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A Stochastic Dynamic Valuation Model for Investment Risks

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Summary

A dynamic valuation model for managing certain assets and liabilities under random interest rates is presented. The model postulates static and dynamic equilibrium conditions. Concerning the question of which discount rate is appropriate for discounting insurance reserves, it is shown how the interest rate risk involved in a guaranteed technical discount rate can be covered by a call-option whose design and price is determined. Furthermore it is shown how the general model can be handled numerically, at least approximately, using computer programs to solve the algebraic moment problem. This opens the way for further applications. The approximate solution using only mean and variance of the processes encountered explains why the Markowitz (1952) approach to portfolio analysis is extensively used in modern portfolio theory.

Résumé

Un Modèle Stochastique d'Evaluation Dynamique pour les Risques de Placement

Nous présentons un modèle d'évaluation dynamique pour la gestion de certains actifs et engagements dans des conditions de taux d'intérêt aléatoires. Le modèle pose comme principe des conditions d'équilibre dynamiques et statiques. En ce qui concerne la question du taux d'escompte approprié pour escompter les réserves d'assurance, nous montrons comment le risque de taux impliqué dans un taux d'escompte technique garanti peut être couvert par une option d'achat dont le but et le prix sont déterminés. De plus, nous montrons comment le modèle général peut être traité numériquement, au moins de façon approximative, en utilisant des programmes informatiques pour résoudre le problème de moment algébrique (algebric moment problem). Ceci ouvre la voie pour d'autres applications. La solution approximative n'utilisant que la moyenne et les écarts des procédés rencontrés explique pourquoi l'approche Markowitz (1952) d'analyse des portefeuilles financiers est énormément utilisée dans la théorie moderne des portefeuilles.

1. Stochastic modelling of interest rates.

The evolution of interest rates is assumed to be described by the stochastic variable D_t representing the discounted value of a unit of money payable at time t. The stochastic variable X_t defined by $D_t=\exp(-X_t)$ describes the force of interest over the time period t. The stochastic variable $R_t=\exp(X_t)$ representing the accumulated value of a unit of money payable at time t is also considered.

There are several ways to model stochastic interest rates depending on the purposes of the practitioner. In continuous time the simplest model to consider for X_t is a Gauss process. This means that X_t has a normal distribution for each t or that R_t has a lognormal distribution. More realistic and important in applications are models in discrete time. Let δ_k be a random force of interest in the k-th valuation period $[k-1,k]$, $k=1,2,\ldots$, and let $D_k=\exp(-\delta_k)$ be the corresponding discount factor. Then one has

$$X_0 = 0, \qquad X_t = \sum_{k=1}^{t} \delta_k, \quad t=1,2,\ldots$$

It is possible to model the future forces of interest δ_k by an ARIMA process (=AutoRegressive-Integrated-Moving-Average process) as done by several authors, for example Pollard(1971), Bellhouse/Panjer(1980/81), Giacotto(1986), Dhaene(1989), Kremer(1990) and others. Using this process it is known that X_t is normally distributed. As alternative, which seems mathematically even more tractable, one can assume that D_1, D_2, ... is a Markov chain with finite state space and homogeneous transition probabilities. This approach is for example taken in Schnieper(1983).

Market or nominal interest rates are positive. Hence one should require that the interest process R_t-1 is non-negative with probability one. However realistic models are already obtained assuming $Pr(R_t<1)$ is small enough. In particular this can be fulfilled when R_t is lognormal.

In this paper we restrict ourselve to a model of random interest in continuous time giving at some places useful comments in discrete time. The development of the theory for alternative models as well as the study of further applications are reported to later.

2. The valuation model.

An Investment Risk Management problem includes two kinds of elements, namely a stream of liability payments L_j to be paid at future times s_j, $j=1,2,\ldots$, which are to be funded by a stream of asset cash flows A_k occuring at times a_k, $k=1,2,...$. Assets are assumed to be non-callable and default-free. Under portfolio we understand the collection of these liability payments and asset cash flows.

