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Bond Duration, Yield to Maturity and Bifurcation Analysis

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Summary

The paper deals with the analytical study of the behaviour of the duration of bonds when the coupon rate, yield to maturity and term to maturity varies, simultaneously or otherwise.

The study of duration as a function of the coupon rate and yield to maturity, leads to the conclusion that the behaviour of this characteristic of a bond is perfectly normal: the duration of a bond is always a decreasing function of the coupon rate and yield to maturity.

On the other hand, I obtain some general conclusions about the behaviour of duration with respect to different terms to maturity, and it appears that duration is an unstable characteristic for discount bonds. My objective then is to determine the bifurcation set between families of bonds, according to the behaviour of their duration: the family of par or premium bonds that form the stable set, and the discount bonds that give rise to the family whose duration behaviour is unstable.

Résumé

Durée d'Obligation, Rendement à l'Echeance et Analyse de Bifurcation

Cet article est consacré à l'étude analytique du comportement de la duration des obligations lorsque le taux du coupon, le rendement à l'échéance et la durée jusqu'à l'échéance varient soit simultanément soit différemment.

L'étude de la duration en tant que fonction du taux de coupon et du rendement à l'échéance, conduit à la conclusion que le comportement de cette caractéristique d'une obligation est parfaitement normal: la duration d'une obligation est toujours une fonction décroissante du taux de coupon et du rendement à l'échéance.

D'un autre côté, j'ai obtenu des conclusions générales sur le comportement de la duration du point de vue de différentes durées jusqu'à échéance, et il semblerait que la durée soit une caractéristique instable pour les obligations audessous du pair. Mon objectif fut alors de déterminer la bifurcation en place entre familles d'obligations, en fonction du comportement de leur duration: la famille d'obligations paritaire ou audessous du pair qui forme l'ensemble stable et les obligations audessous du pair qui engendrent la famille dont le comportement de duration est instable.

INTRODUCTION

The duration of an investment is a characteristic used at times to take decisions for investing capital in specific projects. In particular, in capital markets, rules of thumb based on the duration of bonds and mortgages, and also duration applied to the financial futures markets in hedging against interest rate volatility, are applied to sensitivity analysis, and to the immunization of portfolios, and the calculation of hedge ratios.

About the analytical study on the duration of bonds there exists several works, from among which I can cite the following: (the references are at the end of this paper): Babcock, G.C (1984); Bierwag, G.O., Kaufman, G.G. and Khang, Ch. (1978); Bierwag, G.O. (1987); Bierwag, G.O. and Kaufman, G.G. (1988); Cox, J.C, Ingersoll, J.E. and Ross, S.A. (1979); Fisher, L and Weil, R.L. (1971); Fuller, R.J. and Settle, J.W. (1984); Ingersoll, J.E.(Jr.), Skelton, J. and Weil, R.L. (1978); MCEnally, R.W. (1980); Reilly, F.K. and Sidhu, R.S. (1980); Sulzer, J–R. (1987); Sulzer, J–R et Mathis, J. (1987) and (1988)

These works use empirical methods or else partial (if not general) demonstrations for coming to conclusions about the different behaviour of duration of bonds as function of term to maturity.

The objective in this paper is to study the behaviour of the duration of bonds, from the dynamic point of view, making changes in the coupon rate, yield to maturity and term to maturity.

DEFINITION OF DURATION

The value of an investment defined by the cash flow $F_1, F_2, \ldots F_n$, from which we obtain an annual yield i, is:

$$P = \sum_{t=1}^{n} \frac{F_t}{(1+i)^t}$$

In fact there exists different and infinite combinations $F_1, F_2, \ldots F_n$, that produce the same value P. Then, it is necessary to have a criterion that lets us establish a range of preferences among the investments that produce equal capitalized value. We have the same problem when we know the price P of an investment and the cash flow $F_1, F_2, \ldots F_n$: two different investments can produce the same i; it is necessary to find a valid criterion that permits us to rank these investments.

All of this do not imply that the two criterions capital value and yield to maturity are equivalent, because we know they classify investments differently.

