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Portfolio Insurance

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Summary

Portfolio insurance implies many problems and difficulties.

This contribution scrutinizes three problems. The very first one is:

How to decide to take insurance on the whole portfolio, on some parts of the portfolio. Further more is it good to take any portfolio insurance or a minimum insurance?

Second problem : Is this insurance to be done on a one period or multiperiods analysis to have a strategy that minimise coste?

Third problem is practical achievement : How hedging by interbank long options and replication of these options can be compared and monitored?

Résumé

Assurance de Portefeuille

L'assurance de portefeuille comporte des aspects et des difficultés multiples. Cette contribution examine trois problèmes. Celui qui s'impose entout premier lieu peut s'exprimer ainsi:

Comment décider de prendre une assurance totale ou partielle. Faut-il, d'ailleurs, prendre ou ne pas prendre d'assurance de portefeuille?

Faut-il prendre un minimum d'assurance?

Second problème: cette assurance doit-elle être prise en vue d'une analyse sur une ou plusieurs périodes dans le cadre d'une stratégie visant à minimiser le coût.

La réalisation pratique de l'assurance pose le troisième problème: comment comparer la couverture par des options longues et la "replication" de ces options se compare-t-elle? Comment surveiller ces opérations.

PORTFOLIO INSURANCE

A - STUDY ON A FIXED PERIOD OF THE QUESTION "HOW MUCH TO INSURE ?"

I - Portfolio partial Insurance

We take a portfolio valued W + Z to dissociate risky assets valued Z and non risky assets W . If we valuate utility difference between total utility portfolio and non risky assets utility by taylor developments :

$$U(W+Z)-U(W)=U'(W)(W+Z-W)+U''(W)\frac{(W+Z-W)^2}{2}+\varepsilon=Z\,U'(W)+\frac{Z^2}{2}U''(W)+\varepsilon$$

$$E[U(W+Z)] - U(W) = E(Z)\,U'+E\left(\frac{Z^2}{2}\right)U'' \quad [A1]$$

Expected mathematical value of this difference being valued, suppose now that this portfolio is partially insured at K % for a cost g = Kh . This insured portfolio is valued Wg . Developments of $U(W_g) = U(W-g)$ with respect to g :

can be written $U(W_g) = U(W)-g\,U'(W)-g^2\,U''(W)+\varepsilon$ [A2]

Insurance was brought at price g = Kh ;

for this price $E[U(W\,g)] = U(W_g)$ with [A1] and [A2]

We have then:

$$KE(Z)\,U'(W) + K^2\,\frac{1}{2}E(Z^2)\,U''(W) + g\,U'(W) + g^2\,U''(W) = 0 \quad [A3]$$

$$U'(W)(KE(Z)+g) + U''(W)\left(\frac{E(Z^2)}{2}K^2 + g^2\right) = 0 .$$

$$\text{If } KE(Z) + g = 0,\ g^2 + K^2\frac{E(Z^2)}{2} = 0$$

If $E(Z) = 0$ we find $\frac{U'(W)}{U''(W)} = \frac{K^2\,E(Z^2)+2\,g^2}{2\,g} = K\left(\frac{\sigma_Z^2}{2\,h}+h\right)$

so K is determined.

If utility function is taken from MPT method, it means some drawbacks studied by Clarkson:

$U = E - V\lambda$ is quadratic, this is not very satisfactory.

Clarkson showed that a risk criterion better than variance is $E_z = \int_{-\infty}^{L} (L-x) f(x) dx$ if f(x) is the total return distribution and L minimum goal to reach, but for the previous study a utility fonction of total wealth W + Z is needed $u = \frac{U}{W} = (E - \lambda E_z)$ and $u = \frac{U}{W} = E - \lambda V$ are two possibilities to apply previous calculus.

If a portfolio is totally insured that could mean that insurance utility is much bigger than its cost. In this case, it is not possible to do previous calculus. If a portfolio is partially insured, it means that choosed proportion of insured portfolio is optimum for its price.

Let us take an insured portfolio which gives a guarantee to be valued at initial value at least, even if worst possibilities were to lessen expected values before the period end.

This kind of portfolio insurance lessens variance, but also expected total return because insurance cost is bigger than advantage coming from erased negative total return.

Let us write that expected total return less short term yield is disminished by α % :

$(E-f)(1-\alpha)$, the variance as well $V(1-\varepsilon)$ compared to non insured portfolio (E,V).
E_M , V_M concerns index.

Suppose that this portfolio is indexed up to β % of total value ; remaining assets are without risk. This asset without risk yields : f
Let us look at the maximum utility β for insured and non insured portfolios.

Non insured portfolio utility can be reckoned :

$$U = f + \beta(E_M - f) - \lambda \beta^2 V_M$$
$$U = f + E - f - \lambda V .$$

If we take classical MPT formula $\frac{dU}{d\beta} = 0$

at optimum $E_M - f - 2 \lambda \beta V_M = 0$ $\qquad \beta = \frac{E_M - f}{2 \lambda V_M}$

$$E - f = \beta(E_M - f) = \frac{(E_M - f)^2}{2 \lambda V_M} \qquad V = \beta^2 V_M = \frac{(E_M - f)^2}{4 \lambda^2 V_M}$$

$\frac{E - f}{V} = 2 \lambda$ with portfolio insurance.

