permit hedging against inflation risk and interest-rate risk. Then, if we imagine that financial instruments allowing for hedging against long-term individual income risk and GDP risk such as, for instance, perpetual or Peter Pan futures are traded[1], we can add completeness to the above assumptions about markets. Being markets complete and arbitrage-free, there exists a unique martingale measure Q, equivalent to the objective one, under which, in a nominal setting, all discounted prices are martingales. We assume that, under this risk-neutral measure Q, the state-variables previously introduced stochastically behave as follows:

(2.1) $$de(t) = [a_e + b_e e(t)]\, dt + \sigma_e \sqrt{e(t)}\, dZ_e(t), \qquad a_e, \sigma_e > 0$$

(2.2) $$di(t) = [a_i + b_i i(t)]\, dt + \sigma_i \sqrt{i(t)}\, dZ_i(t), \qquad a_i, \sigma_i > 0$$

(2.3) $$\frac{dp(t)}{p(t)} = (1 - \sigma_p^2)\, i(t)\, dt + \sigma_p \sqrt{i(t)}\, dZ_p(t), \qquad 0 < \sigma_p < 1$$

(2.4) $$r(t) = e(t) + (1 - \sigma_p^2)\, i(t)$$

(2.5) $$\frac{ds(t)}{s(t)} = [a_s + b_s i(t)]\, dt + \sigma_s\, dZ_s(t), \qquad b_s, \sigma_s > 0$$

(2.6) $$\frac{dg(t)}{g(t)} = [a_g + b_g i(t)]\, dt + \sigma_g\, dZ_g(t), \qquad b_g, \sigma_g > 0,$$

where a_e, b_e, σ_e, a_i, b_i, σ_i, σ_p, a_s, b_s, σ_s, a_g, b_g, σ_g are real constants and Z_e, Z_i, Z_p, Z_s, Z_g are standard brownian motions. We assume the infinitesimal increments of any couple of such standard brownian motions to be uncorrelated, except for

(2.7) $$\mathrm{cov}_t[dZ_i(t), dZ_p(t)] = \rho_{ip} dt$$

(2.8) $$\mathrm{cov}_t[dZ_s(t), dZ_g(t)] = \rho_{sg} dt\,,$$

[1] For a description of these particular contracts we refer to Shiller (1996) and to the references there included.

where ρ_{ip} and ρ_{sg} are constants between -1 and 1 and cov_t represents covariance taken with respect to the risk-adjusted measure Q and conditional on information up to time t.

We point out that relations (2.1) to (2.4), besides being compatible with the three-factor model of Cox, Ingersoll and Ross (1985), describe the stochastic behaviour of e, i, p, r under the risk-adjusted measure Q. The same behaviour, but under the objective measure, is described in Moriconi (1995) where, in particular, the drift of $\frac{dp(t)}{p(t)}$ represents, as is natural, the expected inflation rate i(t). In a risk-neutral world, instead, this drift must be adjusted to $(1-\sigma_p^2)\, i(t)$.

Exploiting relations (2.1), (2.2) and (2.4) we can also derive a stochastic differential equation for the nominal interest rate r(t), and rewrite all the relations relevant in the valuation model described in Section 3 in terms of a 5-dimensional standard brownian motion (i.e. with independent components) $Z = (Z_1, Z_2, Z_3, Z_4, Z_5)$:

$$(2.9) \quad dr(t) = [a_r + b_e r(t) + (b_i - b_e)(1-\sigma_p^2) i(t)]\, dt + \sigma_e \sqrt{r(t) - (1-\sigma_p^2) i(t)}\, dZ_1(t) +$$

$$+ \sigma_i (1-\sigma_p^2)\sqrt{i(t)}\, dZ_2(t), \quad \text{where } a_r = a_e + a_i\,(1-\sigma_p^2),\ Z_1 = Z_e,\ Z_2 = Z_i$$

$$(2.10) \quad di(t) = [a_i + b_i\, i(t)]\, dt + \sigma_i \sqrt{i(t)}\, dZ_2(t)$$

$$(2.11) \quad \frac{dp(t)}{p(t)} = (1-\sigma_p^2)\, i(t)\, dt + \sigma_p \sqrt{i(t)}\, \left[\rho_{ip}\, dZ_2(t) + \sqrt{1-\rho_{ip}^2}\, dZ_3(t)\right],$$