Duration is defined by the expression

$$D = \frac{\displaystyle\sum_{t=1}^{n} \frac{tF_t}{(1+i)^t}}{P}$$

or $D = \Sigma t\omega_t$, where $\omega_t = \dfrac{F_t}{P(1+i)^t}$, $t = 1, 2, \ldots, n$. Duration describes the average life of a set of cash flows as determined by their average present value–weighted maturities. In other words, it is the mean payback period, weighted by the present value, of the investment. Clearly, investors always prefer smallest values of duration given the same values for the rest of the parameters of the investment.

DURATION FOR BONDS

Consider a bond with face value C, a semiannual coupon c, and maturity term is $\frac{n}{2}$ years at price C. The price payable today for this bond if the buyer wants to obtain a yield to maturity equal to i, is given by:

$$P = c \,.\, a_{\overline{n|}\,i} + C \,.\, v^n$$

where: $a_{\overline{n|}\,i} = \dfrac{1 - v^n}{i}$, and $v^n = (1+i)^{-n}$

If we take C = 100, and we refer the coupon rate in % , the price of this bond given in percentage, is:

$$P = 100 \,.\, c \,.\, a_{\overline{n|}\,i} + 100 \,.\, v^n$$

Semiannually, the holder of this bond receives the coupon c, and at time n the face value of the bond.

Therefore, the duration of this bond is:

$$D = \frac{\sum_{t=1}^{n} 100 \,.\, c \,.\, v^t \,.\, t \; + \; 100 \,.\, v^n \,.\, n}{P}$$

Substituting for P and manipulating (see appendix 1) results in:

$$D = \frac{1+i}{i} - \frac{n(c-i)+(1+i)}{c(1+i)^n - (c-i)}$$

We want to study D as function of c, n, and, i, i.e., $D = D(c,n,i)$.

DURATION AS FUNCTION OF COUPON RATE c, has the following properties (appendix 2):

i) is continuous and differentiable

ii) When the coupon rate rises infinitely, the duration tends towards

$$\frac{1+i}{i} - \frac{n}{(1+i)^n - 1}$$

iii) Duration is decreasing when coupon rate rises: $\Longrightarrow D'_c < 0$.

iv) All these properties hold for all i and n.

DURATION AS FUNCTION OF YIELD TO MATURITY RATE i:

From appendix 3, we obtain the following properties:

i) Is continous and differentiable.

ii) When i rises infinitely the duration tends to unity.

iii) Duration is decreasing when yield to maturity rises, $D'_i < 0$.

iv) All these properties hold for all c and n.

DURATION AS FUNCTION OF MATURITY TERM

If we consider that duration varies as a function of maturity term, given fixed c and i, the following properties are deduced from appendix 4:

i) Is continuous and differentiable

ii) Duration of zero coupon bond is equal to maturity term.

iii) When maturity term increases infinitely, i.e. when $n \to \infty$, duration tends to

$$\frac{1+i}{i}$$

which is the duration of a consolidated bond.

iv) In a discount bond $(i > c)$, the duration function has a global maximum for the value of n which satisfies the recurrent equation:

$$n^{*} = \frac{i-c}{\delta . c}(1+i)^{-n^{*}} + \frac{1}{\delta} + \frac{1+i}{i-c} \qquad \text{where, } \delta = \ln(1+i)$$

From this value n^{*}, the duration decreases when maturity term increases, but is always above the asymptote: $\frac{1+i}{i}$, i.e., duration of a discount bond is greater than duration of a consolidated bond, from a value of maturity term equal to: $\frac{1+i}{i-c}$

Logically this value of n will only have meaning when $n > 0$, i.e., when $i > c$.

v) Following with discount bonds, when the difference between i and c decreases, with $i > c$, the function $D(n;c,i)$, has a global maximum n^{*}, given by the recurrent equation (iv). This value n^{*}, tends to infinity when $i-c$ tends to zero $(i>c)$:

$$\lim_{i \to c} \left[\frac{i-c}{\delta . c} \cdot v^{n} + \frac{1}{\delta} + \frac{1+i}{i-c} \right] = +\infty$$

vi) When $c > i$, that is, if we consider a premium bond, duration rises (less proportionally) than maturity term n, and the limit when $n \to \infty$ is the duration of a consolidated bond:

$$\frac{1+i}{i}$$

vii) The family of functions $D(n,c,i)$ is structurally stable when $c \geq i$ and is always increasing, concave and has an asymptote when n tends to infinity and therefore D takes lower values than $\frac{1+i}{i}$ which is the duration of a consolidated bond.

