

## **Managing Interest Rate Risk for Long Liabilities**

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### **Summary**

This paper examines the effectiveness of various investment strategies in immunizing a 50-year level annuity certain. Each investment strategy is analyzed over 100 paths of interest rates generated from the one-factor Cox-Ingersoll-Ross stochastic model. It is found that using Macaulay-type risk indexes, such as duration (first order) and convexity (second order), achieves good immunization. An improvement in immunization effectiveness is obtained by using risk indexes that measure price responses to orthonormal-polynomial-based shocks to the term structure of interest rates. It is shown why these new risk indexes, which are not derived from any particular model of interest rate dynamics, are expected to be particularly useful in managing the interest rate risk for long-term liabilities.

### **Résumé**

#### **Gestion du Risque de Taux d'Interet pour les Engagements Longue Duree**

Cet article étudie l'efficacité de diverses stratégies d'investissement pour immuniser un certain de rente nivelé de 50 ans (50 year level annuity certain). Cette stratégie d'investissement est analysée sur 100 chemins de taux d'intérêt générés par le modèle stochastique Cox-Ingersoll-Ross à facteur unique. Il a été découvert que l'utilisation d'indices de risque de type Macaulay, tels que la durée (premier ordre) et la convexité (deuxième ordre), permet une bonne immunisation. Une amélioration de l'efficacité de l'immunisation est obtenue en utilisant des indices de risque qui mesurent les réactions des prix aux chocs orthonormaux-polynomiaux de la structure de taux d'intérêt. Cet article montre pourquoi on s'attend à ce que ces nouveaux indices de risque qui ne sont dérivés d'aucun modèle particulier de dynamique de taux d'intérêt, soient particulièrement utiles pour gérer le risque de taux pour les engagements à long terme.

## Introduction

Life insurance companies in the United States write a substantial amount of business involving long-term payouts to annuitants, usually on some life-contingent form. It is typical for the cash flow obligations arising from portfolios of structured settlement annuities, single premium deferred and immediate annuities, and pension plan termination annuities to extend at least 50 years and often nearly 100 years. Even though a large proportion of the present value of such liabilities arises from the first 20 or 30 years of payments, the long tails do figure importantly in the risk management of the business and in the determination of an appropriate investment strategy and level of surplus to support the liabilities. The situation for pension funds is similar.

Most entities that manage very long "fixed" liabilities use standard immunization techniques to control the asset/liability mismatch risk that derives from fluctuating interest rates. This involves maintaining a close match between asset and liability durations, a condition that is difficult to achieve without using interest rate futures because issuer call provisions are present in most long-term bonds. Full cash flow matching is not possible because non-callable bonds with maturities greater than 40 years do not exist in the United States. Also, other than 30-year U.S. Treasury bonds, the market for long non-callable bonds is not very deep. Recently, many writers of long liabilities have become concerned with the long-run cost of carrying a significant mismatch between asset and liability convexities. They are beginning to take remedial actions in their portfolios that include the purchase of high-convexity instruments such as long-term corporate bonds with investor put options and long-dated call warrants exercisable into long-term non-callable corporate bonds.

This paper reports the results of our studies of interest rate risk management for long-term liabilities. We investigated the following questions by means of computer simulation experiments:

- (1) How well can long liabilities be immunized against interest rate risk?
- (2) How many interest rate risk indexes (duration, convexity, third moment, ...) are necessary to achieve tight immunization?
- (3) How sensitive is the effectiveness of immunization to the frequency of portfolio rebalancing?

In analyzing these questions, we ignored transaction costs for simplicity. Including transaction costs leads to a dissipation of portfolio wealth, but not necessarily to less effective risk control. Also, we assumed that matching portfolios could include both

long and short positions in bonds even though most financial institutions are not permitted to sell bonds short. In practice, matching portfolios generally can be created without taking short positions, and when that cannot be accomplished, derivative instruments such as interest-rate futures can be used.