Notations concerning the stochastic evolution of interest are the same as in the preceding Section. It is always assumed that X_t is normally distributed with mean $\mu(t)$ and variance-covariance $Cov[X_t,X_s]=c(t,s)$. The variance of X_t, equal to $c(t,t)$, is simpler denoted by $\sigma^2(t)$.

Given a discrete model for the yearly random forces of interest δ_k, a partial problem is to find $\mu(t)$, $\sigma^2(t)$ and $c(t,s)$ for $t, s \geq 1$. For the important class of ARIMA processes δ_k, it is known that X_t is normally distributed and in this case $\mu(t)$, $\sigma^2(t)$ and $c(t,s)$ can be generated using Dhaene's(1989) algorithm. In view of this important special case, we always assume that $\mu(t)$, $\sigma^2(t)$ and $c(t,s)$ are known. In the alternative discrete approach using Markov chains, moments of X_t can also be obtained (see Schnieper(1983)).

Consider now the current value at time t of all liability payments given by the stochastic process

$$L(t) = (\sum_j L_j D_{\ell_j})R_t,$$

and the current value at time t of all asset cash flows given by

$$A(t) = (\sum_k A_k D_{a_k})R_t.$$

The underwriting gain $G(t)$ of the portfolio at time t is defined to be the current net value of all asset cash flows and liability payments contained in the portfolio. It is given by

$$G(t) = A(t) - L(t).$$

The underwriting loss $V(t)$ at time t is

$$V(t) = -G(t) = L(t) - A(t).$$

The present value of the portfolio is just the current net value of the portfolio at time $t=0$, that is $G(0)$. We assume that the liability payments are fully funded at the beginning of the investment planning period, that is

$$E[G(0)] = 0. \tag{2.1}$$

Interpreting this condition we say that the portfolio is in a static equilibrium as only present valuation is of concern.

Classical immunization theory operates initially in a deterministic context. Assume that constant forces of interest or forward rates δ_k are given and summarize them in the vector $\underline{\delta}=(\delta_1, \delta_2, \ldots)$. By assumption (2.1) one has

$$G(0) = G(0)(\underline{\delta}) = 0.$$

The <u>classical immunization problem</u> consists to find conditions that guarantee $G(0)(\underline{\delta}+\underline{\epsilon}) \geq 0$ if the assumed force of interest vector $\underline{\delta}$ changes to $\underline{\delta} + \underline{\epsilon}$. In actuarial literature generalizations of the Fisher-Weil immunization theory were recently obtained by Shiu(1988), who also proposes to construct immunized portfolios by the method of Linear Programming. Using his results it is possible to show by contradiction that one cannot in general completely eliminate investment risks. The best thing immunization theory can offer is a minimization of the investment risk by appropriately managing assets and liabilities. For details the reader should consult the abundant amount of recent literature on immunization theory.

In its essence classical immunization theory is a static theory since it refers only to present values. In this paper we consider an alternative dynamic valuation model for investment risks in a stochastic model of interest rates. The proposed method is of dynamic nature since it refers to any time of valuation.

Consider the stochastic process of the underwriting loss. To compensate a loss $V(t)=L(t)-A(t)>0$, the Financial Risk Manager is supposed to retain at time of valuation t an amount $P(t)$, to be determined, which should be financed by the underwriting gain $G(t)$, when positive. Therefore the retained amount must satisfy $0 \leq P(t) \leq G(t)_+$. Let further $U(t)=(P(t)-G(t))_+$ be the stochastic process describing the over-loss, that is the possible loss, which can arise after deduction of the underwriting gain from the retained amount. The resulting stochastic process

$$P(t) - U(t) =: NO(t)$$

is nothing else than the net outcome of the insurer after the amount $P(t)$ has been reserved to cover the financial risk at time t. It turns out to be judicious to require no profit and no loss on net outcomes at time t, that is

$$E[NO(t)] = E[P(t)-U(t)] = 0. \tag{2.2}$$

This equation is interpreted as a <u>dynamic equilibrium</u> portfolio condition. Every solution $P(t)$ to (2.2) defines together with the static condition (2.1) a dynamic valuation model for investment risks. Since a priori different choices of $P(t)$ are not excluded, different valuation models may be of interest. A possible choice is for example

