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Immunization as a Maximin Strategy
The Effects of Transaction Costs and Imperfect Divisibility of Financial Assets

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Summary

Bierwag and Khang proved in 1979 that immunization is a maximin strategy of a game against nature and Khang, in 1983, proved that keeping the duration of a fixed income portfolio equal to the remaining horizon planning period by a continuous rearrangement of that portfolio is also a maximin strategy under a set of hypothesis including some such "perfect divisibility of financial assets" and "absence of transaction costs".

In this paper, it is developed a portfolio selection model following a maximin strategy and using linear programming. From a basic model which includes the hypothesis mentioned above we first relax the hypothesis of perfect divisibility of financial assets and, later on, the absence of transaction costs, making some comments arising from the application of the model.

The most remarkable result is the fact that if we include transaction costs in the model, the maximin strategy may be not to immunize the portfolio in the sense of making its duration equal to the remaining horizon planning period but some sort of partial immunization.

Résumé

L'Immunisation en tant que Stratégie Maximin Les Effets des Coûts de Transaction et de la Divisibilité Imparfaite des Actifs Financiers

En 1979, Bierwag et Khang prouvèrent que l'immunisation est une stratégie maximin d'un jeu contre la nature et en 1983 Khang démontra que garder la duration d'un portefeuille à revenu fixe égale à la durée de planification de l'horizon restant par une réorganisation constante de ce portefeuille, est également une stratégie maximin régie par un ensemble d'hypothèses y compris celles de "divisibilité parfaite des actifs financiers" et "d'absence des coûts de transaction".

Dans cet article, nous développons un modèle de sélection de portefeuille en suivant une stratégie maximin et en utilisant une programmation linéaire. A partir d'un modèle de base qui inclut les hypothèses mentionnées ci-dessus, nous modérons tout d'abord l'hypothèse de la divisibilité parfaite des actifs financiers et plus tard, de l'absence des coûts de transaction en faisant quelques commentaires survenant de l'application du modèle.

Le résultat le plus remarquable est le fait que si nous incluons les coûts de transaction au modèle, la stratégie maximin pourra être non pas d'immuniser le portefeuille dans le sens d'égaler sa durée à la durée de planification d'horizon restant mais une sorte d'immunisation partielle.

I.- INTRODUCTION.

Immunization is a concept that was initially developed by the British actuary F. M. Redington[1] in 1952 when trying to protect the net value of an insurance company against interest rate fluctuations.

Later contributions used the word immunization as follows: to immunize a fixed income portfolio means to guarantee a minimum return for a given period of time - the horizon planning period (HPP)- with independence of the term structure of interest rates movements during that period. The minimum return is the one offered by the market at the beginning of the HPP for a zero coupon bond with maturity at the end of the HPP.

It must be emphasized that immunization protects a fixed income portfolio only against interest rate fluctuations but not against other sorts of risk (as default risk) which must be approached in a different way.

The influence of interest rate changes on the final return of a fixed income portfolio is quite complex:

Let's suppose that just after buying a fixed-income portfolio an increase of interest rates takes place and that there are no more interest rate movements during the remaining HPP.

The increase of interest rates causes immediatly a decrease of portfolio value, but because of the higher

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[1] REDINGTON, F.M.: "Review of the Principles of Life-Office Valuations". Journal of the Institute of Actuaries. vol. 18, 1952.

interest rates the payments generated by the portfolio (such as coupon payments, principal redemptions, etc.) will be reinvested at a higher rate and thus, portfolio value will grow faster. That is why we can not, in principle, forecast which the effect of the interest rates increase on the final portfolio value will be.

Similarly, if, after the purchase of the portfolio, a decrease of interest rates takes place, we have, on the one hand, that the portfolio value will increase but, on the other hand, the payments generated by the portfolio will have to be reinvested at a lower rate. Depending on which of those countereffects is bigger, the portfolio final value will be higher or lower than the value the portfolio would have had if interest rates had not changed.