$$\text{where} \quad dZ_3 = \frac{dZ_p - \rho_{ip}\, dZ_i}{\sqrt{1-\rho_{ip}^2}}$$

$$(2.12) \quad \frac{ds(t)}{s(t)} = [a_s + b_s\, i(t)]\, dt + \sigma_s\, dZ_4(t), \quad \text{with } Z_4 = Z_s$$

$$(2.13) \quad \frac{dg(t)}{g(t)} = [a_g + b_g\, i(t)]\, dt + \sigma_g \left[\rho_{sg}\, dZ_4(t) + \sqrt{1-\rho_{sg}^2}\, dZ_5(t)\right],$$

$$\text{with} \quad dZ_5 = \frac{dZ_g - \rho_{sg}\, dZ_s}{\sqrt{1-\rho_{sg}^2}}\ .$$

To conclude this section, we report the Feynman-Kac representation for the price at time t, $\Pi_t(X_T)$, of a finite-variance contingent-claim that delivers, at a future date T, a random payoff X_T functionally dependent on the values up to time T of the state-variables here introduced:

$$(2.14) \quad \Pi_t(X_T) = E_t\left[X_T\, e^{-\int_t^T r(u)du}\right],$$

where E_t denotes expectation taken with respect to the equivalent martingale measure Q and conditional on information up to time t.

This fundamental equation, expressible in closed form only for very particular and simple forms of the payoff X_T but anyway suitable to be approached via Monte Carlo simulation, will be the building block in all our valuation framework.

3. The valuation model

In this section we describe our model and, in particular, we define the benefits provided by the two alternative schemes. To this end we consider an active employee aged x at the present date, time t, and denote by t_0 the date at which the employee was engaged for the first time, t_1 the 31st of December 1995, t_2 the maturity of the GOB option and T the retirement date, so that $t_0 < t_1 < t < t_2 \leq T$. We fix 1 year as the unit of measure on the time axis and assume, more precisely, that $t_1 - t_0 \geq 18$, $t_2 - t_1 \geq 5$, $T - t_0 \leq 40$. These assumptions and, in particular, the first and the second inequality, identify the first class of transient employees who can opt for receiving a totally defined-contribution pension instead of a totally defined-benefit one.

To simplify the presentation of our model and, at the same time, to avoid going too deeply into the labyrinth of rules governing the Italian Pension Plan, we omit considering some minor details. In particular we do not take into account the fact that the exercise of the GOB option (i.e. the choice of the contributory scheme) may modify the retirement age since the requirements for retirement in the new system are different from those in the old one. Secondly, we disregard the fact that the employees here considered are transient also with respect to a preceding reform (come in force only on the 1rs of January 1993!) and their initial defined benefit is actually composed by two portions, characterized by a different number of years taken into account for computing the average of the last salaries and a different adjustment mechanism for these salaries as well. Thirdly, taking into account that most employees have a salary widely below both the contributory cap in the new system and the retributory cap in the old one, we ignore the presence of both these caps, as well as the presence of a minimum amount guaranteed for the pension instalment ("social" pension). Finally, we act in a theoretical continuous-time world, without considering the fact that any kind of adjustment, in both systems, and/or of interests accumulation, takes actually place with a certain delay and, in particular, stops at the end of the calendar year preceding the retirement (or disability, death, withdrawal) date.

As already declared, we assume neutrality with respect to the demographic risk and independency between the demographic variables and the economic ones introduced in the previous section. We assume moreover that the employee under consideration has a consort aged y (at the present time t), whose survival is independent from his/her life contingencies.

For simplicity, we do not consider either the case in which the employee has or will have children to provide for, or the cases of divorce and remarriage after widowhood.

Let now introduce the following demographic functions, all depending on the age:

- $\ell^{a}(\cdot)$ working life table
- $\ell^{e}(\cdot)$ survival function for the employee
- $\ell^{c}(\cdot)$ survival function for his/her consort
- $\ell^{i}(\cdot)$ survival function for a permanent invalid of the same sex of the employee
- $\mu^{d}(\cdot)$ instantaneous intensity of death for the employee
- $\mu^{i}(\cdot)$ instantaneous intensity of permanent disability for the employee.

As already implicitly said, we assume that retirement takes place with certainty at time T if the employee is still active at the same date, so that we disregard early or delayed retirement.