$$U_i = f + \beta_i (1 - \alpha_i)(E_M - f) - \lambda\ (1 - \epsilon_i)\ \beta_i^2\ V_M$$

$$\frac{d\ U_i}{d\ \beta_i} = 0 = (1 - \alpha_i)(E_M - f) - 2\ \lambda\ \beta_i (1 - \epsilon_i)\ V_M$$

$$\beta_i = \left(\frac{1 - \alpha_i}{1 - \epsilon_i}\right)\left(\frac{E_M - f}{2\ \lambda\ V_M}\right) \quad \text{insured portfolios have } E_i, V_i \qquad \textbf{[A.4]}$$

$$E_i - f = \frac{(1 - \alpha_i)^2}{1 - \epsilon_i}\left(\frac{(E_M - f)^2}{2\ \lambda\ V_M}\right) \quad ; \quad V_i = \frac{(1 - \alpha_i)^2}{1 - \epsilon_i}\ \frac{(E_M - f)^2}{4\ \lambda^2\ V_M}$$

$$\frac{E_i - f}{V_i} = 2\ \lambda\ . \qquad [A.4^{bis}]$$

If we add some portfolios of the same kind insured differently we shall find nevertheless :

$$\frac{E - f}{V} = \sum_{i=1}^{i=n} \quad \frac{E_i - f}{V_i} = 2\ \lambda \quad \text{as for non insured portfolios.}$$

This simple result of MPT method can be compared to results from other utility functions which are much more complicated.

II - <u>Totally Insured portfolio</u>

Let us take a totally indexed portfolio. Index M has an expected value E on period T and a variance V .
This portfolio is valued WM ; its utility

$$U(WM) = U(W) + U'(W)(WM - W) + \frac{U''(W)}{2}(WM - W)^2$$

has an expected value

$$E[U(WM)] = U(W) + W\ U'(W)\ E(M-1) + W^2\ \frac{U''(W)}{2}\ E(M-1)^2\ .$$

This portfolio can be totally guaranteed on period T . If totally insured portfolio utility is
$U(G) = U[W(1-\alpha)] = U(W) - \alpha\ W\ U'(W)$ equal to mathematical expected utility portfolio, it is justified to insure totally. Nevertheless, one cannot be sure that partial insurance could imply higher portfolio utility or not.

$$E[U(WM)] = U(G)$$

can be written

$$W\ U'(W)\ E(M-1) + \frac{W^2}{2}\ U''(W)\ E(M-1)^2 = -\alpha\ W\ U'(W)$$

$$W\ U'(W)(E(M)-1+\alpha) + \frac{W^2}{2}\ U''(W)\ E(M-1)^2 = 0$$

$\frac{U'(W)}{U''(W)} = \frac{W\ E(M-1)^2}{2(1-\alpha-E(M))}$ if $E(M) = 1$, for period T_1 we have for partially insured portfolios with [A.4] :

$$\text{[A.5]} \quad \boxed{\alpha_1 = -\frac{U''(W)}{2\ U'(W)}\ W\ V_M} \quad \beta_1 = \left(\frac{1-\alpha_1}{1-\epsilon_1}\right)\left(\frac{E_M - f}{2\ \lambda\ V_M}\right) \text{ for } \beta_1 = 1$$

it gives ϵ_1. [A.5] formula is nearly a fad, even if assumptions limit validity to horizontal trend market : $E(M) = 1$

$E(M) \neq 1$ for period T_2, so $\alpha_2 \neq \alpha_1$ $\beta_2 \neq \beta_1 = 1$ If, for period T_1 we insure a θ portfolio proportion, its mathematical expected value and variance can be written : $W\ E_W = W\theta E_1 + W(1-\theta)E_M$. $W\theta E_1$ is portfolio part which has a guaranted value at the end of period T_2. For period T_2 we can write :

$$E_W = \theta\ E_1 + (1-\theta)\ E_M \qquad \frac{d\ E_W}{d\theta} = E_1 - E_M = E_2$$

$$\sigma_W^2 = \theta^2\sigma_1^2(1-\theta)^2\ \sigma_M^2 + 2\rho\theta(1-\theta)\ \sigma_1\sigma_M$$

$$\frac{dV_M}{d\theta} = 2\ \sigma_W\ \frac{d\sigma_W}{d\theta} = 2\theta\sigma_1^2 + 2(\theta-1)\sigma_M^2 + 2\rho(1-2\theta)\sigma_1\sigma_M$$

E_2 is mathematical expected loss at the end of period T_2.

If one takes classical MPT utility fonction $U = E_W - \lambda\ V_W$

optimum can be reached for $\frac{dU}{d\theta} = \frac{d\ E_W}{d\theta} - \lambda\ \frac{d\ V_W}{d\theta} = 0$

$$E_2 + 2\lambda\left(\theta\ \sigma_1^2 + (\theta-1)\ \sigma_M^2 + \rho(1-2\theta)\ \sigma_1\ \sigma_M)\right) = 0$$

$$\theta = -\frac{E_2 - 2\ \lambda\ \sigma_M^2 + 2\ \rho\ \lambda\ \sigma_1\ \sigma_M}{2\left(\lambda\ \sigma_1^2 + \lambda\ \sigma_M^2 - 2\ \rho\ \sigma_1\ \sigma_M\right)} < 0$$

means insurance is too costly to make a contract at α_2 fot T_2.

