

On Maximising the Internal Rate of Return for Zero-Coupon Bonds

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Summary

Consider a perfectly competitive bond market where forward bond yield rates are flat and follow an ordinary Brownian motion process without drift. An investor purchases a zero coupon bond (ZCB) for a price to yield a force of interest δ_0 . The investor has the following strategy for selling the bond: sell whenever forward yield rates fall to a certain level below the initial yield rate δ_0 . The investor's the internal rate of return (IRR), is now a random variable. It is proved that the investor expected IRR is maximized by purchasing the ZCB with the longest time to maturity and selling it when the force of interest declines to the level $\delta_0 - 0.3161\sigma\sqrt{t}$ where σ is variance of the Brownian process and t is the remaining time to maturity of the ZCB.

Résumé

Maximisation du Taux de Rentabilité pour les Obligations à Coupon Zéro

Considérez un marché des obligations parfaitement compétitif où les taux de rendement des obligations à terme sont uniformes et suivent un processus de mouvement de type Brownien avec une barrière de réflexion à l'origine. Un investisseur achète une obligation à coupon zéro (ZCB) pour un certain prix, afin qu'elle rapporte une force d'intérêt de δ_0 . L'investisseur a une stratégie de vente de l'obligation à chaque fois que les taux de rendement à terme tombent à un certain niveau en dessous du taux de rendement initial δ_0 . Le taux interne de rentabilité de l'investisseur (IRR), est donc une variable aléatoire. Il est démontré que l'IRR attendu de l'investisseur est maximisé en achetant la ZCB ayant la date d'échéance la plus éloignée et en la vendant lorsque la force de l'intérêt tombe au niveau $\delta_0 - 0,3161\sigma\sqrt{t}$, car σ est une mesure de volatilité du processus Brownien et t est la durée restant jusqu'à l'échéance de la ZCB.

1 Introduction

A market timing strategy, in essence, consists of two decisions: when to buy and when to sell. These decisions are usually based upon the investor's forecasts of market movements. Pfeifer [18] pointed out that market timing is one strategy which portfolio managers may use to attempt to obtain returns in excess of those obtained by unmanaged portfolios. Pfeifer added there is no conclusive empirical evidence supporting the hypothesis that market timing activities have led to significantly higher portfolio returns. In fact, Kane and Marks [11] criticized empirical tests used to measure the performance of portfolio managers who used market timing.

Another type of strategy is selection, i.e., the types of stocks or bonds to maintain in one's portfolio. Fuller [8] compared timing and selection strategies. He concluded that for those who *do not* prefer to buy and hold a well diversified portfolio and keep costs at a minimum (by not expending resources in trying to out-perform the market), market timing yields the greater benefits. Of the two timing decisions, when to buy and when to sell, Fuller stated that the purchase decision is more important.

There are numerous investment strategies for bonds. Some strategies, the so-called "immunization" strategies, attempt to eliminate or reduce the interest rate risks by locking in a rate of return over a designed holding period; see for example Speakes [21], and Bierwag *et al* [5]. Other strategies attempt to maximize the investors internal rate of return (IRR). Barnes, Johnson and Shannon [1] tested six strategies for bonds. They concluded that "passive" strategies out performed "active" strategies (active strategies are strategies aimed at seeking out mispriced bonds), and that the relative success of any particular passive bond portfolio strategy depends on the pattern of short-term and long-term interest rate changes over the period of evaluation. A different type of market timing model for bonds was developed by Benari [3] using a market-implied excess risk premium measure relating long-term to short-term bonds.

Most bond investment strategies rely on the well known results that: (1) bond prices vary inversely with interest rates, and (2) the degree of sensitivity of a bond's price to interest rate fluctuations depends on the bond's duration; see Fuller and Settle [8]. The latter result is reasonable because an unexpected increase (decrease) in interest rates will, on one hand, decrease (increase) the sale price of a bond below (above) its expected level, while

on the other hand it will increase (decrease) the reinvestment income from coupons. As a result, the net effect on the total return depends on the bond's duration.