This paper is organized into four main sections. Section one presents the simulation methodology. Section two describes and discusses the simulation results for various investment strategies. Section three indicates modifications to resolve the difficulties encountered in section two. Section four then presents simulation results for the revised approach. The key findings and conclusions of the paper are then summarized.

## 1. Simulation Methodology

### (a) Liabilities

A 50-year annuity certain with level payments of \$1 every six months (100 payments) was assumed to be representative of long liabilities encountered in life insurance company and pension fund portfolios.

### (b) Assets

The investment portfolio was assumed to consist of long and short positions in zero-coupon and semi-annual coupon non-callable bonds. The bonds available for purchase at any time were assumed to have maturities in integral multiples of six months out to a maximum maturity of 30 years.

### (c) Interest Rates

In order to avoid the pitfalls encountered when non-equilibrium models of the term structure of interest rates are used, we examined only equilibrium models. (An example of such a pitfall is the certainty of losses in a situation where asset duration equals liability duration, but asset convexity is less than liability convexity and only parallel shifts in the term structure are considered.) At the time of writing this paper, our work was complete only for the one-factor Cox-Ingersoll-Ross (CIR) model [1]. We expect to be able to present results at the 2nd International AFIR Colloquium for the two-factor Brennan-Schwartz (BS) model [2].

The one-factor CIR model expresses the term structure of interest rates as a function of the spot (maturity equal to zero) interest rate,  $r$ , which is assumed to follow a square-root diffusion process. Following the notation of the CIR paper [1], we set the four parameters of the model as follows:

$\mu = 0.07$  (mean reversion level of  $r$ )  
 $\kappa = 0.30$  (strength of the mean reversion of  $r$ )  
 $\sigma = 0.10$  (volatility parameter for  $r$ )  
 $\lambda = -0.08$  (market price of interest rate risk)

These parameter values result in a term structure of volatility for par bonds close to that observed for U.S. Treasury bonds during the 1980s. The term structure is positively sloped for  $r \leq 0.0872$ , is humped for  $0.0872 < r \leq 0.0955$ , and is negatively sloped for  $r > 0.0955$ . The simulations were carried out with an initial value of  $r = 0.08$ .

The term structure of interest rates is expressed by the prices,  $P(t)$ , for zero-coupon bonds of maturity  $t$ . In the CIR model,  $P(t)$  takes the following form:

$$P(t) = A(t) \exp[-B(t) r] \quad (1)$$

where the functions  $A(t)$  and  $B(t)$  do not depend on  $r$ , but do depend on the values of the parameters  $\mu$ ,  $\kappa$ ,  $\sigma$  and  $\lambda$ . The spot rates of interest  $\delta(t)$  that correspond to the spot prices  $P(t)$  are given by:

$$P(t) = \exp[-\delta(t) t]$$

Thus, in the CIR model:

$$\delta(t) = [B(t) r - \ln A(t)]/t \quad (2)$$

The one-factor CIR model is unrealistic in a few respects that are not at all damaging to the studies described in this paper, but are nevertheless worth noting. First, changes in yields across the term structure are perfectly (but nonlinearly) correlated, and second, only a single yield curve is possible for each value of  $r$ . The most important reason for wanting to study the two-factor BS model is to overcome these two limitations. Again, we should emphasize that we have chosen to use the CIR model because it is an equilibrium model that is mathematically very tractable.

An exact simulation algorithm is possible for the CIR model. The conditional probability distribution of  $r$  at any point in time, given the initial value of  $r$  at some earlier time, can be expressed as an infinite mixture of gamma distributions with Poisson mixing coefficients [1]. The algorithm is first to draw the appropriate

Poisson variate and then to draw the appropriate gamma variate. APL functions for generating Poisson and gamma variates can be found in Wheeler [3].

The simulation experiments described in this paper used 100 paths of interest rates at monthly intervals for 50 years. This allowed us to study portfolio rebalancing at monthly, quarterly, and semiannual intervals.

