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Nonparallel Yield Curve Shifts and Durational Leverage

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Summary

Modified duration, which reflects portfolio sensitivity to parallel yield curve shifts, can be divided into several partial durations which reflect yield curve sensitivity point by point. A portfolio's vulnerability to general nonparallel shifts is then readily understood and quantified, and often, this sensitivity is found to be orders of magnitude greater than the duration implies.

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Résumé

Derives de Courbes de Rendement Non-Paralleles et Effet de Levier Durationnel

La duration modifiée qui reflète la sensibilité du portefeuille aux translations des courbes de rendement peut être divisée en plusieurs durations partielles qui reflètent la sensibilité de la courbe de rendement point par point. Une vulnérabilité de portefeuille aux decalages globaux non-parallèles est alors facilement comprise et quantifiée; souvent, cette sensibilité est découverte comme étant bien supérieure à ce qu'implique la duration.

As is well known among asset/liability managers, modified duration provides a good summary measure of price sensitivity when the underlying yield curve moves "in parallel." That is, when all yield points move in the same direction and by the same amount. This measure is so good, in fact, that with a little practice most calculations can be done mentally. The simple rule of thumb is that if D stands for the modified duration, the price will change about D% for every 100 basis point (b.p.) change in yields. When D is positive, as for a bond, the price will drop when yields increase, and rise if yields decrease. The reverse happens if D is negative, as would be the case if we sell a bond "short" and invest the proceeds in short term instruments.

As an example, consider a bond valued at $98 with a modified duration of 5 (years). Its value will decrease about 5% if yields rise 100 b.p. That is, it will decrease about $4.90 to $93.10. This estimated price will typically not be exact, since this approximation is only "linear." That is, we are approximating a "rounded" price curve with a straight line. However, we know that for small changes in yields, this curvature or "convexity" error is usually small.

Yield shifts other than 100 b.p. can be accommodated by simple ratios. For example, a 50 b.p. yield increase will decrease price about $2.45, while a 10 b.p. increase will decrease price about $.49. See Bierwag (1987) for an excellent survey of the theory and applications of duration, and an extensive bibliography of the literature.

The purpose of this article is to exemplify the rather different behavior of price when the yield curve does not move in parallel. That is, when the various yield point changes are of different signs and/or different orders of magnitude. In these cases, the behavior of the price function can be very different from what the duration value might imply. That is, the duration approximations can be orders of magnitude in error given "logical" choices for the change in yields one might use.

To simplify the calculations presented here, only the duration estimates will be explored in detail. For generalizations to convexity and more mathematical rigor, the reader is referred to Reitano (1989). Here, the technical details will be minimized to better focus on the intuition underlying the examples.

In order to understand price behavior under nonparallel yield curve shifts, the notions of a partial duration and durational leverage will be introduced. The application of these ideas will then be explored in some detail through several examples. For simplicity, only cash flows without options will be exemplified, though the theory is readily applicable in more general situations.

Partial Durations and Durational Leverage

Let's assume that we have only two fixed cash flows of amounts 10 and 20 at time 5 and 10 years, respectively. Also, assume that the 5 and 10 year annual spot rates are .08 and .10, respectively. We will denote this yield curve by (.08,.10). The price is then calculated:

$$P = 10\ (1.08)^{-5} + 20\ (1.10)^{-10} \tag{1}$$
$$= 14.517.$$

The modified duration of these cash flows is also a present value calculation:

$$D = [5 \times 10\ (1.08)^{-6} + 10 \times 20\ (1.10)^{-11}]\ /\ P \tag{2}$$
$$= 6.999.$$

As noted previously, a parallel increase in the yield curve of 100 b.p. should decrease the price about 7%, to 13.501. An exact calculation shows that for a yield curve increase from (.08,.10) to (.09,.11), the price as in (1) becomes 13.543, for an actual decrease of 6.71%.

