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Some Reflections about a Simplified Algorithm of Portfolio Selection

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Summary

There are a lot of models that simplify the portfolio selection problem; Elton, Gruber and Padberg (3) (4) (5) (6) have developed an algorithm which can be used, without resorting to mathematical programming, with little changes under a variety of index models and constant correlation models.

The simplest model is Sharpe's (12) single index model.

In this paper we investigate the relationship between the optimal portfolio obtained using Elton, Gruber and Padberg's algorithm under the single index model, hereafter SIM, and the optimal portfolio under the single index model with the introduction of a market security, hereafter SIM-MS. The authors (5) underline that the introduction of a market security may be inconsistent with SIM assumptions However, further assumptions make SIM-MS acceptable and it is interesting to note that there is an exact linear relationship between the optimal percentages obtained in SIM case and those obtained in SIM-MS case.

Résumé

Quelques Réflections sur un Algorithme Simplifié de Sélection de Portefeuille

Il existe beaucoup de modèles qui simplifient le problème de sélection de portefeuille; Elton, Gruber et Padberg [3] [4] [5] [6] ont mis au point un algorithme qui peut être utilisé sans avoir recours à une programmation mathématique, avec peu de changements, avec toute une gamme de modèles indexés et de modèles à corrélation constante.

Le modèle le plus simple est le modèle indexé unique de Sharpe [12].

Dans cet article, nous étudions la relation entre le portefeuille optimal obtenu en utilisant l'algorithme Elton, Gruber et Padberg dans un modèle indexé simple, ci-après appelé SIM, et le portefeuille optimal dans un modèle indexé simple avec introduction d'une garantie de marché, ci-après appelé SIM-MS. Les auteurs [5] soulignent que l'introduction d'une garantie de marché pourra ne pas être cohérente avec les suppositions SIM. Cependant, d'autres suppositions rendent SIM-MS acceptable et il est intéressant de noter qu'il y a une relation linéaire exacte entre les pourcentages optimaux obtenus dans le cas du SIM et ceux obtenus dans le cas du SIM-MS.

1. Introduction. The single index model

There are a lot of models (index models or constant correlation models) that simplify the portfolio selection problem. On the one hand these models simplify the amount and type of data needed to solve the portfolio problem; on the other hand under these models it is possible to solve the problem without resorting to mathematical programming.

Elton, Gruber and Padberg [3] [4] [5] [6] have developed an algorithm which can be used with little changes under a variety of index models and constant correlation models.

The simplest model is Sharpe's [12] single index model. This model assumes the existence of a linear relationship between the return on an individual risky asset and the return on a market index, as follows:

1 $$\tilde{R}_i = \alpha_i + \beta_i \tilde{R}_M + \tilde{\varepsilon}_i$$

where α_i and β_i are parameters specific to asset i and $\tilde{\varepsilon}_i$ is a random variable which includes the effect on $\tilde{R}_i$ of all factors different from the market; it is further assumed that:

1a $E(\tilde{\varepsilon}_i) = 0 \quad \forall\, i = 1, \ldots, N$

1b $\mathrm{cov}(\tilde{\varepsilon}_i, \tilde{R}_M) = 0 \quad \forall\, i = 1, \ldots, N$

1c $\mathrm{cov}(\tilde{\varepsilon}_i, \tilde{\varepsilon}_j) = 0 \quad \forall\, i = 1, \ldots, N, j = 1, \ldots, N; i \neq j$

There are no theoretical or equilibrium hypotheses behind this model (Jensen [8]): it assumes only that there is a linear relationship between returns on securities and the market index and this is confirmed by empirical tests. The assumptions that concern the disturbances $\tilde{\varepsilon}_i$ make the model inconsistent (Fama [7]) if we introduce a market security. In fact, if $\tilde{R}_M$ is the return on the market portfolio M that is the portfolio composed by all risky assets in the market, each weighted by the ratio c_i of its total market value to the total market value of all assets, we have:

2 $$\tilde{R}_M = \sum_{i=1}^{N} c_i \tilde{R}_i$$

where N is the number of all risky assets in the market; this means also that:

2a $\quad \tilde{R}_M = \sum_{i=1}^{N} c_i (\alpha_i + \beta_i \tilde{R}_i + \varepsilon_i)$

This implies that $\alpha_M = \sum_{i=1}^{N} c_i \alpha_i = 0$, $\beta_M = \sum_{i=1}^{N} c_i \beta_i = 1$ and $\tilde{\varepsilon}_M = \sum_{i=1}^{N} c_i \tilde{\varepsilon}_i = 0$. It is easy to verify that the conditions $\tilde{\varepsilon}_M = 0$ and 1c, $cov(\tilde{\varepsilon}_i,\tilde{\varepsilon}_j) = 0$ for every $i \neq j$, cannot hold simultaneously. In other words there should be no inconsistency only if $\tilde{\varepsilon}_M \neq 0$ or if there are i and j, with $i \neq j$, for which $cov(\tilde{\varepsilon}_i,\tilde{\varepsilon}_j) \neq 0$.

However, it seems possible to assume that there is a market portfolio with $\alpha_M = 0$, $\beta_M = 0$ and $\tilde{\varepsilon}_M = 0$ (thus $\sigma^2_{\varepsilon M} = 0$, that is zero residual variance) and to assume that only a subset of the assets in the market satisfy all the hypotheses of the single index model (while the remaining assets can have correlated disturbances). This means that the investor may reasonably use the model to select his optimal portfolio if he considers a suitable subset of assets in the market.

If it is assumed that it is possible to invest in portfolios as well as in individual securities, the described situation seems suitable even in the case that the considered portfolio is a market portfolio.

In practice this means that the investor has (or believes that he has) better informations about a subset of securities and he selects his optimal portfolio using this subset; at the same time he buys or sells short a market portfolio (here we speak about a market portfolio not about the market portfolio: the "true" market portfolio will exist only if all investors behave in the same way, have the same information and provide the same values of parameters; in this case the model is inconsistent).

In another way, if it is possible to invest in portfolios, it seems reasonable that the investor selects his optimal portfolio from a subset I of assets in the market and a portfolio P (with $\alpha_P = 0$, $\beta_P = 0$ and $\tilde{\varepsilon}_P = 0$), composed by a subset of securities different from subset I. The assets belonging to I satisfy the assumptions 1, 1a, 1b e 1c with respect to portfolio P. In practice this means that it is possible to invest in a subset I of individual assets and, for example, in a mutual fund. Even in this case there will be an inconsistency if we introduce equilibrium hypotheses (it will be sufficient to assume that all investors, funds' managers too, have the same expectations about returns and that they all selects optimal portfolio following Elton, Gruber and Padberg's algorithm). However the model is not an equilibrium model and it seems reasonable to assume that investors have not the same expectations and so they have different valuation of parameters.

In this paper we investigate the relationship between the optimal portfolio obtained using the Elton, Gruber and Padberg's algorithm under the single index model, hereafter SIM, and the optimal portfolio under the single index model with the introduction of a market security, hereafter SIM-MS (market security means a market portfolio M or a portfolio P with the described features).

In the following paragraph we describe the algorithm in SIM case allowing short sales. In paragraph 3 we describe the algorithm using SIM-MS and allowing short sales. In paragraph 4 we study the relationship between the optimal portfolios obtained in SIM case and in SIM-MS case. In paragraph 5 we examine the results obtained in the two cases disallowing short sales. In paragraph 6 there are some numerical and graphic examples that show how the algorithm works. Conclusions are in the last paragraph.