$$(2.3) \qquad P(t) = g(t)G(t)_+, \quad 0 < g(t) < 1,$$

for a time-dependent factor $g(t)$. In this paper we restrict our attention to the appealing choice

$$(2.4) \qquad P(t) = \min\{ B(t), G(t)_+ \},$$

which means that the Financial Risk Manager retains, when possible, at most the time-dependent constant amount $B(t)$ from the underwriting gain. It is straightforward to check that (2.2) is satisfied if and only if $B(t)$ is solution of the dynamic expected value equation

$$(2.5) \qquad E[G(t)] = E[(G(t)-B(t))_+].$$

Our valuation model is borrowed from the theory discussed in Hürlimann(1990c). One finds there further motivations and justifications for considering it. The actuarial application of Section 3 is also discussed in Hürlimann(1990a/c), but goes farther in its conclusions. The method of Section 4 for solving the general model is not considered in the previous studies.

3. An actuarial application.

Insurance reserves which are invested on the capital market are subject to financial risks. We claim that the valuation model introduced in Section 2 permits to measure the investment risk on insurance reserves and to guarantee under certain conditions technical interest at given rates.

The following elements associated to an insurance portfolio are assumed to be known :

T : time of valuation, i.e. time at which the financial risk is to be covered

${}_TV$: the insurance reserves needed at time T

i_0 : the fixed technical annual rate of interest to which insurance reserves are subject according to insurance conditions

$r_0=1+i_0$: the accumulation factor corresponding to i_0

$R_t=\exp(X_t)$: the accumulated value process such that $E[R_T] > (r_0)^T$

To measure the financial risk involved in this special situation, let us show how to determine the varying capital $B(T)$ in formula (2.4) :

$$P(T) = \min\{ B(T), G(T)_+ \}.$$

In this formula $G(T)$ is the underwriting gain and $B(T)$ is interpreted as the amount needed at time T to cover losses from the investment risk on insurance reserves.

Let first determine the underwriting gain. The initial reserves ${}_0V$ at time $t=0$ define an asset cash flow $A_1={}_0V$ occuring at time $a_1=0$. It is assumed that the insurer invests this amount on the capital market which is subject to interest rate fluctuations following a certain stochastic model. At a future fixed time of valuation $T > 0$ the insurer is faced, according to insurance conditions, with the deterministic liability $V_1={}_TV={}_0V(r_0)^T$ occuring at time $s_1=T$. These are the only elements of our portfolio. In the notations of Section 2 it follows that $A(T)={}_0V.R_T$, $L(T)={}_0V.(r_0)^T$. One obtains

$$(3.1) \qquad G(T) = {}_0V.(R_T - (r_0)^T).$$

According to formula (2.5) the amount $B(T)$ must satisfy the expected value equation

$$(3.2) \qquad {}_0V.(E[R_T] - (r_0)^T) = E[({}_0V.R_T - {}_0V.(r_0)^T - B(T))_+].$$

To simplify set $B(T)={}_0V.b(T)$, where $b(T)$ is the amount needed at time T to cover the investment risk of a unit of the initial reserves. The condition (3.2) transforms to

$$(3.3) \qquad E[R_T] - (r_0)^T = E[(R_T - (r_0)^T - b(T))_+].$$

The right-hand side of this equation is clearly a stop-loss premium. Since this premium is in general strictly positive, a reason for the assumption $E[R_T] > (r_0)^T$ is easily recognized. The stop-loss premium can be evaluated under different assumptions, for example that the process R_t, $t \in [0,T]$, is lognormal distributed with mean $\mu(t)$ and variance $\sigma^2(t)$. In this Section one assumes that $E[R_T]=r^T$ for some expected constant accumulation factor $r > r_0$. This condition simplifies the discussion, but is not essential for the mathematical analysis. Since it is known that in the lognormal case