As it turns out, each term of the summation in (2) provides partial information as to the price sensitivity of these cash flows. The first "partial duration" is defined:

$$D_1 = 5 \times 10\ (1.08)^{-6} / P \tag{3}$$
$$= 2.170.$$

Similarly, the second partial duration is defined:

$$D_2 = 10 \times 20\ (1.10)^{-11} / P \tag{4}$$
$$= 4.829.$$

Obviously, the sum of the two partial durations is just the modified duration. Perhaps somewhat less obvious is that each partial duration quantifies the sensitivity of price to the corresponding yield point. In addition, these values can be used independently or in combination. For the example above with the 100 b.p. parallel shift, the 7% approximation can be decomposed into two additive decreases. First, the price decreased about 2.170% due to the 100 b.p. increase in the 5 year spot rate. Additionally, it decreased about 4.829% due to the 100 b.p. increase in the 10 year spot rate.

For more general yield curve changes, the standard duration approximation for a parallel yield curve shift must be restated with partial durations. Recall that for a parallel shift of i, the resulting price P′ is approximated by:

$$P' \approx P\ (1 - D \cdot \Delta i), \tag{5}$$

where P is the original price and D is the modified duration. Using partial durations, this is generalized to:

$$P' \approx P\,(1 - D_1\,\Delta i_1 - D_2\,\Delta i_2). \tag{6}$$

Here, Δi_1 denotes the change in the 5 year spot rate, and Δi_2 the change in the 10 year spot rate. Because the partial durations sum to D, it is clear that (6) reduces to (5) when the yield change is parallel. That is, when $\Delta i_1 = \Delta i_2$.

For the example above, this partial duration approximation becomes:

$$P' \approx 14.517\,(1 - 2.170\,\Delta i_1 - 4.829\,\Delta i_2). \tag{7}$$

As an example of a nonparallel shift, assume that the 5 year spot rate decreases 100 b.p. to .07, and the 10 year spot rate increases 100 b.p. to .11. The approximation in (7) would then produce a price of 14.131 for a decrease of 2.66%. The exact price with this new yield curve of (.07,.11) is 14.174.

In the same way that the duration approximation can be viewed geometrically as the tangent line approximation to a price curve, the partial duration approximation reflects the tangent plane approximation to a price surface. While the traditional approach models price as a function of only one variable, the parallel shift amount, the partial duration approach models price as a multivariate function. In this multivariate model, the variables are the point by point yield curve shifts. As duration reflects the derivative of a price function, the partial durations reflect its partial derivatives.

For a general yield curve shift, let's denote by $\overline{\Delta i}$ the vector whose components are the point by point yield changes. For the nonparallel shift above, $\overline{\Delta i} = (-.01,.01)$. In the following, we require the notion of the "size" of a yield curve shift, which we will define as the length of the vector $\overline{\Delta i}$. Denoting this length by $|\overline{\Delta i}|$, recall that:

$$|\overline{\Delta i}|^2 = \Delta i_1^2 + \Delta i_2^2, \tag{8}$$

for a two point vector, with the natural generalization to other examples. For the yield curve shift above, we calculate $|(-.01,.01)| = .0141$, or a length of 141 b.p.

Finally, we define the notion of "durational leverage," which will be applied below. Let $\overline{D}$ represent the vector whose components are the partial durations (called the "total duration vector" in Reitano (1989)), and $|\overline{D}|$ denote its length as in (8). The durational leverage is defined as the ratio of the length of $\overline{D}$ to the absolute value of the modified duration D:

$$L = |\overline{D}|/|D|. \tag{9}$$

For the example above, a calculation produces L = .756.

Durational Leverage and Equivalent Parallel Shifts

In general, given any yield curve shift $\overline{\Delta i}$, one can define an "equivalent parallel shift" Δi^E. This is defined so that the traditional duration approximation in (5) with Δi^E equals the partial duration approximation in (6) with the point by point yield shifts. To do this, all that is required is that the duration is nonzero, i.e. $D \neq 0$. In these cases, Δi^E is just a weighted average of the individual shifts:

$$\Delta i^E = a_1 \Delta i_1 + a_2 \Delta i_2, \tag{10}$$

where $a_1 = D_1/D$ and $a_2 = D_2/D$. That is, each weight equals the ratio of the corresponding partial duration to D. Consequently, $a_1 + a_2 = 1$. When the shift is parallel, we have $\Delta i_1 = \Delta i_2$, and this formula reproduces the original shift value as expected.