Let us remark that $\theta = 1-j$ being close to 1. Does it make sense if :

$$E_2 + 2\ \lambda\left(\sigma_1^2(1-j) + j\ \sigma_M^2 + \rho\ \sigma_1\ \sigma_M - 2\ j\ \rho\ \sigma_1\ \sigma_M\right) = 0$$

$$j = \frac{-(E_2 + 2\ \lambda\ \sigma_1^2 + 2\ \lambda\ \rho\ \sigma_1\ \sigma_M)}{2\left(\sigma_M^2 - \sigma_1^2 - 2\ \rho\ \sigma_1\ \sigma_M\right)\ \lambda}$$

$$\frac{d\ E_W}{d\ V_W} = \frac{E_2}{2[\rho(1-2j)\sigma_1\ \sigma_M + (1-j)\ \sigma_1^2 + j\ \sigma_M^2} \qquad \text{[A.7]}$$

to be totally insured ?

to decide on this question : an insurance cost diminished by $\alpha_2\ d\ E_W$ increases variance by $\epsilon_2\ d\ V_W$ and utility changes by $dU = \alpha_2\ d\ E_W - \lambda\ \epsilon_2\ d\ V_W$.

This ratio is very useful to show how a diminished variance implies diminished expected total return. If $j=0$ a Kind of diversification disappears ; so it could be a disadvantage to be totally insured for too long periods. How long to be totally insured is a very difficult question linked to stock market cycle.

III - Minimum insurance for a portfolio

If θ moves closely to 1 , a diversification vanishes ; it is the same if θ moves closely to O $\frac{d\ E_W}{d\theta} = E_1 - E_M = -E_2$

$$\frac{d\ V_M}{d\theta} = 2\ \sigma_W \frac{d\ \sigma_W}{d\theta} = 2\left(\theta\ \sigma_1^2 + (\theta-1)^2\ \sigma_M^2 + \rho(1-2\theta)\ \sigma_1\ \sigma_M\right)$$

$$\frac{d\ E_W}{d\ V_W} = \frac{E_2}{2\left[\theta\left(2\ \rho\ \sigma_1\ \sigma_M - \sigma_1^2 - \sigma_M^2\right) + \sigma_M^2 - \rho\ \sigma_1\ \sigma_M\ \right]}$$

This ratio moves to $$\frac{E_2}{2\left(\sigma_M^2 - \rho\ \sigma_1\ \sigma_M\right)} \quad \text{[A.8]}$$

It is necessary to see for very low value of θ what could be non proportional costs to insured capital which changes this above mentioned ratio

$$\frac{E_2 + \text{non proporional costs}}{2\ \sigma_M(\sigma_M - \rho\ \sigma_1)}$$

Moreover, cost g acceptable for insurance, taking in account utility fonction can be lower than partial guarantee insurance cost. This cost was valued:[A.3].

IV - Variable proportion insurance

When portfolio insurance is done with quoted options, it is modifiable any time. An active management for insurance is possible. Rather good intuition is needed for an active option management be better than a management by chance.

If $E_Z = \theta\ E_1 + (1-\theta)\ E_M$ with θ stochastic variable

E_Z is then a stochastic variable with expected value E and standard deviation σ_Z . The standard stochastic variable is $Z = \frac{E_Z - E}{\sigma_Z}$ for small variations of θ .

$\frac{d\ E_Z}{d\theta} = E_1 - E_M$. Moreover, we saw that :

$$\sigma_W \frac{d\ \sigma_W}{d\theta} = \theta\ \sigma_1^2 + (\theta-1)\ \sigma_M^2 + \rho(1-2\theta)\ \sigma_1\ \sigma_M$$

$$\frac{d\ \sigma_W}{d\theta} = \frac{1}{\sigma_W}\left(\theta\ \sigma_1^2 + (\theta-1)\ \sigma_M^2 + \rho(1-2\theta)\ \sigma_1\ \sigma_M\right)$$

$$z = \frac{d\,E_z}{d\theta} \times \frac{d\theta}{d\,\sigma_M} = \frac{(E_1 - E_M)\quad \sigma_M}{\theta\,\sigma_1^2 + (\theta-1)\,\sigma_M^2 + \rho(1-2\theta)\,\sigma_1\,\sigma_M} \qquad [A.9]$$

This helps finding a distribution law for z coming from distribution of θ valid for small variations of θ .

V - <u>How much to invest and to insure</u>

What would be expected total return of a portfolio which has not a probability higher than P to get an R lower than "R mini" total return on this period.

Take as assumption, that distribution of total return is normal. Higher than R mini total return

Probability can be written : $P_z\left[\frac{R_t - \bar{R}}{\sigma}\right] > \left[\frac{R_{MINI} - \bar{R}}{\sigma}\right]$

$z = (R - \bar{R})/\sigma$.

If Z is a stochastic standard variable, previous condition can be written : $P_z(z > z_{MINI}) < P$. In MPT Plan (R, σ), this condition is valid in the half-plan limited by a right line : $z_0 = \frac{R_{MINI} - \bar{R}}{\sigma}$ $\qquad \bar{R} = R_{MINI} - z_0\,\sigma$ $\qquad$ [A.10]

For several "R mini" this right line can be drawn. They cross the market line linking : M point index representative and riskless assets point F (O,f). Crossing points correspond to portfolios with a mix of index funds and riskless assets. These portfolios have a P probability for expected total return to be equal to R mini. If the manager is not willing to have higher than P probability and that he thinks that his riskless assets proportion is too low, he must do, at least, a partial portfolio insurance.

Normality assumption is not the best, total return distribution are rather lognormal.