#### **(d) Investment Strategy**

The investment strategies we examined are labelled in terms of three pieces of information: first, the types of bonds used (zero coupon or 8% coupon); second, the type and number of risk indexes matched between assets and liabilities (CIR duration only, Macaulay duration only, Macaulay duration and convexity, etc.); and, third, the portfolio rebalancing frequency (monthly, quarterly, or semiannually). The least number of bonds needed to match all of the applicable risk indexes was used. For example, for the investment strategy in which asset and liability durations are matched, two bonds are needed -- most of the market value of the portfolio is invested in the shortest bond having a duration index exceeding the duration index of the liability, and the balance of the portfolio is invested in a one-period bond. When more than one risk index are matched, a laddered portfolio is used. The one-period bond and the 30-year bond are always included in the ladder. Several naive ladder strategies were tested in each simulation experiment. Among these naive ladders, we selected the one which gave the most effective immunization. However, no attempt was made to optimize any given matching strategy in a mathematically precise fashion.

A single simulation experiment consisted of processing a given investment strategy along each of the 100 interest rate paths and recording the key summary financial measure for each path. Processing an investment strategy along any given path of interest rates entailed establishing the initial matching portfolio of assets according to the given strategy, liquidating the portfolio at the end of the first period and making any liability payment then due, re-establishing the appropriate matching portfolio at the beginning of the second period, and continuing in like fashion until the end of the 50-year horizon. The key summary financial measure for each path was chosen to be the initial market value of assets required so that there was neither a surplus nor a deficiency at the end of the 50-year period after all liability payments had been made.

#### **(e) Efficiency of Immunization**

The outcome of a simulation experiment is a probability distribution of initial asset values required to mature the 50-year liability exactly. The initial liability can be valued by using the initial spot prices to discount the stream of 100 future liability

than one represent ultimately profitable outcomes and ratios greater than one represent ultimately unprofitable outcomes. The standard deviation of the distribution of 100 asset/liability ratios measures the efficiency of the immunization. Perfect immunization is represented by all asset/liability ratios equal to unity, giving a probability distribution of zero standard deviation.

## 2. First Simulation Experiments

The simulation results of immunization effectiveness for the first set of strategies we examined are shown in Table 1. As described in section one, such effectiveness (or lack of it) is measured by the size of the standard deviation of the distribution of asset/liability ratios: the smaller the standard deviation, the more effective is the strategy in immunizing the liabilities.

The most obvious point from Table 1 is the extremely high effectiveness of CIR duration<sup>1</sup> as a risk index. Of course, this is a direct result of the interest rate movements following the one-factor CIR model. When the interest rate sensitivity index is precisely tuned to the dynamics of the actual interest rate process, one should get nearly perfect immunization. The trend of the results from semiannual (S) to quarterly (Q) to monthly (M) portfolio rebalancing for the zero-coupon bond strategies suggests that the immunization would be perfect if the rebalancing were continuous.

As stated earlier, the one-factor CIR model is not a good representation of interest rate behavior in the real world. Thus, in actual portfolio management, one could not expect that using a CIR-type duration index would enable highly effective immunization to be achieved. We chose to study the CIR model because it is an equilibrium model that is analytically tractable. Having chosen the CIR model for the interest rate dynamics underlying the simulation studies, our principal purpose in analyzing strategies based on matching the CIR duration indexes of assets and liabilities was to establish a standard of effectiveness against which to measure other immunization strategies.