The effects of changes in interest rates can be made less uncertain by eliminating the reinvestment rate risk inherent in coupon bonds. One such vehicle available for eliminating this risk is the so-called Zero Coupon Bond (ZCB). ZCBs are of relatively recent origin, the first issues appeared in the early 1980's. They are "pure discount" bonds that pay no interest or principal until maturity. Thus a ZCB offers a locked-in yield that is unaffected by fluctuations in interest. ZCBs also offer a call protection because they are generally callable only at par; a call of such a bond will result in a substantial premium to the investor. In spite of this, most corporate ZCBs are issued with a call provision; see, for example, Narayanan and Lim [16] for the conditions under which it will be optimal for a corporation to call such bonds. For a comparison of ZCBs and coupon bonds, see Johnson [10].

Since a default free ZCB will guarantee an investor the yield promised at the time of purchase (if held to maturity), can the investor develop a selling strategy that guarantees an IRR in excess of the promised yield? The objective of this paper is to develop and analyze such a strategy. Clearly the strategy must exploit the inverse relationship between bond prices and interest rates. In fact the strategy used in this paper is one which advises the investor to wait until forward interest rates fall to a certain level below the initial guaranteed yield rate, then sell the ZCB. The distribution of the IRR to the investor resulting from this strategy is investigated. The level of decline that maximizes the expected IRR is obtained. Expected IRR analysis is chosen in this paper (as opposed to net present value analysis) because, as Mc Daniel, Mc Carty and Jessell [15] pointed out, it is the most commonly used discounted cash flow method among practitioners.

2 The Model

Consider an efficient market for ZCBs, i.e., one where there are no transaction costs, no taxes and no restrictions on short sales; all securities are perfectly divisible; information is available to all investors simultaneously; all investors act rationally, and there is no possibility for riskless arbitrage. Bonds are default-free and non-callable, with bond prices being determined solely by

the term structure of interest rates. For simplicity, it is assumed that the term structure is flat, i.e., a flat yield curve, and that any change in market rates result in parallel shifts in the *entire* yield curve.

Let $i(s)$ be the spot rate of interest at time s , and

$$\delta(s) = \ln(1 + i(s)) \tag{1}$$

be the spot force of interest. $\delta(s)$ is assumed to be a diffusion process. In order to simplify the subsequent analysis, a very simple model of the term structure will be used. One such simple model which is consistent with the efficient market hypothesis (see Ingersoll [9, pp. 395-397]) is

$$\delta(s) = \delta_0 + \sigma Z(s), \quad s \geq 0 \tag{2}$$

where δ_0 is the current level of the force of interest (i.e., at time $s = 0$), σ is a measure of the volatility of the $\delta(s)$ process, and $Z(s)$ is standard Brownian motion starting at the origin, i.e., $Z(0) = 0$, $E[Z(s)] = 0$ and $Var[Z(s)] = s$. Karlin and Taylor [12, chapter 7] contains a discussion of the properties of standard Brownian motion.

Unfortunately the $\delta(s)$ process above will eventually become negative with probability 1. To prevent this, a reflecting barrier (in the sense of Karlin and Taylor [12, chapter 7]) is placed at the line $\delta(s) = 0$, i.e., at the s -axis. This ensures that $\delta(s) \geq 0$, however it does not describe its behavior at the boundary. To this end, consider the process

$$\gamma(s) = \delta_0 + \sigma Z(s),$$

which is exactly the same as the initial equation for $\delta(s)$, but without the reflecting barrier. $\gamma(s)$ can be viewed as the "shadow process" of $\delta(s)$. With the reflecting barrier, $\delta(s)$ is defined as

$$\delta(s) = |\gamma(s)|. \tag{3}$$

This is similar to the process defined in Karlin and Taylor [12, chapter 7, pp. 352-354].

Since the bond prices depend *only* on the information contained in the interest rate process $\delta(s)$ and the rates are non-negative, then from Ross [20, theorem 2] no arbitrage is possible.