It is in this context where should be inserted the **Immunization Theorem** developed by Fisher and Weil[2] in 1971.

In this theorem there appears a concept that plays a key role when implementing a immunizing strategy: **portfolio duration.**

Fisher and Weil defined the duration of a fixed-income asset "i" as follows:

$$D^{i} = \frac{\sum_{k=1}^{q^{i}} k \cdot C_{k}^{i} \cdot (1+r_{k}^{-})^{-k}}{\sum_{k=1}^{q^{i}} C_{k}^{i} \cdot (1+r_{k}^{-})^{-k}} \qquad [1]$$

[2] FISHER, L. and WEIL, R.L.: "Coping with the Risk of Interest Rate Fluctuations, Returns to Bondholders from Naïve and Optimal Strategies". Journal of Business. October 1971.

where:

C_k^i is the amount of the k-th payment generated by asset "i"

q^i is the total number of payments generated by asset "i"

r_k^- is the market mean interest rate corresponding to the period [0, k]

D^i is the duration of asset "i"

The portfolio total duration is given by:

$$D = \sum_{i=1}^{n} (V^i/V) \cdot D^i \qquad [2]$$

where:

(V^i/V) is the porportion of asset "i" with respect to the total portfolio value

D^i is the duration of asset "i"

n is the number of different assets included in the portfolio.

Looking at [1] and [2], we can interpret the portfolio duration as a weighted mean term where the weights are the proportion of the present value of each payment generated by the portfolio with respect to its total value. Thus, duration can be considered as a magnitude measured in time units[3].

Once duration has been defined we can enunciate Fisher and Weil's theorem:

[3] In any case we have to point out the fact that duration can be also obtained as an elasticity and so as an adimensional magnitude.

"A portfolio of nonnegative payments is immunized at time t_o if the duration D_{t_o} at time t_o of its promised payments is equal to the length of the desired holding period $T-t_o$"[4] [5]

Throughout this paper we will mean by immunization to keep a portfolio duration equal to the HPP.

With respect to this theorem, it is necessary to point out the following fact: The immunizing duration formula depends on the shape and assumed behaviour of the term structure of interest rates. The different ways in which the term structure rates may change are called by Bierwag "stochastic process"[6] and so we can say that the immunizing duration formula depends on the asumed stochastic process[7].

Therefore, it may happen the stochastic process followed by market interest rates to be different from that initially assumed, in such a way, that duration value used to immunize a portfolio can be significantly different from the one that would have immunized the portfolio. This is called the **stochastic process risk**".

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[4] FISHER,L. and WEIL,R.L.: Op. cit.

[5] In this case, "the desired holding period $T-t_o$" is equivalent to the HPP.

[6] BIERWAG, G.O.:Duration Analysis. Managing Interest Rate Risk. Ed. Ballinger, Cambridge, Massachussets 1987, page 253.

[7] In fact, the duration formula obtained by Fisher and Weil assumes explicitly that the force of interest follows an additive stochastic process or equivalently that the mean interest rate $r_{\bar{k}}$ follows a multiplicative stochastic process. Other alternative duration formulas can be seen in BIERWAG, G.O.; KAUFMAN, G.G. and TOEVS, A.: "Duration: Its Development and Use in Bond Porfolio Management". Financial Analyst's Journal. July-August 1983.

With respect to this point Fong and Vasiceck[8] proved that the more concentrated the payments generated by the portfolio around the end of the HPP are, the smaller the stochastic risk is. So that to minimize such a risk the investor has to choose, among the immunized portfolios, the one with a minimum dispersion.

We have to take into account that as far as the investor has at least three different assets (one of them with a duration bigger than the HPP an another with a duration less than the HPP) the number of combinations to create an immunized portfolio is unlimited. But, as we have just said, the investor will prefer the combinations of assets that generates the most concentrated payments stream around the end of the HPP.