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1 From Equation (1), CIR duration for a t-year zero-coupon bond is derived to be:

$$D_{CIR} = - \frac{1}{P(r,t)} \frac{\partial P(r,t)}{\partial r} = B(t).$$

Table 1

Standard Deviations of Asset/Liability Ratio Distributions  
for  
Various Matching Investment Strategies

Zero Coupon Bonds			
<u>Risk Index(es) Matched</u>	<u>Rebalancing Frequency</u>		
	<u>S</u>	<u>Q</u>	<u>M</u>
$D_{CIR}$	0.00133	0.00092	0.00043
$D_M^{(1)}$	0.04697	0.04700	0.04682
$D_M^{(k)}$ ; k=1,2	0.00953	0.01089	0.01197
$D_M^{(k)}$ ; k=1,2,3	0.00380	0.00445	0.00500
$D_M^{(k)}$ ; k=1,2,3,4	0.00961	0.00999	0.01039
$D_M^{(k)}$ ; k=1,2,3,4,5	0.00174	0.00170	0.00163

  

8% Coupon Bonds			
<u>Risk Index(es) Matched</u>	<u>Rebalancing Frequency</u>		
	<u>S</u>	<u>Q</u>	<u>M</u>
$D_{CIR}$	0.00015	0.00024	0.00013
$D_M^{(1)}$	0.00242	0.00471	0.00591
$D_M^{(k)}$ ; k=1,2	0.00465	0.00480	0.00501
$D_M^{(k)}$ ; k=1,2,3	0.00133	0.00127	0.00129
$D_M^{(k)}$ ; k=1,2,3,4	0.00113	0.00129	0.00138
$D_M^{(k)}$ ; k=1,2,3,4,5	0.00017	0.00034	0.00040

An obvious first attempt in discovering an immunization methodology that is robust against one's ignorance of how interest rates actually behave is to examine the effectiveness of Macaulay-type risk indexes.<sup>2</sup> One good reason for this choice is that such indexes are already in widespread use. A second reason is that studies of changes in the term structure of interest rates have shown that a change in the level of interest rates is the most significant component of change, with changes in the shape of the term structure being of somewhat lesser importance overall [4]. A third reason is that the risks of changes in the slope, curvature, and other shape attributes of the term structure are captured in higher-order Macaulay-type risk indexes [5].

Some people have criticized the use of Macaulay-type risk indexes on the basis that the parallel-shock assumption leads to opportunities for riskless arbitrage. Such criticism is specious. Everyone agrees that a world in which the only possible shocks to the term structure are shape preserving is a non-equilibrium world. That does not prevent one from successfully using in practice a set of risk indexes based on the parallel-shock assumption. At least for situations involving only non-interest-sensitive cash flows, perfect immunization can be achieved regardless of the underlying interest rate dynamics if sufficiently many risk indexes are matched. The important practical question is: How many risk indexes (namely, first, second, third, ..., nth order) are needed in order to achieve effective immunization? In section three, we shall show that the answer depends partly on the length of the liabilities.

Table 1 summarizes the immunization results for investment strategies based on matching various numbers of Macaulay-type risk indexes. The effectiveness of the immunization

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2 For a stream of cash flows,  $l(t)$ , not dependent on interest rates, in an environment characterized by term structure,  $\delta(t)$ , Macaulay duration and convexity are given by:

$$D_M = \frac{1}{L} \int_0^{\infty} t l(t) \exp [-\delta(t) t] dt \quad \text{and} \quad C_M = \frac{1}{L} \int_0^{\infty} t^2 l(t) \exp [-\delta(t) t] dt,$$

$$\text{where } L = \int_0^{\infty} l(t) \exp [-\delta(t) t] dt.$$

$D_M$  and  $C_M$  are first- and second-order risk indexes, respectively, that measure how the present value of the stream of cash flows changes when the term structure experiences a rigid, shape-preserving (parallel) shock. In Table 1,  $D_M^{(k)}$  denotes the Macaulay-type risk index of order  $k$ , given by an integral of the type shown above for  $D_M$  and  $C_M$ , but where the first term in the integrand is  $t^k$ . Thus, using this notation, we have  $D_M = D_M^{(1)}$  and  $C_M = D_M^{(2)}$ .

differs between a given strategy based on zero-coupon bonds and the corresponding strategy based on 8% coupon bonds because those two strategies produce different degrees of mismatch between the higher-order risk indexes of the asset and liability cash flows that have not been matched. What is somewhat surprising about Table 1 is that increasing the frequency of rebalancing sometimes leads to less effective immunization, sometimes to more effective immunization, and sometimes to virtually no change. This is not the anticipated result, and certainly not what was generally observed for the strategy based on matching CIR duration. More surprising about Table 1 is that matching higher-order risk indexes sometimes worsens the effectiveness of immunization. It appears that matching a greater number of Macaulay-type risk indexes does not necessarily improve the immunization effectiveness.