Although formula (10) looks relatively harmless, it is important to note that in general, the a_i weights will neither be positive nor less than one. Consequently, it is possible to have relatively small values of Δi_1 and Δi_2 "leveraged" into a rather large value of Δi^E. In such a case, the resulting change in price will be correspondingly large.

One important application of durational leverage is that it quantifies a relationship between Δi^E and the original shift $\overline{\Delta i}$. Specifically, the absolute value of Δi^E as given in (10) cannot exceed L times the length of $\overline{\Delta i}$. That is,

$$|\Delta i^E| \leq L\,|\overline{\Delta i}|. \tag{11}$$

An equivalent way of expressing this is that:

$$-L\,|\overline{\Delta i}| \leq \Delta i^E \leq L\,|\overline{\Delta i}|. \tag{12}$$

That is, the value of the equivalent parallel shift must be between L times the length of $\overline{\Delta i}$, and -L times this length.

Although it is beyond the scope of this paper, it turns out that the inequalities in (12) are the best possible. That is, given any restriction on $|\overline{\Delta i}|$, one can find shifts of that length so that the equivalent parallel shift is anywhere in this interval desired. So if L is large, even small nonparallel shifts can be "leveraged" into relatively large equivalent parallel shifts. The result of this leverage is that the corresponding change in price will also appear relatively large.

Logically, one might strive to make L close to 0. For then, even large nonparallel shifts will be "scaled down" to relatively small equivalent parallel shifts. As a result, price behavior will be quite stable. Unfortunately, there is a natural limit to this goal. The value of L can be no smaller than $1/\sqrt{m}$, where m is the number of yield points. For the example above, L was equal to .756 which exceeds $1/\sqrt{2} = .707$.

In general, the value of L will be much greater than this minimum when partial durations are both positive and negative. For example, this is often the case for a surplus or net worth position when assets and liabilities are not cash matched. For instance, so-called "barbell" and "reverse barbell" duration matching strategies typically produce very large values of L. In theory, L has no upper bound in such cases. That is, given a price P and a duration D $\neq$ 0, one can find examples of cash flows with these constraints and arbitrarily large durational leverage values.

Restricting our attention to positive partial durations, the situation is a good deal more tame. Below, examples will be given of this more restricted case. Then, a more general example will be explored which exemplifies a barbell strategy to duration matching.

Examples with Positive Partial Durations

For the example in (1) above, the equivalent parallel shift formula becomes:

$$\Delta i^E = .31\,\Delta i_1 + .69\,\Delta i_2. \tag{13}$$

Here, as in the general case for positive partial durations, the equivalent parallel shift value is a simple weighted average. That is, the weights are both positive and less than 1. A simple consequence of this is that in addition to the inequalities in (12), Δi^E always lies between the smallest and largest individual shift values. That is,

$$\min(\Delta i_j) \leq \Delta i^E \leq \max(\Delta i_j). \qquad (14)$$

For the shift $\overline{\Delta i}$ = (-.01,.01) exemplified above, we see from (13) that Δi^E = .0038. That is, for these cash flows this nonparallel shift is equivalent to a parallel yield curve increase of 38 b.p. As another example, the shift (.02,-.01) is equivalent to a parallel yield curve decrease of 7 b.p. In both cases, it is easy to check that the inequalities in (12) are satisfied, recalling that L = .756 in this case.

For a more realistic example, consider a 12% semi-annual 10 year bond. Let's assume for simplicity that the bond yield curve has only 3 "pivotal" points at maturities .5, 5 and 10 years, equal to .075, .09 and .10 respectively. That is, all other semi-annual yields are assumed to be interpolated from these values. In practice, we would usually include additional pivotal points as well, such as those at maturities of 1, 3 and 7 years. As before, we denote this yield curve by (.075,.09,.10). From this yield curve, we can calculate semi-annual spot rates which can be used to price the above bond. A calculation produces a price of about 112.798, and a duration of 6.151 years.