Coming back to the Fisher and Weil's Theorem we have to say that it was enunciated assuming a set of hypothesis including the "perfect divisibility of financial assets" and the "absence of transaction costs".

Further research proved that under those assumptions immunization was a maximin strategy and later on that keeping the portfolio duration equal to the remaining HPP during all that period -Dynamic Immunization Strategy- is also a maximin strategy.

In this paper it is developed a portfolio selection model using linear programming in order to design a maximin strategy but relaxing the hypothesis of perfect divisibility of financial assets and including the

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[8] FONG and VASICECK: "The Tradeoff between Return and Risk in Immunized Porfolios". Financial Analyst's Journal. September-October 1983.

existence of transaction costs. The effects of these two elements are analysed and compared with the existing literature about immunization.

II.- THE BASIC MODEL.

Two theoretical contributions are going to be used to develop this portfolio selection model:

* Bierwag and Khang (1979)[9]: Immunization is a maximin strategy.
* Dantzig (1951)[10]: The maximin solution of a game can be obtained through a linear programme.

According to these results the immunizing strategy can be found through a linear programme.

Initially the model will assume a set of hypothesis that will be relaxed later on:

1. Inital hypothesis.

1. The term structure of interest rates (TSIR) is flat.
2. The TSIR changes follow an additive stochastic process.
3. There is a bound for interest rate fluctuations so that a maximum and a minimum interest rates are considered. This hypothesis doesn't affect the basic

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[9] BIERWAG,G.O. and KHANG, C.: "An Immunization Strategy is a Minimax Strategy". Journal of Finance. May 1979.

[10] DANTZIG, G.B.: A Proof of the Equivalence of the Programming Problem and the Game Problem. Activity Analysis of Production and Allocation. John Wiley, New York, 1951.

model but its introduction is necessary for further enlargements.

4. Financial markets are competitive: individual investor decisions don't affect interest rates which are given exogenously.

5. The stochastic process risk is not considered by investors.

6. Absence of transaction costs.

7. Perfect divisibility of financial assets.

2. Game description.

We are going to describe immunization as the result of a "game against nature".

As we have already said, the target of an immunizing strategy is to guarantee a minimum return over the HPP or, equivalently, a minimum portfolio value at the end of the HPP.

This final portfolio value depends on two elements: the evolution of interest rates and the composition of the portfolio.

We assume that the investor is going to invest an amount of I ptas. and that there are n different fixed-income assets available in the market among which select his portfolio. The portfolio composition is, then, the "strategy" followed by the investor.

We also assume that just after the purchase of the selected fixed income portfolio interest rates may change from its current level (denoted by r_c) to any of the following possible values:

$$r_1, \ldots, r_j, \ldots, r_m \qquad (r_1<\ldots<r_m)$$

remaining unchanged until the end of the HPP.

These possible values of interest rates can be regarded as the "actions" nature can take.

Now we can consider the following n x m payoff matrix:

$$\begin{array}{l|ccccc|}
 & r_1 & \ldots & r_j & \ldots & r_m \\
\hline
\text{Asset } 1 & v_{11} & \ldots & v_{1j} & \ldots & v_{1m} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\text{Asset } i & v_{i1} & \ldots & v_{ij} & \ldots & v_{im} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\text{Asset } n & v_{1n} & \ldots & v_{nj} & \ldots & v_{nm} \\
\hline
\end{array}$$

where v_{ij} denotes the portfolio value at t = T (the end of the HPP) if an amount of I ptas. was invested at t = 0 in asset "i" and just afterwards interest rate became r_j, remaining unchanged until the end of the HPP[11].

According to Bierwag and Khang immunization is the maximin solution of the game described by that payoff matrix.

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[11] If column j corresponds to the value of interest rate r_c, i.e. the initial value of interest rate, all its elements will be equal independently of the asset where investment is made in.

2. Decription of the equivalent linear programme.

As we have already mentioned the maximin strategy of a game as the one described above can be obtained as the solution of a linear programme.