S stock price can be defined $S_{t+\Delta t} = S_t\, e^{\mu \Delta t + z\sigma\sqrt{\Delta t}}$

Total return can be approximated $R_t = \frac{\Delta S_t}{S_t}$ on a period Δt with Z standard variable.

$$R_t = \left(\mu + \frac{\sigma^2}{2}\right)\Delta t + z\,\sigma\,\sqrt{\Delta t} \qquad [A.11]$$

Probability for $R_t \geq R_{MINI}$ can be written

$$P_z\left[\left(\frac{R_t-\bar{R}}{\sigma\sqrt{\Delta t}}\right) \geq \left(\frac{R_{MINI} - \bar{R}}{\sigma\sqrt{\Delta t}}\right)\right] = P$$

or still $P_z(z \geq z_0) = P$ that is to say $\bar{R}_t = R_{MINI} + z\,\sigma\sqrt{\Delta t}$ which cuts plane (R, σ) into two half-planes. Above the line, above mentioned condition is fulfilled, let us remark

$$E\left[\text{Log}\left(\frac{S_t + \Delta t}{S_t}\right)\right] - E[\text{Log}(1+R)] = \mu\,\Delta t \qquad V\left[\text{Log}\left(\frac{S_t + \Delta t}{S_t}\right)\right] = \sigma^2 dt$$

this line cuts market line at a point that can be determined
$\bar{R} = R_{MINI} - z\,\sigma$

$$\bar{R} = f + (E_M - f)\,\beta \qquad \text{[A.12]}$$

if optimal exposure to market for a manager unwilling to have a probability higher than P seems insufficient to him, he has to insure a proportion of his portfolio which is determined optimally as it has been said at the beginning of this contribution. [A.4].

This approach can be used without portfolio insurance, fund manager accepts lower than goal performance with a P probability.

On the contrary, fund manager can insure his portfolio for under-performing risk, after probability being limited to P for insurance contract to be useful to avoid performance below goal, sufficient risk-less assets holding can put a lid on insurance cost.

B - MULTI PERIOD STUDIES TO MINIMISE INSURANCE COST

To make multi-period study, it is necessary to make a global MPT study or to make a precise kind of insurance contract or hedge contract. Some generality is lost to come as near as possible to reality which means to look for low cost formulas. Portfolio insurance up to nine months can be realized by quoted options, daily fluctuations are erased, daily security cost is high specially if volatility is at lofty levels.

Longer than a year options are traded by a few interbank market makers ; they use Black and scholes formula which is not very good for longer than a year options.

In such a low competition market, market makers can take fat margins in Paris for example.

It is possible to work otherwise : to guarantee a capital to a certain maturity can be less costly if maturity is far away. This guarantee to a certain maturity let some leeway for value to fluctuate. That is much less costly than to erase all fluctuations. If maturity is far away, fluctuations can be big so that kind of portfolio insurance can be interesting for long term managed institutions.

The kind of contract limiting this drawback which is interesting for fund management is lower cost.

Initial capital is the lowest boundary of possible values at the end of first and second period. These periods are (12 or) 18 months long so that value fluctuations cannot go much lower than initial capital. So it works as if zero coupon reinbursed at first period end guaranteed this first period maturity. This amount reinvested in zero coupon bonds would be reinbused at the second period end. As duration is always lower than 1,5, value cannot go much lower than initial capital. Insurance contract provides that is cannot be possible to move any fund out of this contract before second period end, for each period capital is indexed on european index (or CAC 40) that is to say relative increment of index is applied to beginning of period value.

To put a lid on insurance cost relative increment can be limited to, say 30 % so if index increased more than 30 % for a period only 30 % will be applied to beginning of period value.

For lack of zero coupon bond with 12 to 18 months maturity, initial capital WZ will be invested in bond portfolio with duration equal to period length. There will be bonds with coupon to be paid in this portfolio. θ is bond exposure $\theta = 1 - I_0$ if it is a 12 months period.
Average yield of the portfolio being : I_0
if it is a 18 months period $\theta \, W \, Z_0 = \left(1 - \frac{3}{2} I_0\right) W \, Z_0$ the remaining portfolio asset will be invested in calls on index with a 12 or 18 months maturity on interbank market. It could be done also by replication. Ordinarily, it will be a total indexation. When volability is high, interbank options are costly. It could be necessary that increment limitation to be diminished from 30 to 25 %. On the contrary, if volatility is

lower, increment could be limited at 35 % possibly. Thanks to this parameter, indexation can be 100 % that is better than otherwise (85 % to 110 % are sometimes used).

As a matter of fact, there is a bond portfolio W with very low risk and a risky asset Z, it is page 1 [A.3] which can be used and not [A.5]

$$\frac{U''(W)}{U'(W)} = \frac{2(g + kE(Z))}{k^2E(Z^2) + 2g^2}$$

g : the admissible cost for investor is determined by utility function and first moments of contingent appreciation. This is good for each period for which admissible cost must be added $g_1 + g_2$.

We can also write as we already did $E_W = (1-I_0)\, I + I_0\, E_1$

neglecting I_0^2 $E_W = I_0(1 + E_1)$

$V_W = I_0\, V_1$ MPT utility function ; $U = I_0(1+E_1) - \lambda\, I_0\, V_1$ [B.1]

If risk measure is Clarkson criterion.

$$U = I_0(1+E_1) - \lambda\, I_0\, E_2 = I_0 - \lambda\, I_0\, E_2 + E_1\, I_0$$

$$U = I_0[1-\lambda\, E_2 + E_1] \qquad [B.2]$$

Now let us take more a stock market approach to study portfolio insurance MPT is not too bad for this.