2. The algorithm using single index model (SIM). The case with short sales allowed

If there is a riskless asset and if short sales are allowed, the problem is:

3 $$\max_{x_1, x_2, \dots, x_n} \frac{E_p - E_0}{\sigma_p} = \max_{x_1, x_2, \dots, x_n} \frac{\sum_{i=1}^{n} x_i E_i - E_0}{\left(\sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij} \right)^{1/2}}$$

$$\text{sub} \quad \sum_{i=1}^{n} x_i = 1$$

where E_p and σ_p = expected return and standard deviation of return on the optimal portfolio;

E_0 = return on a riskless asset;

x_i = optimal percentage invested in the risky asset i;

E_i = expected return on the risky asset i;

$\sigma_{ij} = \text{cov}(\tilde{R}_i, \tilde{R}_j)$.

Using SIM, problem 3 becomes:

3a $$\max_{x_1, x_2, \ldots, x_n} \frac{\sum_{i=1}^{n} x_i E_i - E_0}{\left(\sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \beta_{iM} \beta_{jM} \sigma_M^2 + \sum_{i=1}^{n} x_i^2 \sigma_{\varepsilon i}^2\right)^{1/2}}$$

$$\text{sub} \quad \sum_{i=1}^{n} x_i = 1$$

and this is equivalent to the unconstrained problem:

3b $$\max_{x_1, x_2, \ldots, x_n} \varphi = \frac{\sum_{i=1}^{n} x_i (E_i - E_0)}{\left(\sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \beta_{iM} \beta_{jM} \sigma_M^2 + \sum_{i=1}^{n} x_i^2 \sigma_{\varepsilon i}^2\right)^{1/2}}$$

Elton and Gruber [3] prove that in this case the not normalized optimal percentage invested in asset i (z_i) can be obtained from equation 4:

4 $$\sigma_{\varepsilon i}^2 z_i = \beta_{iM} \left(\frac{E_i - E_0}{\beta_{iM}} - \sigma_M^2 \sum_{j=1}^{n} z_j \beta_{jM} \right)$$

From 4 they derive:

5 $$z_i = \frac{\beta_{iM}}{\sigma_{\varepsilon i}^2} \left(\frac{E_i - E_0}{\beta_{iM}} - \sigma_M^2 \frac{\sum_{i=1}^{n} \frac{\beta_{iM}}{\sigma_{\varepsilon i}^2}(E_i - E_0)}{1 + \sigma_M^2 \sum_{i=1}^{n} \frac{\beta_{iM}^2}{\sigma_{\varepsilon i}^2}} \right) = \frac{\beta_{iM}}{\sigma_{\varepsilon i}^2} \left(\frac{E_i - E_0}{\beta_{iM}} - C^* \right)$$

$\forall\, i = 1, \ldots, n$

and normalizing z_i they obtain the optimal percentage x_i ($x_i = z_i / \sum_{j=1}^{n} z_j$).

The sign of z_i (and of x_i) depends on the sign of the difference:

6 $$\frac{E_i - E_0}{\beta_{iM}} - C^*$$

Thus, considering only assets with positive betas, assets for which the excess return to beta is greater than the cut-off rate C* have positive percentages and assets for which this ratio ($E_i - E_0/\beta_{iM}$) is negative are sold short. When the cut-off rate C* is found it is simple to determine the optimal solution.

C* may be rewritten in the following way (which is not useful to compute C*):

$$7 \qquad C^* = \frac{\beta_{ip}}{\beta_{iM}} (E_p - E_0)$$

where p is the optimal portfolio and $\beta_{ip} = cov(\tilde{R}_i, \tilde{R}_p)/\sigma_p^2$ (*).