Here, rather than use spot rates and the counterpart formula to (2) for duration, we estimate D using (5) and a bond yield shift of 5 b.p. The reason for this is that we want the duration measure in units of the original bond yield curve, not in terms of spot rates, as will be explained below. Specifically, we evaluate P′ in (5) on the yield curve (.0755,.0905,.1005), producing about 112.451. Substituting the exact calculated values of these prices in (5) with Δi = .0005 produces the duration estimate above. A theoretically preferred approach is to also estimate D using a negative shift, Δi = -.0005, and average the two estimates. However, we will not be concerned with this refinement here.

If we calculated the spot-rate-based duration above as in (2), a partial duration would have to be calculated at each spot yield utilized, for a total of 20 in this example. By using the yield curve approach, a partial duration is only needed for each of the input yield points, for a total of 3. These partial durations are produced using the following counterpart to (6):

$$P' \approx P(1 - D_1 \Delta i_1 - D_2 \Delta i_2 - D_3 \Delta i_3). \qquad (15)$$

The trick here is to only shift one of the bond yield rates at a time so that only one of the Δi_j values in (15) is non-zero. We can then solve for the respective D_j. For example, evaluating P′ on the yield curve (.0755,.09,.10) produces about 112.796. Substituting exact price values in (15), and noting that $\overline{\Delta i}$ = (.0005,0,0), we solve for D_1. Similarly, we estimate D_2 and D_3, producing the total duration vector:

$$\overline{D} = (.035,\ .219,\ 5.904). \tag{16}$$

The sum of these partial duration estimates is 6.158, a bit larger than the duration estimate. This estimation "error" can always be reduced to an acceptable tolerance by reducing the size of the shift values. Not surprisingly, most of the yield curve sensitivity of this bond is to the 10 year rate, because that is where most of its cash flow resides. Using these partial durations in (15) produces:

$$P' \approx 112.798(1 - .035\,\Delta i_1 - .219\,\Delta i_2 - 5.904\,\Delta i_3). \tag{17}$$

As an example, assume that the yield curve steepens by $\overline{\Delta i}$ = (-.005,.005,.01). The approximation in (17) calls for a price decrease of 6.00%, while an exact calculation produces a decrease of 5.83%. This shift is equivalent to Δi^E = .0097, which clearly satisfies (14). In addition, a calculation shows that L = .959 for this bond, and (12) is satisfied as well since $|\overline{\Delta i}|$ = .0122, or 122 b.p.

As noted above, we must have $L \geq 1/\sqrt{m}$, so it is natural to inquire as to the structure of the cash flows of minimum durational leverage. As it turns out, minimal leverage is obtained when all partial durations are equal, and hence equal to D/m. It is relatively easy to construct cash flows with this property. Although durational leverage has no upper bound in general, examples with positive partial durations are far more restricted. In particular, it turns out that L can be no greater than 1 in these cases. Hence, positive partial duration examples can never produce real leverage in the sense described above. That is, the size of the equivalent parallel shift cannot exceed the length of $\overline{\Delta i}$ due to (12).

In summary, when partial durations are positive, the durational leverage L must equal or exceed $1/\sqrt{m}$, and be no greater than 1. That is:

$$1/\sqrt{m} \leq L \leq 1. \tag{18}$$

An Example with Mixed Partial Durations

Consider the following example of a "barbell" hedging strategy. Assume that we have a $100 million liability due to be paid in year 5, such as a bullet guaranteed investment contract (GIC). Using the yield curve of (.075,.09,.10) and the valuation procedure above, this liability has a market value of about $63.97 million, and a duration of 4.855 years. For assets, we choose a hedge equal to a combination of the 10 year bond illustrated above, and a "cash" position of 6 month investments. To match durations, we need about 23% of the market value of assets to be short, and 77% in the bond. Let's assume that we purchase a $50 million bond for $56.40 million, and a $17.48 million 6 month discount position for $16.85 million. Assets then total $73.25 million, and have a duration of 4.856 years. We now have a duration matched portfolio.