In what follows we consider two cases: in the first one, treated in Section 3.1, we assume that the maturity of the GOB option coincides with the retirement date, i.e. that $t_2=T$, while in Section 3.2 we deal with the assumption $t_2<T$.

3.1 Valuation of the GOB option with maturity at retirement

Let now consider the case in which the GOB option (of European style) matures at time T, the retirement date, and denote by $B^{r}(T,T)$ the initial instantaneous retirement pension instalment in the old system (i.e. in the defined-benefit scheme). We assume that this quantity is proportional, according to a constant rate α, to the number of years of service, $T-t_0$, and to the arithmetical average of the adjusted salaries earned by the employee in the last n years (with $n \le T-t_0$). We assume moreover that the instantaneous adjustment rate for these last n salaries coincides with the realized inflation rate measured by the variation in the price level, increased by a constant rate β. We have then

$$(3.1.1)\quad B^{r}(T,T) = \alpha \cdot (T-t_0) \cdot \frac{1}{n}\int_{T-n}^{T} s(z)\ \frac{p(T)}{p(z)}\ e^{\beta(T-z)}\,dz\ .$$

In order to define the same quantity in the defined-contribution scheme, we denote by M(k) the value at time k of the accumulated contributions up to that time. We assume that these contributions are proportional to salaries and denote by γ the rate of contribution. We assume moreover that accrued interests are computed at the GDP variation rate. We have then:

$$(3.1.2)\quad M(k) = \gamma \int_{t_0}^{k} s(z)\ \frac{g(k)}{g(z)}\ dz$$

or alternatively, taking into account that at the present date t the value M(t) is obviously known:

(3.1.3) $M(k) = M(t)\frac{g(k)}{g(t)} + \gamma\int_t^k s(z)\,\frac{g(k)}{g(z)}\,dz, \qquad k \geq t.$

The initial instantaneous defined-contribution pension instalment, $C^r(T,T)$, is obtained by multiplying the value of contributions accumulated until retirement, M(T), for a coefficient "transforming" it into a life annuity. This coefficient is a function of the age of the employee at retirement and depends also on his/her sex. Since the sex of the employee under scrutiny is anyway assumed to be known, we consider only the dependence on age and denote by $\delta(\cdot)$ such a function. We have then

(3.1.4) $C^r(T,T) = \delta(x+T-t)\cdot M(T)$.

We assume that, in both systems, the initial instantaneous pension instalment is subsequently adjusted according to the price level, so that its measure at time h, $B^r(T,h)$ ($C^r(T,h)$ respectively), is given by

(3.1.5) $K^r(T,h) = K^r(T,T)\,\frac{p(h)}{p(T)}, \quad K=B,C \text{ and } h > T,$

whose market value at retirement, $\Pi_T(B^r(T,h))$ (and $\Pi_T(C^r(T,h)$ respectively), is obtained by applying the pricing formula (2.14) of Section 2:

(3.1.6) $\Pi_T(K^r(T,h)) = E_T\left[K^r(T,h)\,e^{-\int_T^h r(u)du}\right], \qquad K=B,C.$

The market value at time T of the retirement annuity (reversible to the widow(er)) depends on life contingencies of the employee and his/her consort. If indeed the employee is no longer active at time T being withdrawn, become invalid or dead before this date, the annuity is obviously valueless. If instead the employee is still active at time T, but widow(er), our previous no-remarriage assumption implies that its value depends only on the survival function of the employee. Denoting this value by $B^r_w(T)$ in the defined-benefit scheme and by $C^r_w(T)$ in the defined-contribution one, we have in fact

(3.1.7) $K^r_w(T) = \int_T^{+\infty} \Pi_T(K^r(T,h))\,\frac{\ell^e(x+h-t)}{\ell^e(x+T-t)}\,dh, \qquad K=B,C,$

where $\frac{\ell^e(x+h-t)}{\ell^e(x+T-t)}$ represents the probability that the employee, alive (since active) at time T and aged x+T–t, is still so at time $h \geq T$ (aged x+h–t). Finally, if the employee is still active and married at time T, he/she will receive, at time h > T, the whole pension instalment $B^r(T,h)$

(and $C^r(T,h)$ respectively) if alive, while his (her) widow(er) will receive only a portion, w, of this instalment. We have then in this case, for the price $B^r_m(T)$ (or $C^r_m(T)$) of the pension:

$$(3.1.8)\quad K^r_m(T) = \int_T^{+\infty} \Pi_T(K^r(T,h)) \left\{ \frac{\ell^e(x+h-t)}{\ell^e(x+T-t)} + w \left[1 - \frac{\ell^e(x+h-t)}{\ell^e(x+T-t)}\right] \frac{\ell^c(y+h-t)}{\ell^c(y+T-t)} \right\} dh,$$

$$K=B,C,$$

where $\frac{\ell^c(y+h-t)}{\ell^c(y+T-t)}$ represents the probability that the consort is still alive at time h (aged y+h−t), being alive at time T (aged y+T−t).