Let x_i be the percentage of the initial portfolio value (I) invested in asset "i" (i=1, ... , n). If interest rate shifts from r_o to r_j, the portfolio value at t=T (final of the HPP) is given by:

$$v_{1j} \cdot x_1 + \ldots + v_{ij} \cdot x_j + \ldots + v_{nj} \cdot x_n \qquad [3]$$

If we are seeking to guarantee a minimum portfolio value at t=T (denoted by V) it must be satisfied the following inequality:

$$v_{1j} \cdot x_1 + \ldots + v_{ij} \cdot x_j + \ldots + v_{nj} \cdot x_n \geq V \qquad \text{for all } j \quad [4]$$

i.e. independently of the portfolio composition, its value at t=T must be equal or bigger that the minimum value V. And as the investor wants V to be as big as possible, immunization can be approached through the following linear programme:

(A)

$$\begin{array}{lll} \text{Max} & V & \\ \text{S. to} & \sum_{i=1}^{n} v_{ij} \cdot x_i - V \geq 0 & j = 1, \ldots, m \\ & \sum_{i=1}^{n} x_i = 1 & \\ & x_i \,,\ V \geq 0 & i = 1, \ldots, n \end{array}$$

The programme solution x_i^* indicates the percentage of investment that has to be assigned to buy asset "i" and V^* is the minimum portfolio value guaranteed at the end of the HPP[12].

It may be worth pointing out that the solution of this linear programme is not a corner solution, i.e. unique, but multiple . This is an expected result if we take into account that all immunized portfolios guarantee the final value V^* and that the number of immunized portfolios that can be obtained from three or more different assets is unlimited.

We can rewrite programme A in standard form by adding slack variables:

$$
\begin{array}{lll}
\text{Max} & V & \text{(B)} \\
\text{S. to} & \sum_{i=1}^{n} v_{ij}\cdot x_i - V - h_j = 0 & j = 1, \ldots , m \\
 & \sum_{i=1}^{n} x_i = 1 & \\
 & x_i \ , V \geq 0 & i = 1, \ldots , n
\end{array}
$$

The slack variable h_j can be interpreted as the difference between the actual final value obtained as a consequence of an interest rate change to r_j and the minimum guaranteed value V^*.

[12] The value of V^* is equal to the value of the elements of the column v_{ic}, i.e. the column which includes the hypothesis that interest rate has not changed.

III. ENLARGEMENTS OF THE MODEL.

1.- Stochastic process risk.

According to Fong and Vasiceck the stochastic process risk can be minimized by choosing the portofolio that generates the most concentrated payments stream around the end of the HPP.

So that, in this section we modify programme A in order to make its solution to be the immunized portfolio of minimum dispersion.

The method proposed to solve this problem consists of penalizing the objective function as follows:

For each asset, we calculate the dispersion measure (d_i) proposed by Fong and Vasiceck[13]:

$$d_i = \frac{\sum_{k=1}^{q^i} (t_k^i - T)^2 \cdot C_k^i \cdot (1+r)^{-t_k^i}}{\sum_{k=1}^{q^i} C_k^i \cdot (1+r)^{-t_k^i}} \qquad [5]$$

where:

- q^i is the number of payments until maturity generated by asset "i"
- C_k^i is the amount of the k-th payment generated by asset "i"
- $(t_k^i - T)$ is the difference (days, months, years,...) between the due date of the k-th payment generated by asset "i" and the end of the HPP

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[13] See FONG and VASICECK: Op. cit..

d_i is the choosen dispersion measure of the payment stream generated by asset "i"[14].

This dispersion measure is translated into money units by the following function:

$$b_i = A \cdot d_i \qquad A > 0 \qquad [6]$$

where the value of coefficient A depends on the risk aversion of each individual investor introducing a subjective element in the model. The value of b_i represents the "cost" of investing the total amount I in asset "i" due to the stochastic risk[15].