Thus assets (with positive betas) have positive percentages x_i's if in the opinion of the decision maker they have an expected excess return $E_i - E_0$ higher than the theoretical expected excess return (β_{ip} $(E_p - E_0)$), that is the expected excess return on the security based only on its relationship to the optimal portfolio (Elton, Gruber and Padberg [6]):

(*) equation 7 may be rewritten in the following way:

$$C^* = \frac{\beta_{ip}}{\beta_{iM}} (E_p - E_0) = \frac{cov(\tilde{R}_i, \tilde{R}_p)}{\sigma_p^2} \frac{E_p - E_0}{\beta_{iM}} = \frac{\sum_{j=1}^{n} x_j \, cov(\tilde{R}_i, \tilde{R}_j)}{\sigma_p^2} \frac{E_p - E_0}{\beta_{iM}} =$$

$$= \frac{\sigma_M^2 \beta_{iM} \sum_{j=1}^{n} x_j \beta_{jM}}{\sigma_p^2} \frac{E_p - E_0}{\beta_{iM}}$$

that is $C^* = \sigma_M^2 \beta_p \frac{E_p - E_0}{\sigma_p^2}$.

On the other hand we know from equation 4 that $C^* = \sigma_M^2 \sum_{j=1}^{n} z_j \beta_{jM}$; we also know that the relationship between the optimal percentages z_j and x_j is:

$$z_j = x_j \frac{E_p - E_0}{\sigma_p^2}$$

Thus, we have:

$$C^* = \sigma_M^2 \frac{E_p - E_0}{\sigma_p^2} \sum_{j=1}^{n} x_j \beta_{jM}$$

This means that $C^* = \sigma_M^2 \sum_{j=1}^{n} z_j \beta_{jM} = \frac{\beta_{ip}}{\beta_{iM}} (E_p - E_0)$

8 $E_i - E_0 > \beta_{ip} (E_p - E_0)$

The right-hand side may be considered as a "theoretical expected excess return" if we relate this expression to the right-hand side of the CAPM equilibrium equation ($E_i - E_0 = \beta_{iM} (E_M - E_0)$, independently developed by Lintner, Mossin and Sharpe [11]): in equilibrium the optimal portfolio will be the market portfolio and all risky assets will have an expected excess return $E_i - E_0$ equal to the equilibrium expected excess return, that is $\beta_{iM} (E_M - E_0)$. Now there are no equilibrium hypotheses, so the investor refers only to his optimal portfolio p. Moreover his expectations on an individual security may differ from the expectations based on the relationship between the security and his optimal portfolio.

3. The algorithm using single index model and introducing a market security (SIM-MS). The case with short sales allowed

Asset (n+1) is a market security M, that is a market portfolio or a portfolio with volatility $\beta_{n+1,M} = 1$ and residual variance $\sigma^2_{\varepsilon n+1} = 0$.

Moreover it is assumed that all n individual risky assets satisfy the SIM hypotheses with respect to the market security M (see also paragraph 1).

Applying equation 4 to asset n+1, we have:

9 $$\sigma^2_{\varepsilon n+1} z'_{n+1} = 0 = \beta_{n+1,M} \left(\frac{E_M - E_0}{\beta_{n+1,M}} - \sigma^2_M \sum_{j=1}^{n+1} z'_j \beta_{jM} \right)$$

that is, being $\beta_{n+1,M} = 1$:

10 $$\sum_{j=1}^{n+1} z'_j \beta_{jM} = \frac{E_M - E_0}{\sigma^2_M}$$

Substituting this expression in equation 4, we obtain:

11 $$z'_i = \frac{\beta_{iM}}{\sigma^2_{\varepsilon i}} \left(\frac{E_i - E_0}{\beta_{iM}} - (E_M - E_0) \right) \qquad \forall\, i = 1, \ldots\ldots, n$$

The cut-off rate C*' in this case is equal to the expected excess return on the market $E_M - E_0$.

From equation 10 we can get:

12 $$z'_{n+1}\,\beta_{n+1,M} = z'_{n+1} = \frac{E_M-E_0}{\sigma_M^2} - \sum_{j=1}^{n} z'_j\beta_{jM}$$

This means that we compute the market portfolio's percentage z'_{n+1} after the other percentages.