Now we have all the elements to price, once again by means of the fundamental equation (2.14) of Section 2, the GOB option. In particular we recall that this option is valueless if the employee is no longer active at time T, else he/she will choice the scheme providing the GOB value, given by $\max\{B^r_w(T), C^r_w(T)\}$ in case of widowhood of the employee at time T, $\max\{B^r_m(T), C^r_m(T)\}$ otherwise. In both cases this GOB value can be decomposed in the following alternative ways:

$$(3.1.9)\quad \max\{B^r_k(T), C^r_k(T)\} = B^r_k(T) + \max\{0, C^r_k(T)-B^r_k(T)\}$$

$$= C^r_k(T) + \max\{B^r_k(T)-C^r_k(T), 0\}, \qquad k=w,m.$$

The first decomposition shows up the payoff at maturity ($\max\{0, C^r_k(T)-B^r_k(T)\}$, k=w,m) of what we have called so far GOB option, i.e. the option of Switching to the defined-Contribution scheme, whose value at the present date t, $\Pi_t(SC)$, is given by

$$(3.1.10)\quad \Pi_t(SC) = \frac{\ell^a(x+T-t)}{\ell^a(x)} \left\{ \left[1 - \frac{\ell^c(y+T-t)}{\ell^c(y)}\right] E_t\left[\max\{0, C^r_w(T)-B^r_w(T)\}\, e^{-\int_t^T r(u)du}\right] + \right.$$

$$\left. + \frac{\ell^c(y+T-t)}{\ell^c(y)} E_t\left[\max\{0, C^r_m(T)-B^r_m(T)\}\, e^{-\int_t^T r(u)du}\right] \right\},$$

where $\frac{\ell^a(x+T-t)}{\ell^a(x)}$ represents the probability that the employee, active at the present time t (aged x), is still so at the retirement age x+T−t, while $\frac{\ell^c(y+T-t)}{\ell^c(y)}$ represents the probability that his/her consort, alive at time t (aged y), is still so at time T.

We recall that $\Pi_t(SC)$ measures the extra cost faced by the Italian Government for allowing the employee under scrutiny to choice the new system instead of forcing him/her to stay in the old one. It is quite evident that the same arguments lead to obtain the time t price of the GOB, $\Pi_t(GOB)$, given by

$$(3.1.11)\ \Pi_t(GOB) = \frac{\ell^a(x+T-t)}{\ell^a(x)} \Big\{ \Big[1 - \frac{\ell^c(y+T-t)}{\ell^c(y)}\Big] E_t\Big[\max\{B^r_w(T), C^r_w(T)\}\, e^{-\int_t^T r(u)du}\Big] + \frac{\ell^c(y+T-t)}{\ell^c(y)} E_t\Big[\max\{B^r_m(T), C^r_m(T)\}\, e^{-\int_t^T r(u)du}\Big] \Big\}.$$

We observe that it may be interesting to quantify also the value of the contingent-claim which the employee would have been entitled to if the Government had directly subjected him/her to the new system, leaving however the option of remaining in the old one. The value of this particular option, the option of Remaining in the defined-Benefit scheme, $\Pi_t(RB)$, whose final payoff ($\max\{B^r_k(T)-C^r_k(T), 0\}$, k=w,m) is shown up by the second decomposition in relation (3.1.9), is thus given by

$$(3.1.12)\ \Pi_t(RB) = \frac{\ell^a(x+T-t)}{\ell^a(x)} \Big\{ \Big[1 - \frac{\ell^c(y+T-t)}{\ell^c(y)}\Big] E_t\Big[\max\{B^r_w(T)-C^r_w(T), 0\}\, e^{-\int_t^T r(u)du}\Big] + \frac{\ell^c(y+T-t)}{\ell^c(y)} E_t\Big[\max\{B^r_m(T)-C^r_m(T), 0\}\, e^{-\int_t^T r(u)du}\Big] \Big\}.$$