Let us take two market lines : the first period market line, for example, would be a bear market line, the second period market line would result from major turn in the market and a first leg of bull market.

First market line slope is < 0 .

Second market line slope is > 0 .

During first period there is a low probability that index increment would reach 25 %, but negative total return is a big risk and this risk is increasing with market risk.

Portfolio β will be lower than β of market line intersection during second period. 25 % limitation of increment will be reached easily ; so that this risk is important and increasing with standard deviation of portfolio total return for RMAX limitation probability being lower than P .

If Z is standard variable $z = \frac{R - R_{MAX}}{\sigma}$ the condition can be written $P_r(z_t > z_{MAX}) < P$.

For maximum condition as well as for minimum condition a total return distribution can be taken as normal or lognormal.

$$R_t = R_{MAX} + z\,\sigma \qquad \text{or}$$

There exists a right line

$$R_t = R_{MAX} + z\,\sigma\,\sqrt{\Delta t}$$

which cuts the plane (R,σ) into two half planes. The lowest corresponds to this condition.

$$P_R(z_t > z_{MAX}) < P \;.$$

The line crosses market line at a point of optimal exposure β to market, to avoid that increment limitation be reached more than P times out of hundred.

There is no incentive to take more risk with bigger standard deviation because increment limitation is reached more often. A lower exposure than β would be a mistake in a bull market.

In general, market line intersection with this line is the optimum ; nevertheless on a multi-scenarios basis the resulting representative lines are as many would be optima as scenarios.

$$P_R(z_t > MAX) < P_1$$

$$P_R(z_t < MINI) < P_2$$

For various P values, it is possible to determine portfolio optimal exposure to market, taking in account existing portfolio insurance contracts.
A higher exposure increases insurance cost in such a way that it is not a plus.
A lower exposure would not sufficiently diminish portfolio insurance cost to balance lower expected total return.

C - PORTFOLIO HEDGING AND REPLICATIONS

To erase, on short term, negative total return risk, puts can be bougth with an exercice price corresponding to total return equals to zero on this short term period.

This distribution is truncated because negative total return are erased.
If total return distribution before truncature is normal with a mean R_m and standard deviation σ : expected hedged portfolio total return can be written :

$$\int_0^{+\infty} z\, e^{-\left(\frac{z-z_m}{\sigma}\right)^2} dz$$

less time value of put. Put can be valued :

$$\int_{-\infty}^{0} z\, e^{-\left(\frac{z-z_m}{\sigma}\right)^2} dz \qquad \text{plus time value.}$$

For hedges up to nine months, total value of puts with time value included will be given precisely by Black and scholes formula on which traders work to hedge on longuer period (1 to 3 years) there are some market makers who trade interbank puts up to 5 years. Lack of competition, valuation difficulties linked to Black and scholes formula being inadequate to long puts let market makers take fat margins. So that even two years hedging is costly. Further-more, hedging could be unperfect if volatility is sharphy down, so to depress put value. It is possible to avoid this with look back options that is put without time value that do not react to volatility move. Look back options simply disappear when underlying stock price reach a low limit fixed by contract much lower than actual stock price when contract is signed. This disappearance in case of strong move down does not work in right direction for portfolio insurance.

All in all, no perfect hedge can be done. An hedge is caracterised by its negative correlation to underlying asset. This correlation coefficient will seldom be equal to -1 .

In these conditions, it is necessary to look for other finanancial assets which are rather well anticorrelated with underlying asset. This will not diminish too much anti correlation coefficient. Advantages of non optional asset anti correlated are linked to coefficient stability on rather long span of time. Costs are limited to purchase and sale expenses.

Long US puts on index together with currency options can be used. This is an unstable kind of anti correlated assets. It could be good in global bear markets. Hedging diversification is a remedy to anti correlation instability.

It is more-over necessary to manage actively these quoted hedges to decrease them when anti correlation diminishes, and the other way round.

Transactions costs for daily adjustements are not too high for equities or convertible bonds. It could be too difficult and too costly for long term interbank options.

To take an overall view, we take the MPT method to study mathematical expected total return difference between hedge H and underlying asset A. Let us plot this difference vertically against standard deviation horizontally. Standard deviation can be written

$$\sigma_D^2 = \sigma_A^2 + \sigma_H^2 - 2\ \rho_{AH}\ \sigma_A\ \sigma_H$$

Difference D is a stochastic variable that can be put in standard normed form $z = \dfrac{E_A - E_H}{\sigma}$ this variable is normal or lognormal usually. So that probability for z to be bigger than z_0 is easy to determine. It can be said that probability for z to be lower o equal to z_1 is as easy to find

$$z_1 = \frac{E_A - E_H}{\sigma_0}$$

$$E_A - E_H = z_1\ \sigma_D = z_1\left(\sigma_A^2 + \sigma_H^2 - 2\ \rho_{AH}\ \sigma_A\ \sigma_H\right)^{\frac{1}{2}} \qquad [C.1]$$

for $\rho = 1$, $E_A - E_H = z_1 |\sigma_A - \sigma_H|$

if $p = 5$ % , table gives z_2

for $|\rho| = 1-\varepsilon$ $\qquad E_A - E_H = z|\sigma_A - \sigma_H| + \varepsilon\ z\ K$

representative curve of equation [C.1] crosses Y axis for $\sigma_A = \sigma_H$ This curve asymptots are the straight lines $E_A - E_H = z|\sigma_A - \sigma_H|$ reckoning regularly and ploting hedged portfolio with vertically $E_A - E_H$, horizontally σ_a. Stochastic fluctuations can be seen so that any truly meaning-full hedge anticorrelation decrease will be seen. Charts can be graduated for $\rho = -0,90 \ldots -0,80$ and so on.