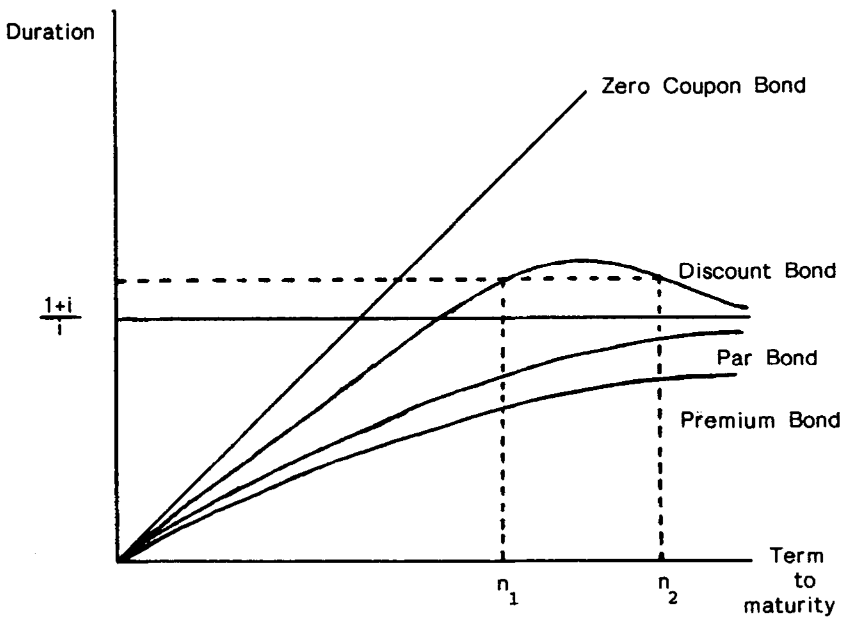
viii) The family $D(n,c,i)$ when $c<i$ has a stable behaviour if the maturity term is less than $\frac{1+i}{i-c}$; when the maturity term is greater than $\frac{1+i}{i-c}$ the duration function will have a maximum and moreover the duration will be

greater than the duration of a consolidated bond. The greater the gap between c and i, the greater the instability.

ix) Finally, the function's family is bifurcated at $c = i$ in two sets diferentiated perfectly: one stable when $c \geq i$ and another unstable when $c < i$, with a strange attractor, in the latter case, that is the duration of a consolidated bond.

OBJECTIONS TO USE OF DURATION IN DISCOUNT BONDS

We can raise several questions about conclusions on the duration of discount bonds. We begin the analysis with i, the yield to maturity. Clearly its value changes with time, and if we consider a period of high interest rates, a lot of bonds will be classified as discount bonds, which means their duration belongs to the unstable family with an unusual behaviour. Then, if we consider the set of discount bonds, we can find bonds with the same coupon and different maturity terms that have the same duration, that is to say:



In the graph we have two discount bonds with the same coupon and different maturity terms, and with equal duration when the maturity term is greater than $\frac{1+i}{i-c}$. (See appendix 4 for more details)

This reasoning raises several unanswered questions and contradictions in the behaviour of discount bond duration:

- How is it that this duration is higher than the duration of a consolidated bond?
- How is it that discount bond duration decreases when the maturity term increases, and moreover in an indefinite form?.
- How is it that there exists two bonds with different terms to maturity and the same duration?
- How is it that duration as a function of n has a global maximum?

It is possible to argue that this reasoning is only true when the yield to maturity rates are very high, and for this reason we arrive at predetermined conclusions. But the anomalous behaviour of the duration of a discount bond occurs when the maturity term is greater than $\frac{1+i}{i-c}$, and we must mention that this variable decreases with i and with the gap $(i-c)$:

$$\frac{\partial}{\partial i}\left[\frac{1+i}{i-c}\right] = -\frac{1+c}{(i-c)^2} < 0$$

$$\frac{\partial}{\partial (i-c)}\left[\frac{1+i}{i-c}\right] = -\frac{1+i}{(i-c)^2} < 0$$

which enables us to say: "the duration of a family of discount bonds has an instable behaviour when the term to maturity is greater than $\frac{1+i}{i-c}$ ". This family will be the larger:

- the greater the yield to maturity, and/or
- the greater the gap $(i-c)$

This last property means that "the probability that a discount bond has a duration whose behaviour is unstable increases if its coupon is smaller (relative to i) and if the maturity term is longer".