But if the proportion of the total portfolio value invested in asset "i" is x_i the total penalization will be given by the term:

$$\sum_{i=1}^{n} b_i \cdot x_i \qquad [7]$$

This term should be introduced in the model by reducing the objective function value.

Thus, the linear programme becomes:

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[14] If asset "i" is a zero coupon bond with maturity at t=T the value of d_i is equal to zero. Obviously an asset with those characteristics is free of stochastic process risk and so, any measure of that risk must take zero value when applied to an asset as the one described.

[15] Other family of functions that could incorporate different attitudes toward risk is $b_i = A \cdot d_i^B$ where B is a parameter that indicates if the riks aversion grows proportionally (B=1), more than proportionally (B>1) or less than proportionally (B<1) with dispersion. The value of this parameter would be also subjective. See ZOIDO MARTINEZ y TERRIENTE QUESADO:"Modelo de Optimización de Renta Fija". Papeles de Economía nº 10. Madrid, 1985.

$$\text{Max} \quad V - \sum_{i=1}^{n} b_i \cdot x_i \qquad \text{(C)}$$

$$\text{S. to} \quad \sum_{i=1}^{n} v_{ij} \cdot x_i - V \geq 0 \qquad j = 1, \ldots, m$$

$$\sum_{i=1}^{n} x_i = 1$$

$$x_i \,, V \geq 0 \qquad i = 1, \ldots, n$$

The solution of this programme will provide the immunized portfolio (duration equal to HPP) of minimum dispersion and, on the whole, it will be unique.

2. Perfect divisibility of investments.

Nowadays, it may be necessary to leave the hypothesis of perfect divisibility of investments if we observe the current financial markets where standardized transactions of a huge volume are common and notes with very high face value are traded. So that, it may happen that it is not possible to invest an exact amount $x_i^* \cdot I$ in asset "i".

In order to solve this problem we will change the decision variable of the model by a new one: the number of units of asset "i" bought. We denote these new variables by N_i.

This change of variable is not only a formal one but it implies a change of the nature of the model which will become a problem of integer linear programming.

Let's define P_i as the current quotation in ptas. at t=0 (the beginning of the HPP) of a unit of asset "i"; then the relationship between x_i and N_i is given by the following equation:

$$x_i = \frac{P_i}{I} \cdot N_i \qquad [8]$$

Then the m first constraints of programme C can be rewritten as:

$$\sum_{i=1}^{n} v_{ij} \frac{P_i}{I} \cdot N_i - V \geq 0 \qquad j = 1, \ldots, m \qquad [9]$$

or alternatively,

$$\sum_{i=1}^{n} v_{ij} \cdot P_i \cdot N_i - I \cdot V \geq 0 \qquad j = 1, \ldots, m \qquad [10]$$

Then, the last constraint of programme C will become:

$$\sum_{i=1}^{n} \frac{P_i}{I} \cdot N_i \leq 1 \quad \text{or} \quad \sum_{i=1}^{n} P_i \cdot N_i \leq I \qquad [11]$$

because due to the integrity of N_i it will be very difficult to find an optimal solution that implies to invest the exact amount I among the N available assets . Of course, we assume that the remaining amount will be, in practice, negligible, but the constraint must be rewritten as an inequality.

In the same way, the objective function will suffer the same change, becoming:

$$V - \sum_{i=1}^{n} b_i \cdot \frac{P_i}{I} \cdot N_i = V - \sum_{i=1}^{n} b_i' \cdot N_i \qquad [12]$$

where $\quad b_i' = b_i \cdot \dfrac{P_i}{I}$

After all those changes, the linear programme, if we follow a maximin strategy, is:

$$\boxed{\begin{array}{lll} & & \text{(D)} \\ \text{Max} & V - \sum_{i=1}^{n} b_i' \cdot N_i & \\ \text{S. to} & \sum_{i=1}^{n} v_{ij} \cdot \frac{P_i}{I} \cdot N_i - V \geq 0 & j = 1, \ldots, m \\ & \sum_{i=1}^{n} P_i \cdot N_i \leq I & \\ & N_i, V \geq 0 & i = 1, \ldots, n \\ & N_i \in N & \end{array}}$$