In general terms, the term structure can be modeled by a diffusion process satisfying the stochastic differential equation

$$d\delta(s) = b(s, \delta(s))ds + a(s, \delta(s))dZ(s) \quad (4)$$

where $b(s, \delta(s))$ and $a(s, \delta(s))$ are the infinitesimal mean and variance respectively; see Karlin and Taylor [13, chapter 15] for more on diffusion processes. For example, Beekman and Shiu [2] used this equation with

$$b(s, \delta(s)) = -\delta(s) \quad \text{and} \quad a(s, \delta(s)) = \sigma,$$

i.e., an Ornstein – Uhlénbeck process, to calculate bond prices. Cox, Ingersoll and Ross [6, p.391, equation (17)] used a mean reverting process with

$$b(s, \delta(s)) = \kappa(\bar{\delta} - \delta(s)) \quad \text{and} \quad a(s, \delta(s)) = \sigma\sqrt{\delta(s)}.$$

A discrete time stochastic model of the term structure can also be used. Bierwag [4] used such a model to provide expressions for bond prices.

For a ZCB with face value of 1 maturing in t years time, its current price (at time 0) is

$$P(0, t) = e^{-t\delta_0}.$$

The *actual* future price of the bond s years from now, $0 < s \leq t$, is

$$P(s, t) = e^{-(t-s)\delta(s)} \quad s \leq t.$$

Note that the actual bond price at time s depends on the *actual* force of interest $\delta(s)$ and not the expected force of interest at s which is δ_0 . Let R_s be the IRR resulting from paying $P(0, t)$ to purchase a ZCB and selling it at s for $P(s, t)$, then

$$P(0, t) = (1 + R_s)^{-s} P(s, t), \quad 0 < s \leq t.$$

Solving for R_s yields

$$\begin{aligned} R_s &= [P(s, t)/P(0, t)]^{1/s} - 1 \\ &= \exp\left[\frac{t}{s}(\delta_0 - \delta(s)) + \delta(s)\right] - 1 \\ &= \exp\left[\delta_0 + \frac{t-s}{s}(\delta_0 - \delta(s))\right] - 1 \end{aligned} \quad (5)$$

$$= \exp\left[\delta_0 - \left(\frac{t}{s} - 1\right)\sigma Z(s)\right] - 1 \quad \text{using eqn. (2)}. \quad (6)$$

Since $0 < s \leq t$, it is clear that the IRR can be increased above $i_0 = e^{\delta_0} - 1$ by selling the bond whenever $Z(s)$ is negative. This points the way to the following strategy for a risk neutral investor:

Strategy 1 *The risk neutral investor who purchased a ZCB at time 0 for a price yielding a force of interest δ_0 , sells the ZCB whenever the market spot force of interest $\delta(s)$, $s > 0$, falls to a certain fixed level below δ_0 .*

If $Z(s)$ is never negative, then $s = t$ and the IRR will be $i_0 = e^{\delta_0} - 1$. If, on the other hand, the bond is to be sold when $Z(s) = -w$, $w > 0$, the expected time it will take for $Z(s)$ to attain the level $-w$ increases as w increases. If w is too large, $\delta(s)$ may never fall to $-w$ before the bond matures. Thus the investor's decision will be to choose the level, $-w$, that $Z(s)$ must fall to in order to maximize his/her expected IRR. Clearly, for *any* model of $\delta(s)$, strategy 1 *guarantees* a minimum IRR of i_0 . This follows directly from equation (5) which does not assume a functional form for $\delta(s)$. Hence the interesting problem is to find the strategy which maximizes the expected IRR in excess of δ_0 . In general, this will be a difficult problem. However, assuming equation (2) holds, the problem yields a simple closed form solution.

In practice, the investor will observe the $i(s)$ process directly from the market and observe the $\delta(s)$ and $Z(s)$ processes indirectly. Thus it is important to be able to express the decline in the $Z(s)$ process in terms of the equivalent decline in the $\delta(s)$ and the $i(s)$ processes. Now $Z(s)$ falling to $-w$ is equivalent to $\delta(s)$ falling to

$$\begin{aligned}\delta(s) &= \delta_0 + \sigma Z(s) \\ &= \delta_0 - w\sigma,\end{aligned}\tag{7}$$

i.e., $\delta(s)$ falling by $w\sigma$ below its initial level δ_0 . From equation (1), this is also equivalent to $i(s)$ falling to the level

$$\begin{aligned}i(s) &= e^{\delta_0 - \sigma w} - 1 \\ &= (1 + i_0)e^{-w\sigma} - 1.\end{aligned}\tag{8}$$

So the investor can equivalently specify the level of decline in terms of the $Z(s)$, $\delta(s)$, or $i(s)$ processes by using w , equation (7), or equation (8) respectively.