Any meaning-full anticorrelation decrease must be tackled by manager who must diminish or sell assets if anticorrelation of these is meaning-fully down. This means more interbank options and higher hedging cost.

Can we compare hedging cost with the cost of replication ? Long term put option replication implies that shorted equities plus risk-less assets have same maturity yield and sensibility that a put of desired maturity.

If this implies a lot of transactions in case of high volatility, this kind of active management can cost more than an interbank put ; usually it is cheaper, specially if volatility is low, even if put price is low in this case. Hedging management when it is not easy to find sufficiently long puts, can mean replication. This could be often necessary on Paris market.

1 - Multi-period hedging

With respect to utility fonction of fund manager we can define optimal β of portfolios having expected return and same variance as hedged portfolios.

An imperfectly hedged portfolio has expected total return fluctuating without trend

$E_M + a_1$ in first period

$E_M + a_2$ in second period

$E_M + a_i$ in i^{th} period with $\sum_1 a_i = 0$

correlatively standard deviation for these periods are equal

$\sigma_M(1+\varepsilon_1)$, $\sigma_M(1+\varepsilon_2)$, $(\sigma_M(1+\varepsilon_i)$

with ε_i and a_i are stochastic variables with zero expected value. They are usually well correlated.

So that with a fund manager defined by his utility function: $U = f + \beta(E_M - f) - \lambda \beta^2 V_M$ f being risk-less rate, $1 - \beta$ proportion of risk-less assets, λ risk aversion coefficient.

β is determinated by utility optimum $\frac{dU}{d\beta} = 0$, writing it for each period.
Let us add these equations

for first period $(E_M + a_1 - f)(2\lambda V_M)^{-1}(1+\varepsilon_1)^{-2} = \beta_M + \Delta\beta_1$

for second period $(E_M + a_2 - f)(2\lambda V_M)^{-1}(1+\varepsilon_2)^{-2} = \beta_M + \Delta\beta_2$

for i^{th} period $(E_M + a_i - f)(2\lambda V_M)^{-1}(1+\varepsilon_i)^{-2} = \beta_M + \Delta\beta_i$

$$n\,\beta_M + \sum_1 \Delta\beta_i = \sum_1 \frac{(1-2\varepsilon_i)(E_M + a_i - f)}{2\lambda V_M} \qquad [C.2]$$

$$\sum_{i=1}^{i=n} (E_M+a_i-f-2\varepsilon_i E_M-2\varepsilon_i a_i+2\varepsilon_i f) = n(E_M-f)-2\varepsilon_i a_i$$

$\varepsilon_i a_i$ second order term is the product of two zero mean series multiplication usually these two series are very correlated. So this sum would be not too different from variance V_e de a

$$n\beta_M + \sum_i \Delta\beta_i \neq \frac{n(E_M-f) - 2V_e}{2\lambda V_M}$$

$$\sum_{i=1}^{i=n} \Delta\beta_i \neq -\frac{V_e}{\lambda V_M}$$

as $\beta_M = \dfrac{E_M - f}{2\lambda V_M}$ for $a = 0$.

To have a good hedge we need $n\beta_M + \sum_i \Delta\beta_i = 0$ or

$$n\beta_M \neq \frac{V_e}{\lambda V_M} \qquad [C.3]$$

Previously hedge monitoring method uses chart plotting of $E_A - E_M = E_M + a$; $\sigma_D = \sigma_m(1+\varepsilon_i)$.

[C.1] equation becomes for $\rho = 1$

$$E_M + a_i = z_i\,\sigma_D$$

$$z_i = \frac{E_M + a_i}{\sigma_M(1+\varepsilon_i)} = \frac{E_M + a_i - \varepsilon_i E_M - \varepsilon_i a_i}{\sigma_M} \qquad [C.4]$$

a and ε being normal stochastic variables with zero expected value, E_M and σ_M being stable certain values, we have to neglect second order $a_i\,\varepsilon_i$ to show that z is a normal stochastic variable with E_M/σ_M expected value. If z goes further than two standard deviations from expected value, the fund manager can think that E_M/σ_M expected value, is evolving. He will do a portfolio adjustement that we could valuate

$$\int_{-\infty}^{-L} \left(z - \frac{E}{\sigma}\right) e^{-\left(z - \frac{E}{\sigma}\right)^2} dz + \int_{L}^{+\infty} \left(z - \frac{E}{\sigma}\right) e^{-\left(z - \frac{E}{\sigma}\right)^2} dz$$

- Estimation of ΔE_t

$$nE_t + n\Delta E_t = (\beta_M+\Delta\beta_1)(E_M+a_1-f)+\ldots(\beta_M+\Delta\beta_i)(E_M+a_i-f) =$$

$$\sum_i \Delta\beta_i E_M - \sum_i f\,\Delta\beta_i + \sum_i a_i\,\Delta\beta_i + \sum_i a_i\,\beta_M + n\,\beta_M\,E_M - n\,f\,\beta_M$$

$$n(E_t+\Delta E_t)-nE_t \neq \sum_{i=1}^{i=n} a_i\ \Delta\beta_i - n\ \Delta E_t - n\ V_e \quad \text{if } a_i\ ;\ \Delta\beta_i \text{ are}$$

totally correlated.