Consider next our net worth or surplus position. A calculation produces a market value of $9.28 million, and a duration of 4.850 years. If the yield curve moves a moderate amount in parallel, the standard approximation in (5) is seen to produce good results. For example, a 50 b.p. parallel increase in the yield curve reduces surplus to $9.07 million, for a decrease of 2.24%. The approximation in (5) would call for a decrease of 2.43% to $9.06 million. For nonparallel shifts, however, the situation is radically different.

For example, the yield curve steepening shift of $\overline{\Delta i}$ = (-.005,.005,.01) decreases surplus by over 15%. Based on the above duration, this decrease might normally be associated only with parallel yield curve increases of over 300 b.p., or "comparably" large nonparallel shifts. As another example, the small and slightly nonparallel shift of $\overline{\Delta i}$ = (.002,.0025,.002) actually increases surplus by .8%, a result not anticipated from the positive duration of 4.850

years. Indeed, naively applying (5) with Δi equal to 20 or 25 b.p. would call for a decrease of 1.0 to 1.2%. Finally, the very small nonparallel shift, $\overline{\Delta i}$ = (-.0002,.0017,-.0018), increases surplus by 12.5% yet has a length of only 25 b.p., implying a significant potential for price sensitivity.

Clearly, the sensitivity of our portfolio to nonparallel shifts can be orders of magnitude greater than its duration implies. Additionally, not even the direction of change is necessarily predictable.

To understand this price behavior, we estimate partial durations as before, producing the total duration vector:

$$\bar{D} = (4.20,\ -35.23,\ 35.88). \tag{19}$$

Note that these partial durations are quite large, suggesting great sensitivity to the yield curve point by point. However, because these values are both positive and negative, they sum to the smaller duration value of 4.85 which greatly disguises this sensitivity. These partial durations give rise to the approximation formula:

$$P' \approx 9.28(1 - 4.2\,\Delta i_1 + 35.23\,\Delta i_2 - 35.88\,\Delta i_3). \tag{20}$$

Substituting the shifts above, we obtain the following:

Shift (b.p.)	Approximation	Exact	
(-50,50,100)	-16.16%	-15.27%	
(20,25,20)	+ .79%	+ .82%	(21)
(-2,17,-18)	+12.53%	+12.53%	

As can be seen, the partial duration approximation in (20) provides good predictions of the surplus changes noted above.

An alternative way of understanding this price behavior is with the notion of equivalent parallel shift. As in (10), we have:

$$\Delta i^E = .87\,\Delta i_1 - 7.27\,\Delta i_2 + 7.40\,\Delta i_3. \tag{22}$$

Because the coefficients of the individual shifts are quite large, even small nonparallel shifts have the potential of becoming leveraged into large equivalent parallel shifts. A calculation produces the following (in b.p.):

Shift	$\lvert\overline{\Delta i}\rvert$	Δi^E	
(-50,50,100)	123	333	
(20,25,20)	38	- 16	(23)
(-2,17,-18)	25	-258	

In each case above, a calculation shows that the inequalities in (12) are satisfied, since the durational leverage of surplus here is 10.40. That is, for this example we have that:

$$-10.40\lvert\overline{\Delta i}\rvert \leq \Delta i^E \leq 10.40\lvert\overline{\Delta i}\rvert. \tag{24}$$

The third shift above, $\overline{\Delta i}$ = (-.0002,.0017,-.0018), is an example of an extreme "negative" shift. That is, a shift for which the negative lower bound in (24) is achieved. Changing the signs in this example produces an extreme "positive" shift.

In general, it is easy to construct such extreme shifts. It turns out that all extreme shifts are proportional to the total duration vector, $\overline{D} = (D_1, D_2, D_3)$, which was used in (10) for the definition of L. In particular, positive multiples of $\overline{D}$ produce all extreme positive shifts, while negative multiples produce all negative shifts. For the example above, allowing for a little rounding, this yield curve shift equals about -.0048% of the total duration vector.