In a way similar to that presented in paragraph 2, asset i (with positive beta) has a positive percentage x'_i if:

13 $$E_i - E_0 > \beta_i\,(E_M-E_0) = \beta_{iM}\,(E_M-E_0)$$

Now the theoretical expected excess return is $\beta_{iM}\,(E_M-E_0)$ instead of $\beta_{ip}(E_p-E_0)$.

In these assumptions it seems easier explaining the meaning of equation 13: the equation $y= \beta_{iM}\,(E_M-E_0)$ is the security market line that is the line where all financial assets are positioned in equilibrium (in equilibrium $E_i - E_0 = \beta_{iM}\,(E_M-E_0)$ for every i and all assets must have $z'_i=0$). However in general assets are positioned above or below the equilibrium line: it is convenient to buy assets above the line and sell short assets below the line.

4. Relationship between SIM case and SIM-MS case

There is an exact relationship between the percentage z'_i obtained in SIM-MS case and the z_i obtained in SIM case. If we call k the ratio between the cut-off rates C*' and C*:

14 $$\frac{C^{*\prime}}{C^*} = \frac{E_M-E_0}{(\beta_{ip}/\beta_{iM})(E_p-E_0)} = k$$

that is $C^{*\prime} = kC^*$, we can rewrite z_i and z'_i in the following way:

15 $$z_i = \frac{\beta_{iM}}{\sigma_{\varepsilon i}^2}\left(\frac{E_i-E_0}{\beta_{iM}} - C^*\right) \qquad z'_i = \frac{\beta_{iM}}{\sigma_{\varepsilon i}^2}\left(\frac{E_i-E_0}{\beta_{iM}} - kC^*\right) \quad \forall i=1, .., n$$

Thus the relationship between z'_i and z_i is:

16 $$z'_i = z_i + \frac{\beta_{iM}}{\sigma^2_{\varepsilon i}}(C^* - C^{*\prime}) = z_i + \frac{\beta_{iM}}{\sigma^2_{\varepsilon i}}(1 - k)\, C^*$$

This means that in SIM case the sign of x_i depends on equation 8 while in SIM-MS case it depends on equation 17:

17 $$E_i - E_0 > k\beta_{ip}(E_p - E_0)$$

If k=1 the optimal solution (the set of percentages) is the same in SIM case and in SIM-MS case; if k<1 assets with positive percentages in SIM case are more than those in SIM-MS case (vice versa if k>1).

We wish to find a rule to establish when k is equal, lower or greater than 1. With this aim the difference C*' - C* may be written in the following way:

18 $$C^{*\prime} - C^* = \sigma^2_M\left(z'_{n+1} + \sum_{j=1}^{n} z'_j\beta_{jM} - \sum_{j=1}^{n} z_j\beta_{jM}\right) =$$

$$= \sigma^2_M\left(z'_{n+1} + \sum_{j=1}^{n}(z'_j - z_j)\beta_{jM}\right) = \sigma^2_M\left(z'_{n+1} + \sum_{j=1}^{n}\frac{\beta^2_{jM}}{\sigma^2_{\varepsilon j}}(C^* - C^{*\prime})\right)$$

From 18 we obtain:

19 $$C^{*\prime} - C^* = \frac{\sigma^2_M\, z'_{n+1}}{1 + \sigma^2_M \sum_{j=1}^{n}\frac{\beta^2_{jM}}{\sigma^2_{\varepsilon j}}}$$

Thus if the percentage of the market security z'_{n+1} is equal to zero then k=1; if that percentage is lower than zero then C*' < C* that is k<1 (vice versa if z'_{n+1} is greater than zero).