- Estimation of ΔV_t , the variance

$$nV_t=(\beta_M+\Delta\beta_1)^2(1+\varepsilon_1)V_M+(\beta_M+\Delta\beta_2)^2\quad(1+\varepsilon_2)V_M+\ldots+(\beta_M+\Delta\beta_i)^2(1+\varepsilon_i)V_M$$

$$n\frac{V_t}{V_M}=n\beta_M^2+\beta_n^2\sum_i \varepsilon_i+2\beta_M\sum_i \Delta\beta_i+nV_{(\beta)}+2\ \beta_M\sum_{i=0}^{i=n}\varepsilon_i\Delta\beta_i+\Sigma\ \Delta\beta_i^2\ \varepsilon_i$$

Neglecting second order terms we can find simple useful formula for $\Delta V_t'/\Delta E_t$ if a_i and $\Delta\beta_i$ totally correlated.

$$\frac{V_t}{V_M}=\left(\frac{2-n}{n}\right)\beta_M^2+V_{(\beta)}=V_{(\beta)}-\left(\frac{V_e}{\lambda\ V_M\ n}\right)^2\left(\frac{2-n}{n}\right) \quad \text{if all } a_i=0,$$

$\varepsilon_i = 0$ we have $\frac{V_t'}{V_M}=\beta_M^2$

$$\frac{\Delta V_t}{V_M}=\frac{V_t-V_t'}{V_M}=2\left(\frac{1-n}{n}\right)\beta_M^2+V_\beta=2\left(\frac{1-n}{n}\right)\left(\frac{V_e}{\lambda\ n\ V_M}\right)^2+V_\beta$$

$$\frac{\Delta V_t}{\Delta E_t}=2\left(\frac{1-n}{n^3}\right)\frac{V_e}{\lambda^2\ V_M}+\frac{V_\beta\ V_M}{V_e} \qquad [C.5]$$

2 - Replication of a put

Let us suppose that to particularize this operation, a specific fund is created with an initial capital equal to the price to be paid to buy the exact kind of put that is to be replicated.

So a fund replicating hundred 3 year puts will have every day the same sensibility than these puts. In a bear market, this put will be replicated by the sales of borrowed equities every day of down market. Proceeds of these sales can be lended on short term market. More precisely $[N(d_1)-1]$ equities are borrowed and sold if the first term of Black and scholes formula could be used for a 3 year period. In this case $V(1-N(d_2))$ would be lended on short term market. V being exercice price of puts.

In a bear market, underlying stock price could be taken equal to $S(1-\alpha)^t$. This price would be used to invoice borrowed equities cost at $\beta\ S(1-\alpha)^t$ per period. So that to unwind a position total cost inclusive of borrowing cost is $S(1-\alpha)^t\ (1+\beta)^t$ that will be approximated $S((1+\beta-\alpha)$. Per contra, amounts produced by equities sale, lended on short term rate supposed equal to γ , are yielding an amount that we shall valuate X

$$X = \gamma\, S(1-c)[(N-N_1)(1+\gamma)^{n-1} + (n_1-n_2)(1+\gamma)^{n-2} + \ldots + N_{n-1} - N]$$

To obtain the result of operation, transaction cost (c %) on equities must be taken in account.

N_0-N_1 equities sold at time	1 for an amount	$(N_0-N_1)(1-c)S$
N_1-N_2	2	$(N_1-N_2)(1-c)S$
⋮	⋮	
$N_{n-1}-N_n$	n	$(N_{n-1}-N_n)(1-c)S$

If unwinding is done at time N, it will cost Y to buy needed equities

$$Y = S(1+c)[(N-N_1)(1+\beta-\alpha)^{n-1} + (N_1-N_2)(1+\beta-\alpha)^{n-2} + \ldots$$

$$\ldots + (N_i-N_{i+1})(1+\beta-\alpha)^{n-i} + \ldots + N_{n+1} - N]$$

replication result X-Y will be :

$$S[(\gamma(1-c)(1+\gamma)^{n-1} - (1+c)(1+\beta-\alpha)^{n-1}(N_0-N_1) + \cdots$$

$$(N_{i-1}-N_i)(\gamma(1-c)(1+\gamma)^{n-i} - (1+c)(1+\beta-\alpha)^{n-i}) + \gamma(1-c)-(1+c)]$$

As γ and $\beta-\alpha$ are small, $\frac{X-Y}{S}$ can be approximated by

$$(\gamma(1-c)+(n-1)\ \gamma^2-(1+c)+(n-1)\alpha+(1-n)\beta)(N-N_1)$$
$$(\gamma(1-c)+(n-2)\ \gamma^2-(1+c)+(n-2)\alpha+(1-n)\beta)(N_1-N_2)+\gamma(1-c)-1-c =$$
$$(N-N_n)(\gamma(1-c)+n\gamma^2-(1+c)+n\alpha - n\beta+\gamma(1-c)-1-c) -$$
$$\sum_{i=1}^{i=n} i(N_{i-1} - N_i)(\gamma^2+\alpha-\beta) \qquad [C.6]$$

Replication result will necessarily be positive in a bear market $(\alpha > \beta)$ this result can be transformed into a total return for the whole period. This total return must be of the same order of magnitude than total return in case of option hedging with 3 year interbank options.
In case of bull market, to replicate a put is not rewarding.