$$(3.4) \qquad E[R_T] = \exp\{\mu(T) + \tfrac{1}{2}\sigma^2(T)\},$$

one obtains through comparison that

$$(3.5) \qquad \mu(T) = T*\ln(r) - \tfrac{1}{2}\sigma^2(T).$$

One observes the following immediate connexion with Black-Scholes theory (Jarrow and Rudd(1983) for fundamentals). The equation (3.3) means that the differential in the expected amount of interest between stochastic and technical interest is equal to the accumulated <u>price of</u>

a <u>call-option</u> evaluated at time T with the interest accumulation factor r. The call, with <u>term</u> T, is on an <u>invested risk capital of one unit</u> and can be exercised at the <u>striking price</u> $b(T)+(r_o)^T$. Without any calculation Black-Scholes formula for the valuation of a call-option leads now to the implicit non-linear equation for $b(T)$:

(3.6) $$r^T N(x) - (b(T)+(r_o)^T)N(x-\sigma(T)) = r^T - (r_o)^T,$$

with $x = \{T*\ln(r) - \ln(b(T)+(r_o)^T)\}/\sigma(T) + \frac{1}{2}\sigma(T)$,
$\sigma(T) = \sqrt{Var[\ln(R_T)]}$ the volatility,
$N(x)$ the standard normal distribution.

In Hürlimann(1990a) this equation is solved numerically for $b(T)$ by iteration in case X_t is a <u>Wiener process with drift</u> such that $\mu(t)=\mu t$, $\sigma^2(t)=\sigma^2 t$, $\mu=\ln(r)-\frac{1}{2}\sigma^2$. Other cases can be solved similarly.

What does the connexion with Black-Scholes formula in the spirit of option pricing theory actually mean? Valued with the expected accumulation factor r let $C_t=r^{-(T-t)}E[(R_t - b(T)-(r_o)^T)_+]$ be the price of the call-option at time $t \in [0,T]$. Then (3.6) tells us that the price of the call-option to be paid at the beginning of the investment period is

(3.7) $$C_o = 1 - (r_o/r)^T.$$

It remains $1 - C_o=(r_o/r)^T$ to be invested on the capital market. In order to guarantee now the technical interest at rate i_o on the insurance reserves, one must have

(3.8) $$r^T = b(T) + (r_o)^T,$$

so that the option can be exercised at r^T if necessary. This condition on the expected accumulation factor r is also sufficient to guarantee the technical interest i_o. Indeed whatever the random interest rate moves the realized accumulated value of the initial investment will at least be $(1-C_o)r^T=(r_o)^T$. Introducing (3.8) in (3.6) one gets immediately

(3.9) $$b(T) = \tfrac{1}{2}(r_o)^T\{ N(\tfrac{1}{2}\sigma(T))/(1 - N(\tfrac{1}{2}\sigma(T))) - 1 \}$$

which allows to calculate the needed expected accumulation factor r to guarantee r_o in function of the technical accumulation factor r_o and the volatility $\sigma(T)$. As special case, if the accumulation factor r is known with certainty to be realized, that is if $\sigma(T)=0$, then $b(T)=0$ as should be.

In other words given any (deterministic) discount rate $\ln(r)$ offered on the capital market, for example the risk free rate in Black-Scholes theory, this result permits to find <u>quantitatively</u> the proper technical discount rate

$\ln(r_0)$, which is necessarily smaller, to be applied without incurring any interest risk. The corresponding qualitative statement seems to be known (announcement of Kozik(1990) to 2nd AFIR Colloquium). Taylor(1990) studies discounting of insurance loss reserves.

4. Solving the general model.

It is clear that the real-life financial world is more complex than the simple situation solved in Section 3. In general a financial problem involves a complicated network of assets and liabilities and the investor uses different financial instruments yielding different rates of return. Using appropriate generalizations the model of Section 2 could probably be refined to apply to more complex real-life problems. We content ourselve to show that the model of Section 2 can be solved numerically, at least approximately.