APPENDIX 1
DURATION FOR BONDS

The price of a bond expressed in percentage is given as:

$$P = 100 . c . a_{\overline{n|}i} + 100\ (1+i)^{-n}$$

where:

- 100 is the face value
- c is the coupon rate (semiannually)
- n is the number of semesters to maturity
- i is the semiannual yield to maturity

and

$$a_{\overline{n|}i} = \frac{1-(1+i)^{-n}}{i} = \frac{1-v^n}{i}$$

The duration of this bond is:

$$D = \frac{100\ \sum_t t.c.v^t + 100n.v^n}{P} = \frac{c\sum_t t.v^t + n.v^n}{c.a_{\overline{n|}i} + v^n} \qquad (1)$$

Let us consider the expression

$$\sum_t t.v^t = v + 2.v^2 + \ldots + n.v^n$$

we can see that it is the present value of an annuity (immediate and ordinary) with n terms varying in arithmetical progression, with the first term being 1 and the difference being 1 too.

Their value is :

$$v + 2\cdot v^2 + 3\cdot v^3 + \ldots + n\cdot v^n =$$
$$= v + v^2 + v^3 + \ldots + v^n + v^2 + v^3 + \ldots + v^n + v^3 + \ldots + v^n + \ldots + v^n =$$
$$= a_{\overline{n|}i} + v.a_{\overline{n-1|}i} + v^2.a_{\overline{n-2|}i} + \ldots + v^{n-1}.a_{\overline{1|}i} =$$

$$= a_{\overline{n}|i} + v\cdot\frac{1 - v^{n-1}}{i} + v^2\cdot\frac{1 - v^{n-2}}{i} + \ldots + v^{n-1}\cdot\frac{1 - v}{i} =$$

$$= a_{\overline{n}|i} + \frac{1}{i}\left[v + v^2 + \ldots + v^{n-1} - (v^n + v^n + \overset{n-1}{\ldots} + v^n)\right] =$$

$$= a_{\overline{n}|i} + \frac{1}{i}\left[v + v^2 + \ldots + v^{n-1} + v^n - (v^n + \overset{n}{\ldots} + v^n)\right] =$$

$$= a_{\overline{n}|i} + \frac{1}{i}\left[a_{\overline{n}|i} - n\,.\,v^n \right] = a_{\overline{n}|i}\left[1 + \frac{1}{i}\right] - \frac{nv^n}{i}$$

Therefore, we obtain:

$$\Sigma\, t\cdot c\cdot v^t = c\left[1 + \frac{1}{i}\right]\cdot a_{\overline{n}|i} - \frac{ncv^n}{i}$$

Substituting this value in eq (1) and operating:

$$D = \frac{c\cdot\left[\frac{1 + i}{i}\right]\cdot a_{\overline{n}|i} - \frac{ncv^n}{i} + n\cdot v^n}{c\,.\,a_{\overline{n}|i} + v^n} =$$

$$= \frac{c\cdot\frac{1 + i}{i}\,.\,a_{\overline{n}|i} + v^n\,.\,\frac{1 + i}{i} - v^n\,.\,\frac{1 + i}{i} - \frac{ncv^n}{i} + n\,.\,v^n}{c\,.\,a_{\overline{n}|i} + v^n}$$

$$= \frac{1 + i}{i} - \frac{v^n + \frac{v^n}{i} + \frac{ncv^n}{i} - nv^n}{c\,.\,\frac{1 - v^n}{i} + v^n} \times \frac{(1 + i)^n\,.\,i}{(1 + i)^n\,.\,i} =$$

$$= \frac{1 + i}{i} - \frac{i + 1 + nc - ni}{c(1 + i)^n - c + i}$$

and hence we have:

$$D = \frac{1 + i}{i} - \frac{n(c - i) + 1 + i}{c(1 + i)^n - c + i}$$

APPENDIX 2

DURATION AS FUNCTION OF COUPON RATE

Let us consider:

$$D(c;\, i,\, n) = \frac{1 + i}{i} - \frac{n.(c - i) + (1 + i)}{c(1 + i)^n - (c - i)}$$

and we study this function when c varies, with i and n fixed

CONTINUITY.