Let us suppose that a bull move comes after a bear market. It is necessary to buy back equities previously borrowed and sold so to diminish sensibility. There are :

$N_n - M_1$ bought back at time	1 for	$(N_n - M_1) S(1+c)$ cost
$M_1 - M_2$	2	$(M_1 - M_2) S(1+c)(1+\delta)$ cost
⋮	⋮	
$M_{m-1} - M_m$	m	$(M_{m-1} - M_m) S(1+c)(1+\delta)^{m-1}$

Proceeds form lended funds capitalised at time will be diminished by

$$\Delta X = \gamma(1-c)[(N_n - M_1)(1+\gamma)^{m-1} + (M_1 - M_2)(1+\gamma)^{m-2} + \ldots + M_{m-1} - M]$$

equities borrowing will be less costly so that capitalised cost of buy back will be at time

$$\Delta Y = S(1+c)[N_n - M_1)(1+\beta+\delta)^{m-1} + (M_1 - M_2)(1+\beta+\delta)^{m-2} + \ldots + M_{m-1} - M]$$

so it is possible as previously to reckon an approximation of overall return and result of the whole period. Total return must be not very different of total return of an option position in the same conditions.

Let us remark that $M_1, \ldots M_m$ can be determined by first term of Black and scholes formula $(N(d_1)-1)$ with some drawbacks for more than one year option replication.
This formula must be modified to take in account dividends of underlying stock ; if this is an index option replication it is not a problem.

To replicate index option puts which is our essential goal and also to replicate an index fund hedged by puts it is not a problem.

In this last case, it is necessary to buy borrowed index fund shares if underlying price is up. On the reverse, equities must be sold and proceeds lended ; all this can be studied in the same way.

3 - Other Replications

To replicate a call, it is necessary to buy equities on borrowed money if stock price is up and to sell them to diminish borrowed amount, if stock prise is down.

To replicate a convertible bond it is necessary to replicate a call and to buy due amount of bonds with duration equal to duration of the bond part of convertible bond.

The most precise method consists to determine every day convertible bond sensibility from good approximation formulas as Merton formula : $W = e^{-bT} S\, N(d_1) - e^{-\ T} V\, N(d_2)$

$$b = d/S \qquad d_2 = d_1 - \sigma\sqrt{T}\ .$$

There exist also price approximation formulas for long term calls and warrants. For example $(W+a)^2 = L^2 + S^2$ (Kassouf) with W warrant price) more convenient, less precise.

S underlying stock price L ; convertible bond floor a ;

$$2(W+a)dW = 2\,Sds$$

$$\frac{dW}{W+a} = \frac{dS}{S} \quad \text{(sensibility)} \qquad [C.7]$$

On a chart of W and S it appears that the tangent slope determines shares number to be hold. One can show that total return of such a replication will not be far from convertible bond total return.

4 - <u>Call replication</u>

In a moderate bear market, to sell calls that lose value as time elapses and market drifts down can be a rather protective strategy. For long lasting languid bear market that are not conducive to any quick drop, this could be best.

To replicate a not too long call it is necessary to buy $N(d_1)$ shares and to borrow $N(d_2)\ V$. V being exercise price. Sensibility of such a position will be what comes from Black and scholes formula delta. In case of a low volatility, total return will be somewhat better than total return of call position this can be reckon easily.

A position replicating a call will hold :

N_0 shares of underlying at time t_0 which give coupons for an amount $N_0 S_0\ d$, N_1 give $N_1 S_1 d$, N_{n-1} give $N_{n-1} S_{n-1}\ d$,

during period 1 : $(N_1 - N_0)$ shares were bought at $(N_1 - N_0)\ S_1$; equities purchase cost are to be capitalised at f ; also at time t_n :

$$X = (N_1 - N_0)\ S_1 (1+f)^{n-1} + (N_2 - N_1)\ S_2 (1+f)^{n-2} + \ldots + (N_n - N_{n-1}) S_n$$

So this position can be valued at unwinding time (n) $N_n\ S_n$. It was $N_0\ S_0$ at the beginning of this position. Coupons are to be capitalised at f

$$Y = d[N_1\ S_1 (1+f)^{n-1} + N_2\ S_2 (1+f)^{n-2} + \ldots + N_n\ S_n]$$

$$X = \frac{Y}{d} - N_o\, S_1 (1+f)^{n-1} - (1+f)^{-1}(1-\alpha)\, \frac{Y}{d} + N_n\, S_n \left(\frac{1-\alpha}{1+f}\right)$$

if $S_1 = (1-\alpha)\, S_{1-1}$.

Position result can be written easily

$R = Y - X + N_n S_n - N_o S_o$, if $S_t = (1-\alpha) S_{t-1}$ we find

$$R = Y\left(1 - \frac{1}{d} + \left(\frac{1-\alpha}{1+f}\right)\frac{1}{d} + N_o\, S_1 (1+f)^{n-1} + N_n\, S_n \left(\frac{f-\alpha}{1+f}\right)\right)$$

$$R = \frac{Y}{1+f}\left(- \frac{f+\alpha}{d} + 1+f + N_o\, S_1 (1+f)^{n} + N_n\, S_n (f-\alpha)\right) \qquad [C.8]$$

All these formulas are sufficiently easy to manipulate that it is possible to take in account a serie of market moves.

5 - To conclude on this point

A fund devoted to a long option replication has to be created with an initial capital equal to time t_o option price multiplied by the number of options to be replicated.

This fund is defined by the option number and the option maturity to be replicated.

At maturity, this fund must have as value the exercise price mutiplied by number of options to be replicated.

It must be self financing ; no extra capital is needed during replication period. This fund can borrow and lend without constraints.

Imperfect replication in a volatile market could be worse than long options hedging.