In the next section, we suggest an explanation for some of these puzzling results, and, in the process, we develop an approach that is expected to yield more intuitive results. The new approach is tested by simulation in the fourth section of the paper. Before leaving this section, however, we should point out that the effectiveness of the immunization achieved by matching Macaulay risk indexes for assets and liabilities is quite high. Even in such a rudimentary form, immunization is a good risk management tool.

### 3. Toward a Better Approach

Immunization risk indexes are always based on some assumption as to the form of shock in interest rates. First, consider an arbitrary change in the term structure from  $\delta(t)$  to  $\delta(t) + \Delta(t)$ . Assume the change,  $\Delta(t)$ , can be expanded in a Taylor series about  $t=0$ .

$$\Delta(t) = \sum_{k=0}^{\infty} \Delta_k t^k \quad (3)$$

where  $\Delta_k$  times  $k!$  is the  $k$ th derivative of  $\Delta(t)$  with respect to  $t$ , evaluated at  $t=0$ . The low-order  $\Delta_k$  can be characterized simply:  $\Delta_0$  is the shock to the level of the term structure,  $\Delta_1$  is the shock to the slope of the term structure (at  $t=0$ ), and  $\Delta_2$  is the shock to the curvature of the term structure (at  $t=0$ ).

We are interested in determining how the present value,  $L$ , of a stream of cash flows,  $l(t)$ , changes when  $\delta(t)$  changes to  $\delta(t) + \Delta(t)$ . This can be analyzed by expanding  $L$  in a multivariate Taylor series involving the variables  $\Delta_0, \Delta_1, \dots$ . Coefficients in the Taylor series expansion of  $L$  involve partial derivatives of  $L$  with respect to the various  $\Delta_k$ . By using Equation (3) above, it can be seen that all such coefficients, apart from constants, are Macaulay-type risk indexes -- specifically, all are present-value-weighted time moments of the stream of cash flows  $l(t)$ . Thus, immunizing against a shock in  $\Delta_k$  involves matching Macaulay-type risk indexes. The effectiveness of immunization is

essentially a question of how many  $\Delta_k$  are important, and hence, how many risk indexes need to be matched. That, in turn, depends on the interest rate process (namely, the forms of  $\Delta(t)$  possible) and the length of the cash flow stream to be immunized (namely, how large  $t$  can get).

Now consider the special case of the CIR interest rate process underlying the simulation results presented in this paper. Using Equation (2) for a shock in the short-term rate from  $r$  to  $r+R$  gives:

$$\Delta_{\text{CIR}}(t) = R [B(t)/t]$$

Apart from a scale factor,  $R$ , which is the size of the shock to short-term (maturity equal to zero) rates, the function  $B(t)/t$  fully describes the maturity dependence of the shock to the CIR term structure. It is a monotonically decreasing function of  $t$ , is equal to unity at  $t=0$ , and behaves like  $1/t$  as  $t$  approaches infinity. We analyzed the Taylor series expansion of  $B(t)/t$  about  $t=0$  and found that many terms are needed to describe the function accurately for large  $t$ . This should not be surprising. A Taylor series expansion in powers of  $t$  might have a finite radius of convergence,  $t_0$ , and thus converge very slowly for  $t$  near  $t_0$ . Even if  $t_0$  is infinite, the expansion will likely converge very slowly for large  $t$  because the terms  $t^k$  for  $k>0$  approach infinity as  $t$  approaches infinity. Thus, it can take a large number of terms in the expansion to achieve the cancellations necessary before a function such as  $\Delta(t)$  (which is bounded as  $t$  approaches infinity) becomes satisfactorily approximated on a large interval of positive  $t$ .