Is Leverage Convexity?

One open question is whether durational leverage is really "convexity" in disguise. That is, might it be the case that the effect of durational leverage can be captured by adding a convexity adjustment to the duration approximation? The answer in general is no. For the surplus example analyzed above, one calculates the convexity to be equal to 140.52. Using this value improves the approximation for the 50 b.p. parallel increase from 2.43%

to 2.27%, a very good estimate for the actual decrease of 2.24%. For nonparallel shifts, the traditional approach is not even well-defined because it is not known which value of Δi should be used. Certainly, the equivalent parallel shift analysis demonstrates that the proper value of Δi for reproducing the partial duration approximation can be radically different from the original shift values.

What about higher order derivatives? Since duration and convexity reflect the first two derivatives of price, might it be the case that better approximations can be found with higher order derivatives? That is, approximations which will capture the effect of durational leverage? Again, the answer in general is no. The reason is simply that a nonparallel shift can produce a price which is outside the range of possible values produced by parallel shifts. For example, consider the following price function with three cash flows and a flat yield curve:

$$P(i) = 20 - 20\,(1 + i)^{-1} + 11\,(1 + i)^{-2}. \qquad (25)$$

A standard calculus argument shows that for any parallel yield curve shift, the resulting price can be no smaller than 10.909. However, the nonparallel yield curve shift to (.0905,.0902) produces a price of 10.884, which is below this minimum. Consequently, the parallel shift model is simply not adequate to capture the potential price of these cash flows.

Although it is beyond the scope of this paper, it should be noted that given a price and duration value, one can construct examples with arbitrarily large durational leverage using only three cash flows. Using four or more cash flows, one can also fix convexity.

Summary and Final Comments

In the above sections, we explored a model for the sensitivity of price to nonparallel yield curve shifts. When partial durations were positive, the price function was seen to behave fairly consistently with its modified duration, D. In such cases, the equivalent parallel shift was within the numerical range of individual yield shifts, and the resultant price change appeared reasonable given the value of D. However, by using the partial durations, these price changes were seen to be not only reasonable, but accurately predictable. These positive partial duration results are generally applicable to fixed income asset portfolios and cash matched net worth or surplus positions. Positive cash flows do not necessarily guarantee positive partial durations, so a calculation is usually necessary. For example, the 5 year bullet GIC liability above has a total duration vector, $\overline{D} = (-.45, 5.30, 0)$.

When partial durations are both positive and negative, the situation changes radically. The price function can then change in ways which appear totally inconsistent with the given value of D. In particular, neither the direction of price change nor its order of magnitude, would generally be predictable. In such cases, price sensitivity was "leveraged" by the interaction of the partial durations and the individual yield curve shifts. These more general results are readily applicable to more general net worth positions.

It should be emphasized that as in the illustrations above, it is not necessary to calculate a partial duration for each cash flow. While it is theoretically valid to do so, the calculations soon become unwieldy and provide little insight to complex portfolios. Instead, it is better to base the partial durations on the smaller number of "observed" yield points on the typical yield curve. In general, there are only 10 or so such points, since yields at other maturities are typically interpolated. Consequently, although there may be many cash flows, all yield sensitivities will emanate from changes in these basic yield points. That is, only 10 or so partial durations will generally be needed for all analyses.

In addition, assets with options can readily be analyzed using partial duration techniques. In this case, partial durations are again estimated using (15) or an appropriate generalization. The only difference is that the values of P and P′ are developed using an option pricing model, rather than with simple present value calculations.

For generalizations and more details on these and other applications of this theory, the reader is referred to Reitano (1989).

REFERENCES

Bierwag, G.O. Duration Analysis: Managing Interest Rate Risk. Cambridge, MA: Ballinger Publishing Company (1987).

Reitano, R.R. "A Multivariate Approach to Duration Analysis," ARCH, 1989 (Vol. 2).

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