3. Transaction costs.

Up to now, the model has only tried to solve the problem of selecting the initial portfolio if we are going to follow a maximin strategy. Under the hypothesis of absence of transaction costs a continuous portfolio rearrangement has to be made in order to keep it immunized. That is the meaning of the **Dynamic Global Portfolio Immunization Theorem** enunciated by Khang in 1983:

> "Consider an investor who has a planning period of length m. Suppose the forward interest rates structure shifts up or down by a stochastic shift parameter at any time and an indetermined number of times during the investor's planning period. If the investor follows the Strategy in such a world, then the investor's wealth at the end of his or her planning period will be no less than the amount anticipated on the basis of the forward interest rate structure observed initially (or at time 0). Furthermore, the investor's wealth at time m will actually be greater than the amount anticipated initially if at least one shock takes place during the planning period"[16].

The question arising now is weather this result is still valid when transaction costs are included in the model.

Thus, in this section we are going to discuss the selection of the optimal portfolio following a maximin strategy once a given period of time has passed since the initial portfolio was bought, but taking into account that portfolio rearrangements imply a loss due to transaction costs.

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[16] KHANG, C.: "A Dynamic Global Porfolio immunization Strategy in the World of Multiple Interest Rate Changes: A Dynamic Immunization and Minimax Theorem". Journal of Financial and Quantitative Analysis. September, 1983.

We have then that the new optimal portfolio will depend on the initially selected one and the costs that portfolio rearrangements causes.

Let's suppose that transaction costs are a percentage α of the assets value traded in any rearrangement (we assume this percentage is the same for all assets[17]).

Let N_i^- be the number of units of asset "i" of the initial portfolio. Then it is clear that if we rearrange the portfolio composition we will incur the following transaction costs:

$$\sum_{i=1}^{n} \alpha \cdot P_i \cdot |N_i - N_i^-| \qquad [13]$$

where $|N_i - N_i^-|$ represents the number of units of asset "i" bought and sold when restructuring the portfolio.

However, in a linear programme we can not include variables in absolute value, so we have to make the following changes:

First, we add "n" new constraints in the model:

$$N_i - N_i^- = M_i' - M_i'' \qquad i = 1, 2, \ldots, n \qquad [14]$$

$$M_i' \, , \, M_i'' \in N$$

[17] In practice that cost may differ depending upon what sort of asset is traded or even the intermediate with who we operate. In this case we can simply substitute coefficient α by the corresponding coefficients α_i.

Second, these n new constraints imply the addition of $2 \cdot n$ new integer variables that will also appear in the objective function in order to introduce transaction costs.

The new objective function is:

$$V - \sum_{i=1}^{n} b_i' \cdot N_i - \sum_{i=1}^{n} \alpha \cdot P_i \cdot (M_i' + M_i'') \qquad [15]$$

Taking into account that positive values of M_i' and M_i'' penalize the value of the objective function we can ensure that, at least one of these two new variables, will take zero value at the optimal point. Because of this reason it is guaranteed that

$$|N_i - N_i^-| = (M_i' + M_i'') \qquad [16]$$

and so the term

$$\sum_{i=1}^{n} \alpha \cdot P_i \cdot (M_i' + M_i'') \qquad [17]$$

includes all transaction costs derived from a possible portfolio rearrangement.

Nevertheless, for the value of the objective function to have an economical interpretation, transaction costs must be valuated at time T and not at the moment at which the rearrangement takes place. To solve this problem we have to redefine coefficient α multiplying it by the accumulation factor $(1+i_c)^T$. Anyway we will keep the notation α for transaction costs.