To conclude this subsection, we remark that the linear homogeneity of all the quantities defined in relations (3.1.5) to (3.1.8) with respect to $K^r(T,T)$ with, in particular, the same coefficients for K=B and C, could lead to choice the GOB value simply by comparing the initial instalments $B^r(T,T)$ and $C^r(T,T)$ respectively. In order to price the GOB option (or the GOB value), and not simply to choice the more favourable system, it is anyway necessary to proceed along the lines described above. It is however possible, exploiting relations (3.1.5) to (3.1.9) and after some algebraic manipulations, to rewrite the pricing formulae (3.1.10) to (3.1.12) in the following ways:

$$(3.1.13)\ \Pi_t(SC) = \frac{\ell^a(x+T-t)}{\ell^a(x)} E_t\Big[\max\{0, C^r(T,T)-B^r(T,T)\}\, e^{-\int_t^T r(u)du} \cdot E_T\Big[\int_T^{+\infty} c(h)\,\frac{p(h)}{p(T)}\, e^{-\int_T^h r(u)du}\, dh\Big]\Big]$$

$$(3.1.14)\quad \Pi_t(GOB) = \frac{\ell^a(x+T-t)}{\ell^a(x)} E_t\Big[\max\{B^r(T,T), C^r(T,T)\}\, e^{-\int_t^T r(u)du} \cdot$$

$$\cdot E_T\Big[\int_T^{+\infty} c(h)\, \frac{p(h)}{p(T)}\, e^{-\int_T^h r(u)du}\, dh\Big]\Big]$$

$$(3.1.15)\quad \Pi_t(RB) = \frac{\ell^a(x+T-t)}{\ell^a(x)} E_t\Big[\max\{0, B^r(T,T)-C^r(T,T)\}\, e^{-\int_t^T r(u)du} \cdot$$

$$\cdot E_T\Big[\int_T^{+\infty} c(h)\, \frac{p(h)}{p(T)}\, e^{-\int_T^h r(u)du}\, dh\Big]\Big]$$

with $c(h) = \dfrac{\ell^e(x+h-t)}{\ell^e(x+T-t)} + w\left[1 - \dfrac{\ell^e(x+h-t)}{\ell^e(x+T-t)}\right] \dfrac{\ell^c(y+h-t)}{\ell^c(y)}$.

We point out that the numerical valuation of formulae (3.1.13) to (3.1.15) requires the computation of two-stages iterated conditional expectations under the risk neutral measure Q: in the first stage the expectation E_T is conditional on information up to the future time T, while in the second one E_t is conditional on information up to the present date t. Monte Carlo simulation should presumedly be the best way to approach this problem, even if we expect to get good results only after having consumed a considerable quantity of CPU time.

3.2 Valuation of the GOB option with maturity before retirement

In this subsection we consider the case in which the GOB option matures at time $t_2 < T$. As in the previous subsection, the option value depends first of all on life contingencies of the employee and his/her consort at maturity t_2. In particular it is valueless if the employee is no longer active at t_2, while it differs according to the facts that

(a) the employee is still active but widow(er) at t_2,

(b) the employee is still active and married at t_2.

Moreover, the exercise of the option at t_2 could imply different benefits not only in case of retirement (at time T) but also in cases of death, permanent disability or withdrawal occurred between times t_2 and T. To this purpose we disregard withdrawal because we deem it reasonable enough to assume that, if the employee changed job, he/she would remain member of the same Pension Plan, i.e. of the National Pension Plan, compulsory for all workers[2].

Let now start with considering the retirement benefits. The instantaneous pension instalment and its adjustment mechanism are of course the same described in the previous subsection so that, in particular, M(T), $B^r(T,h)$ and $C^r(T,h)$ with $h \geq t$, given by relations (3.1.1) to (3.1.5), remain unchanged. In order to quantify the final payoff of the GOB option we need however to price these instalments at maturity t_2 (instead of T), so that we have:

$$(3.2.1)\quad \Pi_{t_2}(K^r(T,h)) = E_{t_2}\Big[K^r(T,h)\, e^{-\int_{t_2}^{h} r(u)du}\Big], \qquad K=B,C.$$