In the notations of Section 2 we can assume without loss of generality that $a_k=k$, $s_j=j$, $j, k=1,\dots,n$, by setting if necessary some A_k or L_j equal to zero. Net cash flows at time j are then defined by

$$(4.1) \qquad N_j = A_j - L_j, \quad j=1,\dots,n.$$

The process of the underwriting gain of the portfolio at time t is then given by

$$(4.2) \qquad G(t) = (\sum_{j=1}^{n} N_j D_j) R_t = \sum_{j=1}^{n} N_j \exp(X_t - X_j).$$

According to (2.5) one has to solve the equation

$$(4.3) \qquad E[G(t)] = E[(G(t) - B(t))_+]$$

under the assumption that $E[G(t)]>0$, which means that positive expected values of net cash flows are expected in the future. Clearly $G(t)$ takes also negative values, otherwise $B(t)=0$ is a trivial solution.

For fixed time of valuation t we approximate the random variable $G(t)$ by a discrete random variable $X(n)$ concentrated on n atoms $x_i \in \mathbb{R}$, $-\infty < x_1 < x_2 < \dots < x_n < \infty$, with weights $y_i \in \mathbb{R}$, $i=1,\dots,n$. For this one solves the algebraic moment problem of order n :

$$(4.4) \qquad \sum_{i=1}^{n} y_i (x_i)^k = m_k, \quad k=1,\dots,2n-1,$$

where the moments

$$(4.5)\qquad m_k = E[G(t)^k]$$

are given by the formulas (4.6) and (4.7) below. This can be done following the algorithm of the Appendix and using computer programs. Under the assumption that the process X_t is normally distributed with <u>known</u> mean $\mu(t)$ and variance-covariance $c(t,s)$, the moments of $G(t)$ are calculated as follows :

$$(4.6)\qquad m_k = E[(\sum_{j=1}^{n} N_j D_j)^k R^k]$$

$$= \sum_{\substack{\Sigma k_j = k,\\ k_j \geq 0}} k!(\prod_{j=1}^{n} N_j^{k_j}/k_j!)E[D_1^{k_1}\ldots D_n^{k_n} R_t^k],$$

where the moments in the sum are obtained using the characteristic function $\Phi(\underline{t})$ of the multivariate normal random vector $\underline{X}=(X_t, X_1, X_2, \ldots, X_n)$ evaluated at $\underline{t}=-i(k, -k_1, -k_2, \ldots, -k_n)$. One gets the formula (cf. Grimmett and Stirzaker(1982), 5.8, pp.104-105) :

$$(4.7)\qquad E[D_1^{k_1}\ldots D_n^{k_n} R_t^k] = E[\exp(kX_t - \sum_{j=1}^{n} k_j X_j)]$$

$$= \exp\{ k\mu(t) - \sum_{j=1}^{n} k_j \mu(j) + \tfrac{1}{2}k^2\sigma^2(t) + \tfrac{1}{2}\sum_{j=1}^{n} k_j^2 \sigma^2(j)$$

$$-k\sum_{j=1}^{n} c(t,j)k_j + \sum_{i<j} c(i,j)k_i k_j \}.$$

The solution $B=B(t)$ to (4.3) is thus approximated by the solution $B(n)$ to the equation

$$(4.8)\qquad E[(X(n) - B(n))_+] = E[X(n)].$$

Since $G(t)$ takes on negative values, one increases the order of approximation n until $X(n)$ takes on negative values, that is until at least $x_1 < 0$. Then $B(n)$ is evaluated using the following explicit result.

<u>Lemma</u>. Under the above assumptions one has

$$(4.9)\qquad B(n) = x_m - 1/\{1-F(x_m)\}.\{x_1 + \sum_{k=1}^{m-1} (1-F(x_k))(x_{k+1}-x_k)\}$$

where m is the greatest integer such that the curly bracket is negative and $F(x)=Pr(X(n)\leq x)$.

<u>Proof</u>. For brevity set $X=X(n)$, $B=B(n)$. Then (4.8) is

equivalent with

$$E[X] = E[(X-B)_+] = E[X] - \int_0^B x dF(x) - B(1 - F(B)).$$

Assume that m, depending on B, is chosen such that

$$x_m \le B < x_{m+1}.$$

Then one must have

$$B(1 - F(x_m)) = - \sum_{i=1}^{m} y_i x_i$$

$$= (1 - F(x_m))x_m - x_1 - \sum_{k=1}^{m-1} (1-F(x_k))(x_{k+1}-x_k),$$

which is formula (4.9) (last equality by induction on m). The condition on m is straightforward.