If the denominator could vanish , $\Rightarrow c(1+i)^n - (c-i) = 0$, we obtain the following value of c:

$$c = -\frac{i}{(1 + i)^n - 1} = -\frac{1}{s_{\overline{n|}i}} < 0,$$

therefore, $D(c;i,n)$ **is always a continuous function of c.**

ASYMPTOTE

When $c \to \infty$:

$$\lim_{c\to\infty}\left[\frac{1 + i}{i} - \frac{n.(c - i) + (1 + i)}{c(1 + i)^n - (c - i)}\right] = \frac{1 + i}{i} - \frac{n}{(1 + i)^n - 1}$$

VARIATION OF THE FUNCTION

The derivative with respect to c is:

$$\frac{\partial D}{\partial c} = \frac{n[c(1 + i)^n - c + i]-[n(c - i) + 1 + i]\,[(1 + i)^n - 1]}{[c(1+i)^n - (c - i)]^2}$$

and hence:

$$\frac{\partial D}{\partial c} = - \frac{(1 + i)^n.(ni - 1 - i) + (1 + i)}{[c(1 + i)^n - (c - i)]^2}$$

Obviously the sign of this derivative depends only on the sign of the numerator. The following reasoning is sufficient to demonstrate that it is always negative:

$$- [(ni - 1 - i)\cdot(1 + i)^n + (1 + i)] < 0, \iff$$

$$\iff (1 + i)^n\cdot ni - (1 + i)^{n+1} + (1 + i) > 0 \iff$$

$$\iff n > \frac{(1+i)^{n+1} - (1+i)}{i.(1+i)^n} \iff n > (1+i)\cdot a_{\overline{n}|i}$$

This condition is true, because n is greater than the actual value of an annuity due with n terms. Therefore, the duration is a decreasing function of c.

APPENDIX 3
THE DURATION AS FUNCTION OF YIELD TO MATURITY

We obtain the variation of the duration as function of i, in a general case when the cash flow of the investment is a continuous function of time t denoted by $F(t)$; then we suppose that δ is the yield to maturity rate compounded continuously. The price of this investment is:

$$P = \int_0^n F(t) \cdot e^{-\delta t}\, dt$$

The duration is:

$$D = \frac{\int_0^n t \cdot F(t) \cdot e^{-\delta t}\, dt}{\int_0^n F(t) \cdot e^{-\delta t}\, dt}$$

To see how D varies as function of the yield to maturity it is necessary to determine the sign of the derivative $\frac{\partial D}{\partial \delta}$.

This derivative is:

$$\frac{1}{P^2}\left[\int_0^n F(t)e^{-\delta t}dt \int_0^n -t^2F(t)e^{-\delta t}dt - \int_0^n tF(t)e^{-\delta t}dt \int_0^n -tF(t)e^{-\delta t}dt\right]$$

The sign of this derivative depends only on the expression within parenthesis. Operating on this expression we obtain:

$$-\int_0^n \int_0^n F(t)e^{-\delta t}F(s)s^2e^{-\delta s}.dtds + \int_0^n \int_0^n tF(t)e^{-\delta t}sF(s)e^{-\delta s}.dtds =$$

$$= -\int_0^n \int_0^n F(t)e^{-\delta t}\left[F(s)s^2.e^{-\delta s} - tsF(s)e^{-\delta s}\right]dt.ds =$$

$$= -\int_0^n \int_0^n F(t)e^{-\delta t}.s.F(s)e^{-\delta s}\,[s - t]\,dt.ds =$$

Integrating on the two triangles that form the integration domain (square), I obtain:

$$-\int_0^n ds \int_{t=0}^s F(t)F(s)e^{-\delta t}e^{-\delta s}s(s-t)dt - \int_0^n ds \int_{t=s}^n F(t)F(s)e^{-\delta t}e^{-\delta s}s(s-t)dt =$$