The behavior noted in the previous section regarding the effectiveness of immunization as a function of the number of risk indexes matched likely arises because an insufficient number of terms in the Taylor series expansion of  $\Delta_{\text{CIR}}$  were immunized. Higher-order risk indexes need to be matched in order to immunize effectively the full maturity dependence of  $\Delta_{\text{CIR}}$ . This leads us to ask: Can we find a better immunization methodology in which just a few risk indexes will be needed and which can be expected to work well for long liabilities, both in the CIR world and in practice?

The slow convergence of the Taylor series expansion of  $\Delta(t)$  can be overcome by getting rid of the infinite maturity interval through a change of variable that maps the interval  $0 \leq t \leq \infty$  onto a finite interval, say  $0 \leq x \leq 1$ . The transformation,  $x = t/(t+t_0)$ , with scale parameter  $t_0$ , accomplishes this. The transformed function  $\tilde{\Delta}(x) = \Delta(t)$ , if it is at all well-behaved, can be approximated adequately by a low-order polynomial in  $x$ , suggesting that excellent immunization efficiency should be attained by using only a few risk indexes. Rather than fitting the transformed function  $\tilde{\Delta}(x)$  by means of a Taylor series about  $x=0$ , we believe it is preferable to use the method of orthonormal polynomials, because the error of fit will then decrease as each next higher-order

polynomial is included in the fit. This suggests that including higher-order risk indexes based on the higher-order orthonormal polynomials should improve the effectiveness of the immunization as well as the goodness of fit.

Tilley [6] has described how to apply the method of orthonormal polynomials to fit U.S. Treasury yield curve data. Let  $q_k(x)$ ,  $k=0,1, \dots$ , denote the orthonormal polynomials on the unit interval described in that paper.<sup>3</sup> Then, the transformed version of Equation (3) can be expressed as:

$$\tilde{\Delta}(x) = \sum_{k=0}^{\infty} \alpha_k q_k(x), \quad (4)$$

where  $\alpha_k$  are the fitting coefficients. Tilley has labelled the  $k=0,1,2$ , and 3 components as level, tilt, warp, and undulation, respectively, by reference to the characteristic shapes of the respective  $q_k(x)$ , and has suggested that risk indexes be defined to correspond to shocks having these shapes. Specifically, the  $k$ th component risk index,  $R_k$ , is defined by shocking the term structure from  $\tilde{\delta}(x)$  to  $\tilde{\delta}(x) + \epsilon q_k(x)$  and then computing the elasticity of the response of the present value,  $L$ , of the stream of cash flows,  $l(t)$ , to the shock  $\epsilon q_k(x)$ :

$$R_k = - \lim_{\epsilon \rightarrow 0} \{ L[\tilde{\delta}(x) + \epsilon q_k(x)] - L[\tilde{\delta}(x)] \} / \epsilon L[\tilde{\delta}(x)] \quad (5)$$

In practice, one would evaluate  $R_k$  in  $t$ -space rather than  $x$ -space by making a shock  $\epsilon q_k(t/t_0)$  to the term structure  $\tilde{\delta}(t)$ . Equation (5) has been expressed in  $x$ -space because it is symbolically more elegant there than it is in  $t$ -space. The risk index  $R_0$  is the same as the Macaulay duration  $D_M$  and measures the response of the present value of a cash flow stream to a shock in the level of the term structure. The risk indexes  $R_1$ ,  $R_2$ , and  $R_3$  measure the responses of the present value of a cash flow stream to shocks in the tilt, warp, and undulation, respectively, of the term structure. The indexes  $R_k$  for  $k>0$  depend on the scale parameter  $t_0$ . Because  $x = 1/2$  when  $t=t_0$ , and because  $q_1(1/2) = 0$ , the scale parameter  $t_0$  can be considered to be the "pivot" point of the term structure, and should be chosen accordingly.