Strategy 1 belongs to the "buy-low, sell-high" family of investment strategies. However it does not deal with the "when-to-buy" decision, only the

“when-to-sell” decision. Once a ZCB is acquired, this strategy suggests the best time to sell it so as to maximize the investor’s expected IRR. This strategy will appeal to investors with a flexible planning horizon. If such an investor purchases a bond and its price quickly increases by, say 50%, such an investor will sell the bond and take the “profit” without unduly worrying about the reinvestment rate risks. On the other hand, an investor with a fixed planning horizon may not find “buy-low, sell-high” strategies very appealing because of the interest rate risks inherent in such strategies. Such an investor may prefer an immunization strategy.

3 The Main Results

Strategy 1 is based on the random stopping time $T(w)$, the first time that $Z(s) = -w$, i.e.,

$$T(w) = \inf\{s : Z(s) = -w \mid Z(0) = 0, w > 0\}. \quad (9)$$

It is well known, see Karlin and Taylor [12, chapter 7, equation (3.4)], that the random variable $T(w)$ has pdf $g(s, w)$, as a function of s , given by

$$g(s, w) = \frac{w}{\sqrt{2\pi s^3}} \exp\left[-\frac{w^2}{2s}\right], \quad s > 0.$$

This result is based on an unrestricted random walk, i.e., one without barriers. Fortunately, since the $\delta(s)$ process has its reflecting barrier at the s -axis and its “selling” (or absorbing) barrier is, from equation (7), at $\delta_0 - w\sigma > 0$, then the reflecting barrier will not affect the distribution of $T(w)$ because it is placed *below* the absorbing barrier. This means the reflecting barrier can never be reached before selling the bond; see Figure 1. Since the bond must be redeemed at time t , the event $T(w) > t$ implies that the investor held the bond to its maturity.

Define a new stopping time $\tau(w, t)$ as the time at which the bond is sold or matures,

$$\tau = \tau(w, t) = \text{Min}\{T(w), t\}. \quad (10)$$

Clearly τ has a mixed distribution: τ is continuous over the range $(0, t)$ with pdf $g(s, w)$ and has a mass point at $\tau = t$ when $T(w) > t$. The probability

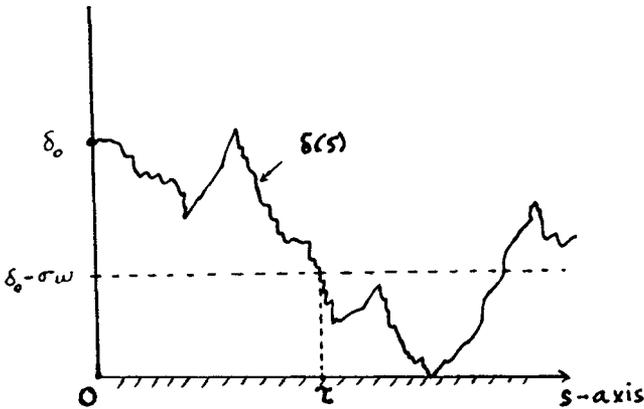


Figure 1: Absorbing and reflecting barriers

at this mass point is

$$\begin{aligned} \Pr[\tau = t] &= \int_t^\infty g(s, w) ds \\ &= 1 - G(t, w) \end{aligned}$$

where

$$G(t, w) = \int_0^t g(s, w) ds.$$

Figure 2 shows a bond sold at the random time τ . Notice that the investor must wait until $\delta(s)$ falls to a level $\delta_0 - \sigma w$, $w > 0$.

Let R_τ be the IRR resulting when the bond is sold at the random time τ (based on $Z(\tau) = -w$). From equation (6)

$$1 + R_\tau = \exp\left[\delta_0 + \left(\frac{t}{\tau} - 1\right)\sigma w\right]$$

where it is understood that $\tau = t$ implies the bond is held to maturity. Let Y_τ be the force of IRR in excess of δ_0 , i.e.,

$$\begin{aligned} Y_\tau &= \ln(1 + R_\tau) - \delta_0 \\ &= \sigma w \left(\frac{t}{\tau} - 1\right) \geq 0. \end{aligned}$$