In practical applications of the above constructive result one will increase the order of approximation n until $|B(n+1) - B(n)|$ is sufficiently small.

Special values of the order of approximation lead to interesting interpretations. Look for example at the case $n=2$. The algebraic moment problem is solved by the atoms

$$(4.10) \qquad x_1=m_1 - \tfrac{1}{2}\sigma(\sqrt{4+\gamma^2} - \gamma), \quad x_2=m_1 + \tfrac{1}{2}\sigma(\sqrt{4+\gamma^2} - \gamma),$$

with the weights

$$(4.11) \qquad y_1=\tfrac{1}{2}(1 + \gamma/\sqrt{4+\gamma^2}), \quad y_2=\tfrac{1}{2}(1 - \gamma/\sqrt{4+\gamma^2}),$$

where σ^2, γ are the variance and the skewness of the process $G(t)$. A non-trivial solution to (4.8) is only obtained when $x_1<0$ and is given by $B(2)=-(y_1/y_2)x_1$. One shows easily that $B(2)$ is then maximized when $\gamma=2(B(2)-m_1)/\sigma$. This leads after straightforward calculation to the general inequality

$$(4.12) \qquad B(t) \le (1/4).(\mathrm{Var}[G(t)]/E[G(t)]),$$

which is derived alternatively and without restriction from the inequality of Bowers(1969). In particular this best upper bound explains why the Markowitz(1952) approach to portfolio analysis, that is mean-variance analysis, is extensively used in modern portfolio theory (cf. for example Müller(1988) for a recent survey on this subject).

Appendix : A remark on the algebraic moment problem.

For the convenience of the reader and in order to be self-contained, we reproduce with minor changes the second Section of Hürlimann(1990b) useful for the practical implementation of numerical algorithms to solve the algebraic moment problem.

Given is a real random variable X whose first 2n-1 moments are supposed to be finite and known. We consider the approximation of X by a discrete random variable concentrated on n real mass points x_i with weights $y_i=\Pr(X=x_i)$, $i=1,\dots,n$. This approximation problem is solved by the algebraic moment problem which consists to solve the system of equations

$$\text{(A.1)} \qquad \sum_{i=1}^{n} y_i (x_i)^k = E[X^k] = m_k, \quad k=0,1,\dots,2n-1$$

by given moments m_k of the random variable.

According to Mammana(1954) this algebraic problem can be solved as follows. The mass points $x_1, \dots, x_n$ are the distinct real zeros of the polynomial equation

$$\text{(A.2)} \qquad p_n(x) = \begin{vmatrix} 1 & x & x^2 & \dots & x^n \\ 1 & m_1 & m_2 & \dots & m_n \\ & & & \dots & \\ m_{n-2} & m_{n-1} & m_n & \dots & m_{2n-2} \\ m_{n-1} & m_n & m_{n+1} & \dots & m_{2n-1} \end{vmatrix} = 0$$

where $|\ |$ means the determinant. The weights are found from the linear system (A.1) using Cramer's rule. One obtains

$$\text{(A.3)} \qquad y_i = \left| D_i / D \right|, \quad i=1,\dots,n,$$

where

$$D = s_n(x_1,\dots,x_n) \prod_{i<j} (x_i-x_j),$$

$$D_i = s_n(x_1,\dots,\hat{x}_i,\dots,x_n) \prod_{\substack{i<j \\ i,j\neq k}} (x_i-x_j) \cdot \sum_{j=1}^{n} (-1)^{j-1} m_j s_{n-j}(x_1,\dots,\hat{x}_i,\dots,x_n)$$

In these expressions the s_n's are the elementary symmetric functions and $\hat{x}_i$ means that the variable x_i has to be dropped in the corresponding elementary symmetric function. It is worthwile to write down the solutions for the special instances n=1,2,3.