The market values at t_2 of the retirement annuity in the defined-benefit and defined-contribution schemes are then, in case (a):

$$(3.2.2)\quad K^r_w(t_2) = \frac{\ell^a(x+T-t)}{\ell^a(x+t_2-t)} \int_T^{+\infty} \Pi_{t_2}(K^r(T,h))\, \frac{\ell^e(x+h-t)}{\ell^e(x+T-t)}\, dh, \qquad K=B,C,$$

where $\dfrac{\ell^a(x+T-t)}{\ell^a(x+t_2-t)} \cdot \dfrac{\ell^e(x+h-t)}{\ell^e(x+T-t)}$ represents the probability that the employee, being active at time t_2, remains so until the retirement date T and then survives at least until time h. In case (b) we have indeed:

$$(3.2.3)\quad K^r_m(t_2) = \frac{\ell^a(x+T-t)}{\ell^a(x+t_2-t)} \int_T^{+\infty} \Pi_{t_2}(K^r(T,h)) \Big\{ \frac{\ell^e(x+h-t)}{\ell^e(x+T-t)} + $$

$$ + w\Big[1 - \frac{\ell^e(x+h-t)}{\ell^e(x+T-t)}\Big] \frac{\ell^c(y+h-t)}{\ell^c(y+t_2-t)} \Big\}\, dh, \qquad K=B,C,$$

where $\dfrac{\ell^c(y+h-t)}{\ell^c(y+t_2-t)}$ represents the probability that the consort is still alive at time h being so at time t_2.

Let now imagine that the employee becomes invalid at time t_i between t_2 and T. In this case we assume that the initial instantaneous pension instalment in the defined-benefit scheme, $B^i(t_i,t_i)$, is computed as though the employee had worked until the retirement age, i.e.

$$(3.2.4)\quad B^i(t_i,t_i) = \alpha \cdot (T-t_0) \cdot \frac{1}{n}\int_{t_i-n}^{t_i} s(z)\, \frac{p(t_i)}{p(z)}\, e^{\beta(t_i-z)}\, dz\ ,$$

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2 Even if this were not true and the employee definitively left the Plan, one could assume that the benefit at withdrawal (consisting for instance in a refund of contributions) is the same in both schemes, so that the value of the GOB option is not affected by it (but of course the GOB value is!).

with $n \le t_i - t_0$. In the defined-contribution scheme we assume instead that the value $M(t_i)$ of contributions accumulated until time t_i, given by relation (3.1.2) or (3.1.3) of the previous subsection, is increased by $T-t_i$ annual contributions computed on the arithmetical average of the last m salaries (with $m \le t_i - t_0$), salaries adjusted at the same rate used for computing the analogous average in the defined-benefit scheme. Moreover, we assume that this value is transformed into an annuity using the same function δ introduced in Section 3.1 but computed, if more favourable to the employee, at a fixed age ξ (instead of $x+t_i-t$). We have then in this case, for the initial instantaneous pension instalment $C^i(t_i,t_i)$:

$$(3.2.5)\quad C^i(t_i,t_i) = \delta(\max\{\xi, x+t_i-t\}) \cdot \Big[M(t_i) + \gamma(T-t_i)\frac{1}{m}\int_{t_i-m}^{t_i} s(z)\,\frac{p(t_i)}{p(z)}\,e^{\beta(t_i-z)}\,dz\Big].$$

The initial instalment is then, in both schemes, adjusted at

$$(3.2.6)\quad K^i(t_i,h) = K^i(t_i,t_i)\,\frac{p(h)}{p(t_i)}\,, \qquad K=B,C \quad \text{and} \quad h > T\,,$$

whose value at time t_2 is given by

$$(3.2.7)\quad \Pi_{t_2}(K^i(t_i,h)) = E_{t_2}\Big[K^i(t_i,h)\,e^{-\int_{t_2}^{h} r(u)du}\Big], \qquad K=B,C.$$