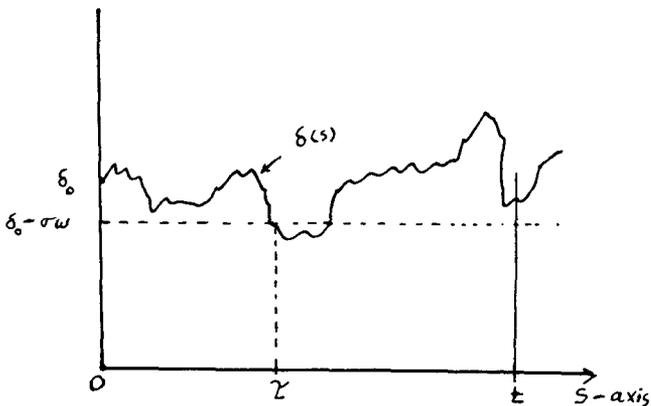


Figure 2: Time of sale

Recall that strategy 1 implies Y_τ must be non-negative. Clearly Y_τ has mixed distribution with a mass point at 0, i.e.,

$$\begin{aligned} Pr[Y_\tau = 0] &= Pr[\tau = t] \\ &= 1 - G(t, w), \end{aligned}$$

and pdf $f(y, w)$ given by

$$f(y, w) = \left[2t\pi\sigma^2 \left(1 + \frac{y}{\sigma w} \right) \right]^{1/2} \exp\left[-\frac{w^2}{2t} \left(1 + \frac{y}{\sigma w} \right) \right] \quad y > 0.$$

The random variable

$$X_\tau = 1 + \frac{1}{\sigma w} Y_\tau$$

has pdf $h(x, w)$ given by

$$h(x, w) = \frac{\theta^{1/2} x^{-1/2} e^{-\theta x}}{\sqrt{\pi}} \quad x > 1,$$

where

$$\theta = \frac{w^2}{2t}. \tag{11}$$

So X_τ has a truncated gamma distribution in the region $x > 1$, implying that Y_τ is positively skewed. This corroborates Pfeifer's statement "..... even if the market return distributions were approximately normal, the return distribution produced by market timing will be positively skewed"; see Pfeifer [18, p. 452].

Let $\mu_k(w, t)$ be the k^{th} moment of the excess force of IRR, i.e.,

$$\mu_k(w, t) = E[Y_\tau^k] = \int_0^t (\sigma w (\frac{t}{s} - 1))^k g(s, w) ds. \quad (12)$$

Clearly, the investor has control over the variables w (by waiting until $Z(s) = -w$) and t (by initially purchasing a ZCB with t years left to maturity). Thus w and t are the investor's decision variables. Since the investor wants to maximize $\mu_1(w, t)$, expressions for the first partial derivatives of $\mu_1(w, t)$ are needed. To this end the following well known and easily derived expression for the derivative of an integral is provided: If

$$I(t) = \int_{a_1(t)}^{a_2(t)} f(x, t) dx$$

then

$$\frac{dI}{dt} = f(a_2, t) \frac{da_2}{dt} - f(a_1, t) \frac{da_1}{dt} + \int_{a_1}^{a_2} \frac{\partial}{\partial t} f(x, t) dt \quad (13)$$

assuming all of the relevant derivatives exist.

From equations (12) and (13),

$$\frac{\partial}{\partial t} \mu_1(w, t) = \int_0^t \frac{\sigma w}{s} g(s, w) ds > 0 \quad (14)$$

and

$$\frac{\partial^2}{\partial t^2} \mu_1(w, t) > 0.$$

So $\mu_1(w, t)$ is an increasing convex function of t , implying that the investors expected excess yield above the initial force of yield δ_0 increases at an increasing rate with the length of time to maturity of the bond. This is due, in part, to the fact that a bond's price volatility increases with its duration, and the duration of a ZCB is its time to maturity; see Fuller and Settle ([8], equation (6)). To summarize,

Result 1 *An investor in an efficient bond market for zero coupon bonds with the intention of purchasing these bonds then selling them according to Strategy 1. In order to maximize his/her expected IRR, the investor must purchase ZCBs with the longest time to maturity.*

This result supports the assertion of Lloyd and Haney [14] that the greater the time spent in the market, the lower the risk of an investor not achieving the long-run market rate of return for any given level of diversification.