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3 For completeness, the first appendix of the Tilley paper [6], which describes these orthonormal polynomials, has been reproduced (with small modifications) as an appendix to this paper.

By using Equation (5), one can derive the following simple expression for the price responses,  $R_k$ , of a zero-coupon bond of maturity  $t$  :

$$R_k(t) = q_k(t/t+t_*) t \quad (6)$$

Equation (6) is valid for any assumption regarding interest rate dynamics. From Equation (6), one can see that all  $R_k$  are measured in units of time (for example, in years). For a given zero-coupon bond with maturity  $t > 1$ , the magnitude of the  $k$ -th-order risk index does not increase without bound as  $k$  increases, unlike the situation for Macaulay-type indexes. Unlike Macaulay-type indexes, the sign of the risk indexes  $R_k$  for  $k \geq 1$  can be either positive or negative depending on the value of the ratio  $t/t_*$ .

The unit-interval orthonormal-polynomial technique is used in the next section to improve upon the effectiveness of the 50-year liability immunization, without having to assume specific knowledge of the underlying interest rate process (in other words, without knowing that, in this paper, it is the one-factor CIR process).

#### 4. Second Simulation Experiments

A scale parameter  $t_* = 5$  years was used to transform the infinite maturity interval  $0 \leq t \leq \infty$  to the finite interval  $0 \leq x \leq 1$  through the change of variable  $x = t/t_*$ . Further simulation experiments were carried out using zero-coupon and 8% coupon bonds and the orthonormal-polynomial risk indexes defined in Equation (5). The results are shown in Table 2.

The results in Table 2 largely fulfill the expectations developed in Section 3. Only a small number of risk indexes is needed in order to achieve highly effective immunization. When the first four orthonormal-polynomial-based risk indexes are matched in the semi-annual rebalancing strategies, the immunization is essentially as good as when the CIR duration index alone is matched. Including higher-order risk indexes generally improves the immunization.

#### Summary and Conclusions

The three questions posed in the introductory section of this paper have been addressed. The simulation experiments have shown that long liabilities can be well immunized against interest rate risk. Assuming perfect knowledge of the stochastic dynamics of the term structure, one can construct a risk index (or multiple risk indexes in a multiple-factor model) that enables perfect immunization in the limit of continuous rebalancing. More practical, however, was our study of Macaulay-type risk indexes and our derivation of a sequence of new risk indexes. Although our work has demonstrated that

Table 2

Standard Deviations of Asset/Liability Ratio Distributions  
for  
Various Matching Investment Strategies

Zero-Coupon Bonds			
<u>Risk Index(es) Matched</u>	<u>Rebalancing Frequency</u>		
	<u>S</u>	<u>O</u>	<u>M</u>
$R_0$	0.04697	0.04700	0.04682
$R_k; k=0,1$	0.00180	0.00200	0.00216
$R_k; k=0,1,2$	0.00200	0.00231	0.00249
$R_k; k=0,1,2,3$	0.00116	0.00157	0.00188
$R_k; k=0,1,2,3,4$	0.00013	0.00017	0.00021

  

8% Coupon Bonds			
<u>Risk Index(es) Matched</u>	<u>Rebalancing Frequency</u>		
	<u>S</u>	<u>O</u>	<u>M</u>
$R_0$	0.00242	0.00471	0.00591
$R_k; k=0,1$	0.00053	0.00049	0.00046
$R_k; k=0,1,2$	0.00031	0.00034	0.00039
$R_k; k=0,1,2,3$	0.00006	0.00007	0.00011
$R_k; k=0,1,2,3,4$	0.00006	0.00006	0.00005

good risk control can be achieved by matching Macaulay risk indexes for assets and liabilities, better results were obtained by matching the new risk indexes. The new risk indexes are based on shocks to the term structure of interest rates that have the shape of the unit-interval orthonormal polynomials defined in this paper.