All those changes increase the problem dimension due to the addition of new variables (M_i' and M_i'') and new constraints ($N_i - N_i^- = M_i' + M_i''$). The linear programme is then:

$$
\begin{array}{lll}
 & & \text{(E)} \\
\text{Max} & V - \sum_{i=1}^{n} b_i' \cdot N_i - \sum_{i=1}^{n} \alpha \cdot P_i \cdot (M_i' + M_i'') & \\
\text{S. to} & \sum_{i=1}^{n} v_{ij} \cdot \dfrac{P_i}{I} \cdot N_i - V \geq 0 & j = 1, \ldots, m \\
 & \sum_{i=1}^{n} P_i \cdot N_i \leq I & \\
 & N_i - M_i' + M_i'' = N_i^- & i = 1, \ldots, n \\
 & N_i,\ M^i,\ M^{i\prime},\ V \geq 0 & i = 1, \ldots, n \\
 & N_i,\ M_i',\ M_i'' \in N &
\end{array}
$$

The solution of this linear programme provides the maximin strategy taking into account transaction costs.

The most important results that arise from the application of this model is the fact that depending on the level of transaction costs, i.e. the value of α, immunization may be different from the maximin strategy and so diverging from the **Dynamic Minimax Theorem** according to which "the dynamic immunization stratey (or the Strategy) is also a minimax strategy", i.e. "the minimum rate of return over the planning period associated with any other strategy is lower than that associated with the dynamic immunization strategy"[18].

In any case we don't mean that the former theorem is wrong but taking into consideration transaction costs , the maximin strategy is some sort of "partial immunization": we would only immunize the portfolio if the transaction costs are not bigger than the possible loss derived from not adopting an immunizing strategy.

IV.- FINAL REMARKS.

We have described an alternative method for managing fixed income asset portfolios under the viewpoint of immunization . That has been possible thanks to theoretical contributions of several authors.

The basic model developed in section II has been enlarged in section III in order to bring it up to reality through the relaxation of some of the initial hypothesis.

The most important results are obtained when introducing transaction costs in the portfolio rearrangement process:

First, as we have already said, transaction costs may affect immunization in the sense that only if the possible loss derived from not following the immunizing strategy are less than transaction costs, the maximin strategy will be to immunize.

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[18] KHANG, C.: Op. cit.

Additionally, transaction costs may affect to the dispersion of the optimal portfolio as follows: if the dispersion is not penalized enough it may not be convenient to rearrange the portfolio to obtain the immunized portfolio with minimum dispersion.

The linear programme E can be enlarged if it is intended to extend the model applicability . For example diversification can be introduced in the model limiting the participation of an asset or group of asset in the optimal portfolio.

Another possibility could be to state the model including the hypothesis of perfect divisibility of financial assets. In this case the linear programme would be:

$$
\begin{array}{lll}
\text{Max} & V - \sum_{i=1}^{n} b_i \cdot x_i - \sum_{i=1}^{n} \alpha \cdot P_i \cdot (M'_i + M''_i) & (E') \\
 & & \\
\text{S. to} & \sum_{i=1}^{n} v_{ij} \cdot x_i - V \geq 0 & j = 1, \ldots, m \\
 & & \\
 & \sum_{i=1}^{n} x_i = 1 & \\
 & & \\
 & x_i - M'_i + M''_i = \bar{x}_i & i = 1, \ldots, n \\
 & & \\
 & x_i,\ M'_i,\ M''_i,\ V \geq 0 & i = 1, \ldots, n
\end{array}
$$

This model can be extended parametrizing the coefficient α in the objective function, working out the maximin strategy depending on the value of α. In this way we could obtain the value of α from which immunization is not a maximin strategy.

Another result is the period of time that must pass to rearrange the portfolio if we are following a maximin strategy. That period depends,in any case, on many factors such as interest rate evolution , new asset issues during the HPP and, of course, the transaction costs.

Finally we have to remember that the results of the application of this model depends on its assumptions and so that they will be valid only if reality does not differ too much from those assumptions.

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