Assuming that the disability annuity is reversible to the widow(er), and taking into account that disability may occur in any moment between t_2 and T, the market values at time t_2 of the annuity are, in cases (a) and (b) respectively:

$$(3.2.8)\quad K^i_w(t_2) = \int_{t_2}^{T} \frac{\ell^a(x+t_i-t)}{\ell^a(x+t_2-t)}\,\mu^i(x+t_i-t)\int_{t_i}^{+\infty} \Pi_{t_2}(K^i(t_i,h))\,\frac{\ell^i(x+h-t)}{\ell^i(x+t_i-t)}\,dh\,dt_i\,, \qquad K=B,C$$

$$(3.2.9)\quad K^i_m(t_2) = \int_{t_2}^{T} \frac{\ell^a(x+t_i-t)}{\ell^a(x+t_2-t)}\,\mu^i(x+t_i-t)\int_{t_i}^{+\infty} \Pi_{t_2}(K^i(t_i,h))\Big\{\frac{\ell^i(x+h-t)}{\ell^i(x+t_i-t)} + w\Big[1-\frac{\ell^i(x+h-t)}{\ell^i(x+t_i-t)}\Big]\frac{\ell^c(y+h-t)}{\ell^c(y+t_2-t)}\Big\}\,dh\,dt_i\,, \qquad K=B,C,$$

where $\frac{\ell^a(x+t_i-t)}{\ell^a(x+t_2-t)}\,\mu^i(x+t_i-t)\,dt_i$ represents the probability that the employee, active at time t_2, remains so until time t_i and then becomes invalid between times t_i and t_i+dt_i, while $\frac{\ell^i(x+h-t)}{\ell^i(x+t_i-t)}$ represents the probability that an invalid alive at time t_i (aged $x+t_i-t$) is still so at time h (aged $x+h-t$).

In case of death of the employee at time t_d, between t_2 and T, nothing is due if the employee was no longer married at time t_2. If instead there is still a consort at death, we assume that the initial instantaneous pension instalment in the defined-benefit scheme, $B^d(t_d,t_d)$, is equal to the portion w of the amount computed in the same way as in the retirement case, i.e., being $n \leq t_d - t_0$

$$(3.2.10)\quad B^d(t_d,t_d) = w \cdot \alpha \cdot (t_d - t_0) \cdot \frac{1}{n} \int_{t_d-n}^{t_d} s(z)\, \frac{p(t_d)}{p(z)}\, e^{\beta(t_d-z)}\, dz\ .$$

In the defined-contribution scheme we assume instead that the initial pension instalment, $C^d(t_d,t_d)$, is computed by applying the function δ to the age ξ, if greater than $x+t_d-t$, and multiplying the value so obtained for the value of contributions at death, $M(t_d)$, and for the reduction factor w. We have therefore:

$$(3.2.11)\quad C^d(t_d,t_d) = w \cdot \delta(\max\{\xi,\, x+t_d-t\}) \cdot M(t_d)\ .$$

Subsequently this instalment is adjusted, in both schemes, at

$$(3.2.12)\quad K^d(t_d,h) = K^d(t_d,t_d)\, \frac{p(h)}{p(t_d)}\ , \quad K=B,C \quad \text{and} \quad h > T\ ,$$

with value at time t_2:

$$(3.2.13)\quad \Pi_{t_2}(K^d(t_d,h)) = E_{t_2}\Big[K^d(t_d,h)\, e^{-\int_{t_2}^{h} r(u)du}\Big]\ , \qquad K=B,C.$$

The market value at time t_2 of the corresponding annuity is thus given by

$$(3.2.14)\quad K_m^d(t_2) = \int_{t_2}^{T} \frac{\ell^a(x+t_d-t)}{\ell^a(x+t_2-t)}\, \mu^d(x+t_d-t) \int_{t_d}^{+\infty} \Pi_{t_2}(K^d(t_d,h))\, \frac{\ell^c(y+h-t)}{\ell^c(y+t_2-t)}\, dh\, dt_d\ , \quad K=B,C,$$

where $\dfrac{\ell^a(x+t_d-t)}{\ell^a(x+t_2-t)}\, \mu^d(x+t_d-t)\, dt_d$ represents the probability that the employee, active at time t_2, remains so until time t_d and then dies between times t_d and t_d+dt_d.

We have at last all the elements to value, at the present date t, the GOB option. To this end we observe that the market value at time t_2 of all future benefits is given, in the event (a) (employee active and widow(er)), by the sum of the retirement and the disability annuity, in the event (b) (employee active and married) by the sum of retirement, disability and widow(er) annuity. Denoting these conditional values by B_w, C_w and B_m, C_m respectively, we have indeed:

$$(3.2.15)\quad K_w(t_2) = K_w^r(t_2) + K_w^i(t_2), \qquad\qquad K=B,C$$

(3.2.16) $K_m(t_2) = K^r_m(t_2) + K^i_m(t_2) + K^d_m(t_2)$, K=B,C.