The theoretical analysis we have presented indicates why the use of a small number of Macaulay-type risk indexes (duration and convexity, for example) can enable relatively short-term fixed liabilities to be immunized well. It also suggests why more Macaulay type indexes might be needed to immunize long-term fixed liabilities very tightly. Our work demonstrates why fewer of the unit-interval orthonormal-polynomial-based risk indexes should be needed to achieve excellent immunization, regardless of the length of the liabilities.

As to the interest rate dynamics underlying the simulation studies, only the one-factor CIR model was examined here, but it is reasonable to expect that excellent results for the new risk indexes will also be found when the two-factor BS model is used. Moreover, the analysis of historical movements of the U.S. Treasury yield curve by Tilley [6] strongly suggests that the new risk indexes will perform well in practice.

### Appendix

A good reference for the material in this appendix is Morse and Feshbach [7].

The orthonormal polynomials  $q_n(x)$  introduced in section three are given by  $q_n(x) = (2n+1)^{1/2} Q_n(x)$  for  $n=0,1, \dots$ , where the  $Q_n(x)$  are related to the well-known Legendre polynomials,  $P_n(x)$ , through the equation  $Q_n(x) = P_n(1-2x)$ . Assuming a unit weight function, the Legendre polynomials are mutually orthogonal on the interval  $[-1,1]$ , and the polynomials  $Q_n(x)$  are mutually orthogonal on the interval  $[0,1]$ , as expressed by:

$$\int_0^1 Q_m(x) Q_n(x) dx = 0, \quad \text{for } m \neq n$$

The normalization integrals for the  $Q_n(x)$  are:

$$\int_0^1 [Q_n(x)]^2 dx = (2n+1)^{-1}$$

The  $Q_n(x)$  take on the following values at the boundaries:  $Q_n(0)=1$  and  $Q_n(1)=(-1)^n$ . The polynomials  $Q_n(x)$  satisfy the following recursion relation:

$$(n+1)Q_{n+1}(x) + nQ_{n-1} = (2n+1)(1-2x)Q_n(x)$$

The first eight orthogonal polynomials  $Q_n(x)$  are:

$$Q_0(x) = 1$$

$$Q_1(x) = 1-2x$$

$$Q_2(x) = 1-6x+6x^2$$

$$Q_3(x) = 1-12x+30x^2-20x^3$$

$$Q_4(x) = 1-20x+90x^2-140x^3+70x^4$$

$$Q_5(x) = 1-30x+210x^2-560x^3+630x^4-252x^5$$

$$Q_6(x) = 1-42x+420x^2-1680x^3+3150x^4-2772x^5+924x^6$$

$$Q_7(x) = 1-56x+756x^2-4200x^3+11550x^4-16632x^5+12012x^6-3432x^7$$

#### References

- [1] J. Cox, J. Ingersoll, and S. Ross. "A Theory of the Term Structure of Interest Rates." *Econometrica* 53, 1985.
- [2] M.J. Brennan and E.S. Schwartz. "A Continuous Time Approach to the Pricing of Bonds." *Journal of Banking and Finance* 3, 1979.
- [3] R.E. Wheeler. "Random Variate Generators." *ARCH* 1974.1.

- [4] G.J. Harris. "Multi-factor Duration and Immunization." Applied Research 6, J.P. Morgan Investment Management Inc., 1986.
- [5] D. Chambers, W. Carleton, and R. McEnally. "Immunizing Default-Free Bond Portfolios with a Duration Vector." Journal of Financial and Quantitative Analysis 23, 1988.
- [6] J.A. Tilley. "A Stochastic Yield Curve Model for Asset/Liability Simulations." Proceedings of 1st AFIR International Colloquium 2, 1990.
- [7] P.M. Morse and H. Feshbach. Methods of Theoretical Physics Part I. McGraw-Hill Book Company, 1953.