$$=-\int_0^n ds \int_{t=0}^s F(t)F(s)e^{-\delta t}e^{-\delta s}s(s-t)dt - \int_0^n dt \int_{s=0}^t F(t)F(s)e^{-\delta t}e^{-\delta s}s(s-t)ds =$$

$$=-\int_0^n ds \int_{t=0}^s F(t)F(s)e^{-\delta t}e^{-\delta s}s(s-t)dt - \int_0^n ds \int_{t=0}^s F(t)F(s)e^{-\delta t}e^{-\delta s}t(s-t)dt =$$

$$= -\int_0^n ds \int_{t=0}^s F(t)F(s)e^{-\delta t}e^{-\delta s}.(s^2 - st + t^2 - st).dt =$$

$$= -\int_0^n ds \int_{t=0}^s F(t)F(s)e^{-\delta t}e^{-\delta s}.(s-t)^2.dt\ ,$$

expression whose sign is evidently negative

The above demonstration is a general reasoning on the behavior of duration with respect to the yield to maturity. We can apply this result to the case of bonds, taking into account the following:

i) The cash flow function $F(t)$ is an impulse (or delta) function defined by

$$F(t) = \begin{cases} c & \text{if } 1 \leq t \leq n-1 \\ 1+c & \text{if } t=n \end{cases}$$

ii) It is possible to find the yield to maturity rate i from the equivalence equation

$$1+i = e^{\delta} \text{ hence } i = e^{\delta} - 1$$

what it gives the relationship between the rate i compound semiannually and the rate δ compound continuously.

APPENDIX 4

DURATION AS FUNCTION OF TERM TO MATURITY

Let us consider

$$D(n;c,i) = \frac{1+i}{i} + \frac{n(i-c) - (1+i)}{c(1+i)^n + i - c}$$

and we want to study this function when n varies, c and i bein fixed.

CONTINUITY

To make the denominator of the right hand expression of $D(n)$ vanish, we need:

$$c(1+i)^n + i - c = 0 \quad \Rightarrow \quad (1+i)^n = \frac{c-i}{c}$$

$$n \cdot \log(1+i) = \log(c-i) - \log c \qquad n = \frac{\log(c-i) - \log c}{\log(1+i)}$$

If $c \leq i$, there exists no n.
If $c > i \Rightarrow n < 0$. Therefore $D(n;\ c,\ i)$ is a continuous function of n.

ASYMPTOTE

The limit as n tends to infinity, is:

$$\lim_{n\to\infty}\left[\frac{1+i}{i} + \frac{n(i-c) - (1+i)}{c(1+i)^n + i - c}\right] = \frac{1+i}{i}$$

which is the duration of a consolidated bond. It means that this is the greatest value that the duration could reach.

To know if this function cuts the asymptote, we must equate $D(n)$ to the duration of a consolidated bond $\frac{1+i}{i}$. Denoting by ν the intersecting value, we have:

$$\frac{1+i}{i}+\frac{\nu\cdot(i-c)-(1+i)}{c(1+i)^{\nu}+(i-c)}=\frac{1+i}{i} \quad\Longrightarrow\quad \frac{\nu\cdot(i-c)-(1+i)}{c(1+i)^{\nu}+(i-c)}=0 \quad\Longrightarrow$$

$$\Longrightarrow\quad \nu\cdot(i-c)-(1+i)=0 \quad\Longrightarrow\quad \nu=\frac{1+i}{i-c}$$

an expression that is valid only when $i > c$, i.e.,we consider a discount bond.

CRITICAL POINTS

The partial derivative with respect to n is:

$$\frac{\partial D}{\partial n}=\frac{(i-c)\left[c(1+i)^{n}+i-c\right]-\left[n(i-c)-(1+i)\right]c(1+i)^{n}\ln(1+i)}{\left[c(1+i)^{n}+i-c\right]^{2}} \qquad (1)$$