Special cases

(a) $n=1$: $x_1=m_1$, $y_1=1$

(b) $n=2$: $-p_2(x) = (m_2-m_1^2)x^2 + (m_1m_2-m_3)x + m_1m_3-m_2^2 = 0$,

$$x_1=\tfrac{1}{2}(s_1 + \sqrt{s_1^2 - 4s_2}), \quad x_2=\tfrac{1}{2}(s_1 - \sqrt{s_1^2 - 4s_2}),$$

$$s_1=(m_3-m_1m_2)/(m_2-m_1^2), \quad s_2=(m_1m_3-m_2^2)/(m_2-m_1^2), \tag{A.4}$$

$$y_1=(m_1x_2-m_2)/x_1/(x_2-x_1), \quad y_2=(m_2-m_1x_1)/x_2/(x_2-x_1)$$

(c) $n=3$: $-p_3(x) = s_0x^3 - s_1x^2 + s_2x - s_3 = 0$, where

$$s_3 = \begin{vmatrix} m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \\ m_3 & m_4 & m_5 \end{vmatrix} \qquad s_2 = \begin{vmatrix} 1 & m_2 & m_3 \\ m_1 & m_3 & m_4 \\ m_2 & m_4 & m_5 \end{vmatrix}$$

(A.5)

$$s_1 = \begin{vmatrix} 1 & m_1 & m_3 \\ m_1 & m_2 & m_4 \\ m_2 & m_3 & m_5 \end{vmatrix} \qquad s_0 = \begin{vmatrix} 1 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{vmatrix}$$

To determine one zero of this cubic polynomial equation, use Newton's algorithm and then solve a quadratic equation. Alternatively apply Cardano's formulas to obtain explicit expressions for the zeros :

Let $t_i=s_i/s_0$, $i=1,2,3$, $p=t_2-t_1^2/3$, $q=t_1t_2/3-t_3-2t_1^3/27$,

$D=-4p^3-27q^2>0$, $\cos(\Phi)=-3\sqrt{3/2}.q/\sqrt{-p}$. The mass points are

$$x_i = z_i + t_1/3, \quad i=1,2,3, \text{ where}$$

$$3z_1 = 2\sqrt{-3p}.\cos(\Phi/3),$$

$$3z_2 = -2\sqrt{-3p}.\cos(\Phi/3+\pi/3), \tag{A.6}$$

$$3z_3 = -2\sqrt{-3p}.\cos(\Phi/3-\pi/3).$$

The weights are calculated from the formulas

$$y_1 = [m_3 - m_2(x_2 + x_3) + m_1x_2x_3]/x_1/(x_1 - x_2)/(x_1 - x_3),$$
$$y_2 = -[m_3 - m_2(x_1 + x_3) + m_1x_1x_3]/x_2/(x_1 - x_2)/(x_2 - x_3),$$
$$y_3 = [m_3 - m_2(x_1 + x_2) + m_1x_1x_2]/x_3/(x_1 - x_3)/(x_2 - x_3).$$

(d) $n > 3$:

In general, since $p_n(x)=0$ has n distinct real zeros, one can apply any numerical method to solve such equations. Our numerical experience with the Newton-Maehly algorithm, which is a suitable modification of the ordinary Newton algorithm, was satisfying and is summarized as follows :

n=1,2,3 : the method has no numerical defects,
n=4,5 : no numerical defects were detected using the "double" precision floating-point arithmetic,
n=6,...,15 : satisfying results in "quadruple" precision floating-point arithmetic,

Remark. For $n>3$ the coefficients of the normalized polynomial

$$p_n(x) = x^n - s_1x^{n-1} + s_2x^{n-2} \pm \ldots + (-1)^n s_n = 0$$

can be found by solving the linear system of equations

$$\text{(A.7)} \qquad \sum_{i=0}^{n} (-1)^i m_{k+i} s_{n-i} = 0, \quad k=0,1,\ldots,n-1,$$

where $s_0=1$. This follows from the representation (A.2) in determinant form.

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