The level w that maximizes $\mu_1(w, t)$ can similarly be found:

$$\begin{aligned} \frac{\partial}{\partial w} \mu_1(w, t) &= \int_0^t \sigma \left(\frac{t}{s} - 1 \right) \left(2 - \frac{w^2}{s} \right) g(s, w) ds \\ &= 2\sigma \int_{\theta}^{\infty} (y/\theta - 1)(1 - y) \frac{y^{-1/2} e^{-y}}{\sqrt{\pi}} dy \end{aligned} \quad (15)$$

by the change of variable $y = w^2/2s$ and θ defined in equation (11). To find the optimum level w_t^* , the equation

$$\frac{\partial}{\partial w} \mu_1(w, t) = 0 \quad (16)$$

must be solved. Notice that equation (15) contains only one unknown, θ . Clearly an optimum level for w exists if and only if a positive solution θ^* to equation (16) exists. It should be noted that θ^* will not depend on σ , the volatility parameter. This is because σ is not a part of the $Z(s)$ process. If a θ^* exists, then $w_t^* = \sqrt{2\theta^* t}$ will be the optimum level of decline for the standard Brownian motion process $Z(s)$, not for the $\delta(s)$ process.

Define $g(\theta)$ as

$$g(\theta) = \int_{\theta}^{\infty} \left(\frac{y}{\theta} - 1 \right) (y - 1) y^{-1/2} e^{-y} dy.$$

Clearly, if $\theta > 1$, then $(y/\theta - 1) > 0$ and $(y - 1) > 0$, so $g(\theta) > 0$. Hence no root $\theta > 1$ exists for the equation $g(\theta) = 0$. By using the change of variable to x where $y = 2\theta/(1 - x)$,

$$g(\theta) = \sqrt{2\theta} \int_{-1}^1 \frac{(1+x)(2\theta - 1 + x)}{(1-x)^{7/2}} \exp\left[\frac{-2\theta}{1-x}\right] dx.$$

Now $g(1) > 0$ and as $\theta \rightarrow 0$, $g(\theta)$ is negative. So $g(\theta)$ has a root in the region $(0, 1)$; the root at zero must be excluded. Solving this equation yields the root (to 7 significant digits)

$$\theta^* = 0.04995716, \tag{17}$$

resulting in

$$w_t^* = 0.3161\sqrt{t}. \tag{18}$$

From equation (7), the optimum level of the decline in $\delta(s)$ below its initial level lies on the square root boundary, i.e., for fixed t , $\delta(s)$ must decline to

$$\delta(s) = \delta_0 - 0.3161\sigma\sqrt{t}. \tag{19}$$

This result, which is new to the finance literature, can be summarized as follows:

Result 2 *An investor in a perfectly competitive ZCB market who decides to purchase a ZCB with time to maturity of t years, and who sells according to Strategy 1, will maximize the expected IRR by selling whenever the market spot force of interest is $\delta_0 - 0.3161\sigma\sqrt{t}$ for fixed t .*

Figure 3 shows the optimum selling boundary $\delta_0 - \sigma\sqrt{2\theta^*t}$. A ZCB is bought at time 0 with t years left to maturity. The bond is not sold at time N but is sold at time T because the line AB represents the level $\delta_0 - \sigma\sqrt{2\theta^*t}$ at which the bond is to be sold, and T is the first that $\delta(s)$ reaches the line AB. Note that the selling line AB is fixed for each time to maturity. Since ZCBs are usually issued for terms of 30 years or less, the maximum level of decline will be $\sigma\sqrt{60\theta^*} = 1.73\sigma$ (this applies to a 30 year term). For example, with a 30 year ZCB and $\sigma = 1.5\%$, the investor must wait until the spot force of interest falls by 2.596% before selling.