After all, obviously, the different results obtained in SIM case and in SIM-MS depends only on the percentage of the new asset n+1 (the market security) in the second case; however, it is interesting to note that the optimal percentage z_i obtained in SIM case is the sum of the optimal percentage z'_i obtained in SIM-MS case and of $\eta z'_{n+1}$, where η depends only on the parameters involved in the model:

20 $z_i = z'_i + \eta \, z'_{n+1}$

$$\text{where } \eta = \frac{\beta_{iM}}{\sigma^2_{\varepsilon i}} \; \frac{\sigma^2_M}{1 + \sigma^2_M \sum_{j=1}^{n} \frac{\beta^2_{jM}}{\sigma^2_{\varepsilon j}}}$$

5. The algorithm under SIM or SIM-MS disallowing short sales

Elton and Gruber [3] demonstrate that in SIM case the percentage z_i can be obtained in the following way:

21 $$z_i = \frac{\beta_{iM}}{\sigma^2_{\varepsilon i}} \left(\frac{E_i - E_0}{\beta_{iM}} - \sigma^2_M \sum_{j=1}^{h} z_j \beta_{jM} \right) = \frac{\beta_{iM}}{\sigma^2_{\varepsilon i}} \left(\frac{E_i - E_0}{\beta_{iM}} - C^*_h \right) \quad \forall\, i = 1, \ldots, h$$

where h is the assets' number of the optimal portfolio and the cut-off rate C^*_h is given by:

22 $$C^*_h = \sigma^2_M \frac{\sum_{i=1}^{h} \frac{\beta_{iM}}{\sigma^2_{\varepsilon i}} (E_i - E_0)}{1 + \sigma^2_M \sum_{i=1}^{h} \frac{\beta^2_{iM}}{\sigma^2_{\varepsilon i}}}$$

In practice it is simple to find assets included in the optimal portfolio: they are ranked by their excess return to beta $(E_i - E_0)/\beta_{iM}$ (descending order if β_{iM} is positive). Then C^*_h is computed for h=1, 2, ... and so on. The procedure stops when for all assets from 1 to h it is verified that $(E_i - E_0)/\beta_{iM} > C^*_h$ and for all assets from h+1 to n it is verified that $(E_i - E_0)/\beta_{iM} < C^*_h$. Computing C^*_h is simplified thanks to the fact that the optimal cut-off rate is the maximum value of C^*_h if we compute it for every h from 1 to N

(Cheung, Kwan, Yip [2]; Kwan [9]). This means that C^*_h is always higher than the cut-off rate C* computed with short sales allowed. The idea is always the same: assets have positive percentages if E_i-E_0 is greater than β_{ip_h} $(E_{p_h}-E_0)$ (where p_h is the optimal portfolio in this situation).

In SIM-MS case the optimal portfolio is composed by the same assets with positive percentages in the case with short sales allowed because C*' is a constant with respect to the number of assets considered. The only change is due to the change in the normalization factor.

Now, considering the relationship between SIM case and SIM-MS case, we can start from the relationship between C* and C*'. The possibilities are C*=C*' or C*<C*' (we exclude C*>C*' because this means $z'_{n+1}<0$ and now short sales are not allowed).

When C*=C*' we have $z'_{n+1} = 0$ thus $C^*_h \geq C^*$; this means that the number of assets in SIM case is not greater than that in SIM-MS case.

When C* < C*', we consider the equation:

$$23 \qquad C^*_h = C^* - \sigma^2_M \sum_{j=h+1}^{n} z_j\beta_{jM}$$

from 23 we derive:

$$24 \qquad C^{*\prime} - C^*_h = C^{*\prime} - C^* + \sigma^2_M \sum_{j=h+1}^{n} z_j\beta_{jM} = \frac{\sigma^2_M\, z'_{n+1}}{1 + \sigma^2_M \sum_{j=1}^{n} \frac{\beta^2_{jM}}{\sigma^2_{\varepsilon j}}} + \sigma^2_M \sum_{j=h+1}^{n} z_j\beta_{jM}$$

Thus we have $C^{*\prime} < C^*_h$ if it is:

$$25 \qquad z'_{n+1} < - \sum_{j=h+1}^{n} z_j\beta_{jM} \left(1 + \sigma^2_M \sum_{j=1}^{n} \frac{\beta^2_{jM}}{\sigma^2_{\varepsilon j}} \right)$$

In this case (disallowing short sales) we were not able to find an explicit relationship between z_i and z'_i.