The roots of n^* , if they exist, will be the solutions of the equation:

$$(i-c)\left[c(1+i)^{n^*}+i-c\right]-\left[n^*(i-c)-(1+i)\right]\cdot c(1+i)^{n^*}\cdot\ln(1+i)=0$$

$$c(i-c)(1+i)^{n^*}+(i-c)^2-c\delta n^*(i-c)(1+i)^{n^*}+c\delta(1+i)^{n^*+1}=0$$

Operating:

$$-c\delta n^*(i-c)(1+i)^{n^*}+c\delta(1+i)^{n^*+1}=-c(i-c)(1+i)^{n^*}-(i-c)^2 \qquad (2)$$

Multiplying by $(1+i)^{-n^*}=v^{n^*}$, carrying $c\delta(1+i)^{n^*+1}$ to the right hand, and changing the sign:

$$c\delta n^{*}(i-c) = c\delta(1+i) + c(i-c) + (i-c)^{2}.v^{n^{*}}$$

To deduce n^{*}, we obtain the recurrence equation

$$n^{*} = \frac{i-c}{\delta.c}(1+i)^{-n^{*}} + \frac{1}{\delta} + \frac{1+i}{i-c} \tag{3}$$

with $\delta = \ln(1+i)$ and $i > c$.

To prove the existence of this root n^{*}, it is necessary that the derivative verifies Bolzano's Theorem, that is to say: provided that the derivative is a continuous function, it must an increasing function and after that a decreasing function. To study the increasing function I choose the intersection between the function $D(n)$ and the duration of a consolidated bond, and the result is:

$$\frac{1+i}{i} - \frac{\nu(i-c) - (1+i)}{c(1+i)^{\nu} + i-c} = \frac{1+i}{i} \Longrightarrow \nu = \frac{1+i}{i-c} \quad \text{with } i>c$$

To substitute n for ν in the derivative:

$$\frac{\partial D}{\partial n}(\nu; i>c) = \frac{i-c}{c(1+i)^{\nu} + i-c} > 0$$

always, which implies that the derivative is increasing at the point $n = \nu$.

The sign of the derivative at the boundary value as $n \to \infty$, is:

$$\text{sign}\left[\lim_{n\to\infty} \frac{\partial D}{\partial n}(\nu; i>c)\right] = \text{sign}\left[-(i-c)c\delta\right] < 0$$

then, the function is decreasing. Therefore, we demonstrate the existence (at least) of a root in eq (3).

To discover if this value is a maximum or a minimum, we must take into account the following reasoning.

Let us consider the function:

$$y = \frac{f(x)}{g(x)}$$

The first derivative is:

$$y' = \frac{f'.g - f.g'}{g^2}$$

To find the critical points, it is necessary that the numerator

$$f'(x)\cdot g(x) - f(x)\cdot g'(x) \tag{4}$$

vanishes. The roots x^* of this equation fulfil:

$$f'(x^*)\cdot g(x^*) - f(x^*)\cdot g'(x^*) = 0$$

The second derivative y'', is:

$$y'' = \frac{[f''g + f'g' - g''f - g'f'].g^2 - 2gg'[f'g - g'f]}{g^4}$$

The value of this derivative at the point $x = x^*$, is

$$y''(x^*) = \frac{f''(x^*)g(x^*) - g''(x^*)f(x^*)}{g^2(x^*)}$$

We can see that: i) the sign of this expression is the same as the sign of their numerator, and ii) this numerator is the derivative of the expression (4).

In our case, to find out if n^* is a maximum or a minimum, we follow a similar argument: we calculate the derivative of the numerator in eq (1) written in the form:

$$c(i-c)(1+i)^{n} + (i-c)^{2} - c\delta n(i-c)(1+i)^{n} + c\delta(1+i)^{n+1}$$

The derivative with respect to n is:

$$c(i-c)(1+i)^{n}\delta - c\delta(i-c)(1+i)^{n} - c\delta n(i-c)(1+i)^{n}\delta + c\delta(1+i)^{n+1}\delta$$

This expression is simplified to:

$$\delta\left[-c\delta n(i-c)(1+i)^{n} + c\delta(1+i)^{n+1}\right] \qquad (5)$$

Substituting for eq.(2) into eq.(5), we obtain:

$$\delta\left[-c(i-c)(1+i)^{n^{*}} - (i-c)^{2}\right]$$

the sign of this equation is negative taking into account that $i > c$, and therefore the duration of the discount bond has a global maximum.

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