The probability that the $\delta(s)$ process reaches the fixed barrier σw_t^* below its initial level, and before time t , i.e., the probability that the ZCB with maturity t is sold before its maturity, is

$$\begin{aligned} \Pr\{T < t \mid w = \sqrt{2\theta^*t}\} &= \int_0^t g(s, \sqrt{2\theta^*t}) ds \\ &= \int_{\theta^*}^{\infty} \frac{r^{-1/2} e^{-r}}{\sqrt{\pi}} dr \\ &= 0.7519 \end{aligned}$$

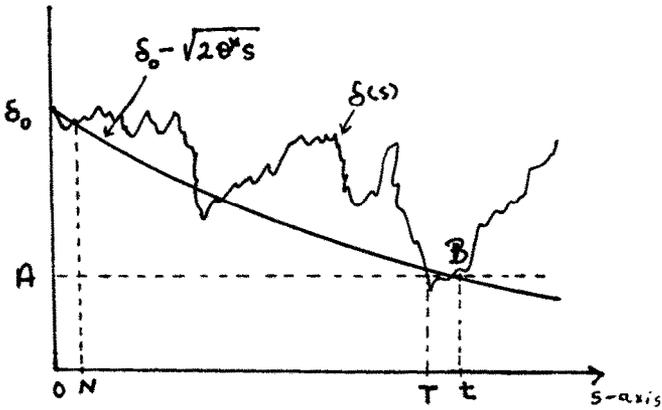


Figure 3: square root boundary

independent of t . So the investor has a 75% chance of realizing the maximum expected gain $\mu_1(w_t^*, t)$ under strategy 1!

An explicit expression for $\mu_1(w_t^*, t)$ is easily obtained as follows:

$$\begin{aligned}
 \mu_1(w_t^*, t) &= \sigma w_t^* \int_0^t (t/s - 1) g(s, w_t^*) ds \\
 &= \sigma \sqrt{2\theta^* t} \int_{\theta^*}^{\infty} (\tau/\theta^* - 1) \frac{\tau^{-1/2}}{\sqrt{\pi}} e^{-\tau} d\tau \\
 &= 2.9\sigma\sqrt{t},
 \end{aligned} \tag{20}$$

Similarly, the variance of Y_τ can be found:

$$\begin{aligned}
 \mu_2(w_t^*, t) &= (\sigma w_t^*)^2 \int_0^t (t/s - 1)^2 g(s, w_t^*) ds \\
 &= 2\theta^* t \sigma^2 \int_{\theta^*}^{\infty} (\tau/\theta^* - 1)^2 \frac{\tau^{-1/2}}{\sqrt{\pi}} e^{-\tau} d\tau \\
 &= 28.1\sigma^2 t,
 \end{aligned}$$

giving

$$\text{Var}[Y_\tau] = 19.7\sigma^2 t. \tag{21}$$

Finally as an example, an investor purchasing a ZCB with 10 years left to maturity in a market with $\sigma = 1\%$ can expect a yield of 9.17% in excess of the initial yield to maturity δ_0 , with a standard deviation of 14.04%.

4 Summary

A perfectly competitive zero-coupon bond market was considered. It had a flat term structure for interest rates, with any changes in market rates resulting in parallel shifts in the entire yield curve. A risk neutral investor in such a market was assumed to have a selling strategy which required him/her to sell the bond whenever the market rates fall to a certain fixed level. When the spot force of interest follows Brownian motion without drift, it was proved that the investor can maximize his/her expected internal rate of return by following these steps:

- step 1 choose ZCBs with longest remaining time to maturity (t^*);
- step 2 sell them when the force of interest falls by $0.3161\sigma\sqrt{t^*}$ below the level the investor's guaranteed rate of return.

These steps to maximize the expected force of IRR are new to the finance literature. The expected mean and variance of the excess force of IRR are given in equations (20) and (21).

5 Comments

The objective of this paper has been to derive an explicit selling strategy for ZCBs. To this end a simple diffusion model of the term structure of the interest rate process was used. Other diffusion models, see equation (4), need to be considered, e.g., (1) an Ornstein-Uhlenbeck (O-U) process with a reflecting barrier at the origin, or (2) the mean reverting process. Unfortunately, the task of finding the optimum level of decline in the interest rates which maximizes $\mu_1(w, t)$ for ZCBs will, in general, be a difficult one. For the O-U process, the stopping times T and τ (equations (9) and (10)) do not have pdf's that are explicitly available. Nobile, Ricciardi and Sacerdote [17], and Ricciardi and Sato [19] have discussed the problems inherent in finding the first passage time density of the O-U process. For more general

Gaussian processes, obtaining first passage probabilities for T and τ are exceedingly difficult; see Durbin [7].

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