The GOB value at time t_2 is then given, in case (a), by

(3.2.17) $\max\{B_w(t_2), C_w(t_2)\}$

or, in case (b), by

(3.2.18) $\max\{B_m(t_2), C_m(t_2)\}$.

This value can be decomposed in order to show up the final payoff of the GOB option, i.e. of the option of Switching to the defined-Contribution scheme, $\max\{0, C_k(t_2)-B_k(t_2)\}$, k=w,m, or of the option of Remaining in the defined-Benefit scheme, $\max\{B_k(t_2)-C_k(t_2), 0\}$, whose prices at the present date t are given by:

$$(3.2.19)\ \Pi_t(SC) = \frac{\ell^a(x+t_2-t)}{\ell^a(x)} \left\{ \left[1-\frac{\ell^c(y+t_2-t)}{\ell^c(y)}\right] E_t\left[\max\{0, C_w(t_2)-B_w(t_2)\}\, e^{-\int_t^{t_2} r(u)du}\right] + \right.$$

$$\left. + \frac{\ell^c(y+t_2-t)}{\ell^c(y)} E_t\left[\max\{0, C_m(t_2)-B_m(t_2)\}\, e^{-\int_t^{t_2} r(u)du}\right] \right\}$$

$$(3.2.20)\ \Pi_t(RB) = \frac{\ell^a(x+t_2-t)}{\ell^a(x)} \left\{ \left[1-\frac{\ell^c(y+t_2-t)}{\ell^c(y)}\right] E_t\left[\max\{B_w(t_2)-C_w(t_2)), 0\}\, e^{-\int_t^{t_2} r(u)du}\right] + \right.$$

$$\left. + \frac{\ell^c(y+t_2-t)}{\ell^c(y)} E_t\left[\max\{B_m(t_2)-C_m(t_2), 0\}\, e^{-\int_t^{t_2} r(u)du}\right] \right\}.$$

The time t price of the GOB, $\Pi_t(GOB)$, is instead given by

$$(3.2.21)\ \Pi_t(GOB) = \frac{\ell^a(x+t_2-t)}{\ell^a(x)} \left\{ \left[1-\frac{\ell^c(y+t_2-t)}{\ell^c(y)}\right] E_t\left[\max\{B_w(t_2), C_w(t_2))\}\, e^{-\int_t^{t_2} r(u)du}\right] + \right.$$

$$\left. + \frac{\ell^c(y+t_2-t)}{\ell^c(y)} E_t\left[\max\{B_m(t_2), C_m(t_2)\}\, e^{-\int_t^{t_2} r(u)du}\right] \right\}.$$

We point out that also in the case dealed with in this subsection the numerical valuation of formulae (3.2.19) to (3.2.21) requires the computation of two-stages iterated conditional expectations under the risk-neutral measure Q: in the first stage we need to compute E_{t_2}, stochastic at the present date t; in the second one we "sum up" by computing E_t.

4. Concluding remarks

In this paper we have presented a valuation model suitable for pricing contingent-claims intervening in pension schemes and applied it to a particular contingent-claim (GOB option) recently introduced in Italy by a new legislation reforming the National Pension Plan. The GOB option is an option to exchange a pension annuity computed on the grounds of a defined-benefit scheme for another one computed instead according to a defined-contribution scheme. Lacking any information about style and maturity of such an option we have considered, in the paper, two possible cases, in both of which the GOB option is of European style. In the first case we have assumed its maturity to coincide with the retirement date, so that only benefits at retirement are involved. In the second case we have analysed an option with maturity before retirement, so that also disability, withdrawal and early-death benefits are involved, besides the retirement ones. As for the economic framework, we have assumed to act in a complete and arbitrage-free market, where we have modeled, directly under the equivalent martingale measure and consistently with the multifactor model of Cox, Ingersoll and Ross (1985), the following state-variables: real and nominal interest rates, expected inflation rate, price level, individual salary and Gross Domestic Product.

As expected, given the complexity of the particular contingent-claims under scrutiny, the valuation formulae here obtained are not expressed in closed form, so that a numerical approach is called for. Our next step is then to complete the paper with the discussion of numerical techniques and presentation of related numerical results. An interesting topic for future research consist, on one hand, in considering alternative assumptions about the economic framework and, in particular, in analysing the valuation problem under the more realistic assumption of incomplete markets. On the other hand, it would be challenging to consider the case in which the GOB option is of American style.

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