6. A numerical example

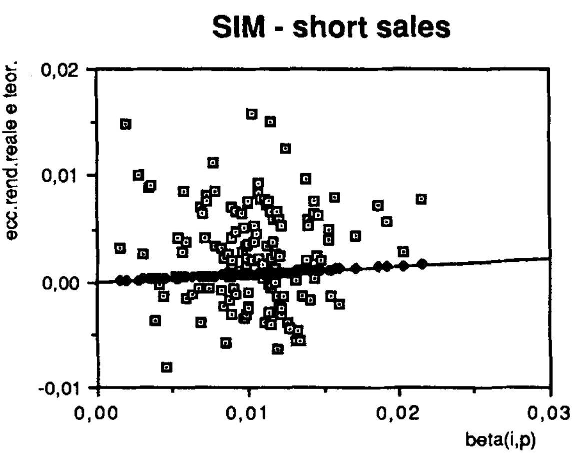
We must underline that the example is useful only to show (from a numerical and graphic viewpoint) how the algorithm works: this is because we have not verified the consistency of data with the theoretical hypotheses of SIM-MS model.

We considered weekly returns (logarithms of price relatives) of 138 assets quoted on Milan Stock Exchange during 1989. The returns on the market security are returns on Comit, a global market index.

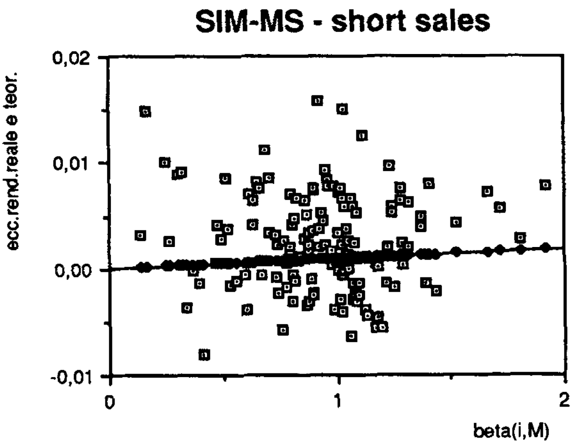
The parameters β_i and $\sigma^2_{\varepsilon i}$ were obtained by a linear regression between returns on individual assets and returns on Comit index. Expected returns are arithmetical means of returns. The return on a riskless asset is return on government bonds.

The results obtained are very similar; in fact $k \cong 1.111$ $(C^{*\prime} \cong 0.000933$, $C^* \cong 0.000840)$ In SIM case assets with positive percentages are 85; in SIM-MS case they are 84 (obviously 84 assets are the same in any case).

The reported graphics 1 and 2 clarify the meaning of the algorithm:

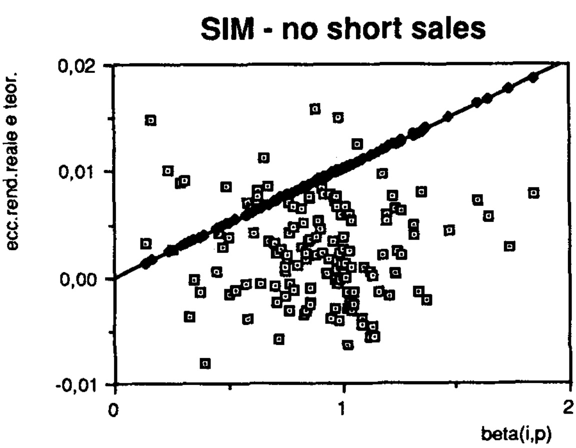


graphic 1

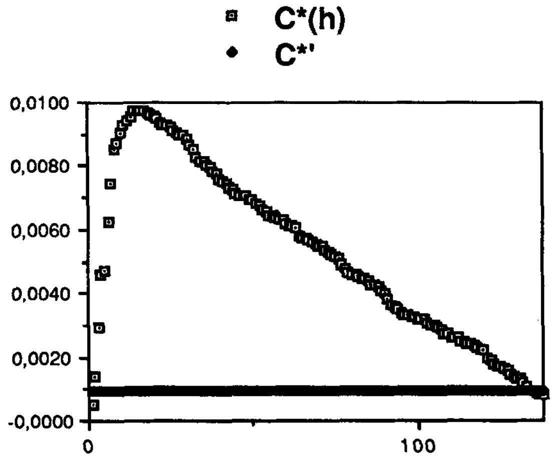


graphic 2

Disallowing short sales, in SIM case the optimal portfolio (graphic 3) is composed by 16 assets while in SIM-MS case it is composed by 84 assets (graphic 2). In fact in SIM case the cut-off rate is very high (C^*_h ≅ 0.00978, while C*' ≅ 0.000933). In graphic 4 C^*_h is reported for h from 1 to 138.



graphic 3



graphic 4

Conclusions

There is an exact relationship between the optimal portfolios obtained using Elton, Gruber and Padberg's algorithm under the single index model or under the single index model with the introduction of a market security. In fact there is a linear relationship between the unnormalized percentage z_i obtained in the SIM case and the percentage z'_i obtained in SIM-MS case: z_i is the sum of z'_i and of $\eta z'_{n+1}$, where η depends on parameters involved in the algorithm and z'_{n+1} is the market security's percentage. It is not surprising that the differences between z_i and z'_i depends on the percentage z'_{n+1} of the new asset; however it is interesting to note that there is an functional relationship between percentages. Moreover, in the second case it seems easier to understand the economical meaning of the algorithm.

Anyway, we must underline that the results can be considerably different disallowing short sales.

References

[1] Sherman C. Cheung and Clarence C. Y. Kwan. "A note on simple for optimal portfolio selection", Journal of Finance 43 (March 1988), pp. 241-245

[2] Sherman C. Cheung, Clarence C. Y. Kwan and P. C. Y. Yip. "A note on optimal portfolio selection under stable paretian distributions", Decision Sciences 16 (Fall 1985), pp. 435-41

[3] Edwin J. Elton, Martin J. Gruber and Manfred W. Padberg. "Simple criteria for optimal portfolio selection", Journal of Finance 31 (December 1976), pp. 1341-57

[4] Edwin J. Elton, Martin J. Gruber and Manfred W. Padberg. "Simple rules for optimal portfolio selection: the multi group case", Journal of Financial and Quantitative Analysis 12 (September 1977), pp. 329-345

[5] Edwin J. Elton, Martin J. Gruber and Manfred W. Padberg. "Simple criteria for optimal portfolio selection: the multi-index case". Portfolio theory 25 years later. E. J. Elton and M. J. Gruber (eds.), North-Holland, 1979, pp. 7-19

[6] Edwin J. Elton, Martin J. Gruber and Manfred W. Padberg. "Modern portfolio theory and investment analysis", John Wiley & Sons, 1984

[7] Eugene F. Fama. "Risk, return and equilibrium: some clarifying comments", Journal of Finance 23 (March 1968), pp. 29-40

[8] Michael C. Jensen."Tests of capital market theory and implications of the evidence", Handbook of Financial Economics, North Holland Publishing Company, 1979

[9] Clarence C. Y. Kwan. "Portfolio analysis using single index, multi-index, and constant correlation models: a unified treatment", Journal of Finance 39 (December 1984), pp. 1469-83

[10] Richard Roll. "A critique of the asset's pricing theory's tests. Part I: on past and potential testability of the theory", Journal of Financial Economics 4 (1977), pp. 129-176

[11] William F. Sharpe. "Capital asset prices: a theory of capital market equilibrium under conditions of risk". Journal of Finance 19 (1964), pp. 425-442

[12] William F. Sharpe. "Portfolio theory and capital markets", Mac-Graw Hill, New York, 1970