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# A Note on the Variables Used to Describe the Term Structure of Interest Rates

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## Summary

In this paper is tried to establish the relationships among the variables that are usually used to describe the Term Structure of Interest Rates (TSIR), from the point of view of the postulates of the mathematics of finance. It is also analysed the difference between variables defined on a continuous and discrete framework.

Once those relationships are established, we describe the TSIR with the accumulation factor, the logarithmic accumulation factor, the logarithmic density, the rate of interest (as well as their mean values) and the force of interest, distinguishing a flat TSIR, a positively sloped TSIR and a negatively sloped TSIR.

Afterwards, we analyse the effects of changes of the TSIR, due to additive and multiplicative shocks on different variables, comparing their equivalences or differences.

## Résumé

## Remarque sur les Variables Utilisées pour Décrire la Structure de Durée des Taux d'Intérêt

Cet article tente d'établir les relations entre les variables qui sont généralement utilisées pour décrire la Structure des Taux d'Intérêt (TSIR), du point de vue des postulats des mathématiques financières. Il analyse également les différences entre les variables définies dans un cadre continu et discret.

Lorsque ces relations sont établies, nous décrivons la TSIR avec le facteur d'accumulation, la densité logarithmique, le taux d'intérêt (ainsi que leurs valeurs moyennes) et la force de l'intérêt, différenciant une TSIR uniforme, une TSIR à pente positive et une TSIR à pente négative.

Puis, nous analysons les effets des changements de la TSIR, dus à des chocs additifs ou multiplicatifs sur des variables différentes, en comparant leurs équivalences ou différences.

## 1.- CHARACTERIZATIONS OF THE TERM STRUCTURE OF INTEREST RATES (TSIR)

From the viewpoint of the Mathematics of Finance the TSIR can be characterized using different variables. So that the relationship amcng those variables must be clearly stated to guarantee its correct use in modeling financial problems.

As a first example of this situation, we can often find the use of continuous variables for dealing with continuous payments streams and the use of discrete variables for coping with discrete ones, when both can be used in any case.

Another example is the treatment of the shifts or shocks in the TSIR: one shock defined over one variable may be equivalent to a different shock over other variable, and the same shock defined over different variables implies different behaviours of the TSIR.[1]

Let

$$[t_o,t_1] \quad [t_1,t_2] \quad [t_2,t_3] \quad . \ . \quad [t_{j-1},t_j] \quad . \ . \quad [t_{n-1},t_n]$$

be the consecutive intervals over which we want characterize the TSIR. We define a real variable function f, bigger than 1, over each interval, known as accumulation factor:

$$f\ (t_{j-1},t_j) > 1 \qquad \text{defined in} \quad [t_{j-1},t_j]$$

so that one money unit due at $t_{j-1}$ is equivalent to $f\ (t_{j-1},t_j)$ money units due at $t_j$.[2]

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[1] See, for instance: Bierwag, Kaufman and Toevs, 1983, pages 145-147; Vasicek and Fong, 1982, pages 341-344.

[2] See Gil, L. 1987, pages 64-66; McCutcheon and Scott, 1987, pages 13-14.

The following condition holds:

$$f(t_m,t_q) = f(t_m,t_{m+1}) * \ldots * f(t_{q-1},t_q) \qquad t_m < t_q$$

and:

$$f(t_o,t_n) = f(t_o,t_1) * \ldots * f(t_{n-1},t_n) \qquad [1]$$

The result of [1] is the amount M due at $t_n$ in which the monetary unit due at $t_o$ is transformed during $[t_o,t_n]$, if we use the corresponding accumulation factors for each subperiod.

Analogously, we define the discount factor as:

$$0 < f^{-1} (t_{j-1},t_j) < 1 \qquad \text{defined in } [t_{j-1},t_j]$$

The discount factor satisfies a similar relationship to [1]. We are only going to work with the accumulation factors.

Taking logarithms in [1], we have:

$$\ln f (t_o,t_n) = \Sigma \ln f(t_{j-1},t_j) = \Sigma \Phi (t_{j-1},t_j) \qquad [2]$$

where $\Phi (.) = \ln f (.)$ it is called the logarithmic accumulation factor [3]. It follows:

$$M = \exp [ \Sigma \ln f(t_{j-1},t_j)] = \exp [ \Sigma \Phi(t_{j-1},t_j)]$$

$$f(t_m,t_q) = \exp \Phi (t_m,t_q) \qquad t_m < t_q \qquad [3]$$

We define a new variable $r (.) = f (.) - 1$ as the excess of the accumulation factor over 1. So:

$$f (.) = 1 + r (.) \qquad [4]$$

$$\Phi (.) = \ln [1 + r(.)]$$

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[3] Gil, L. op. cit. page 65.

then:

$$M = [1 + r(t_o,t_1)] * . . * [1 + r(t_{n-1},t_n)]$$

Obviously, r(.) is the increment produced over one money unit during the period (.).

Let $\tau$ (rate of interest) be the result of dividing r(.) by the interval length:

$$\tau (.) = r (.) / (.)$$

The variable $\tau(.)$ is the price for one money unit per time unit.

If the intervals are unitary ones, $\tau(.)$ and r(.) have the same value:

$$\tau (t,t+1) = r(t,t+1) / (1) = r (t,t+1)$$

that is why in so many cases they are both mixed up.

We call logarithmic density [4] the result of dividing the logarithmic accumulation factor by the interval length:[5]

$$\Theta (.) = \Phi(.) / (.)$$

then, from [3]:

$$\ln f(t_s,tq) = \Theta (t_s,t_q) * (t_q-t_s) \qquad [5]$$

$$f(t_s,tq) = \exp [ \Theta (t_s,t_q) * (t_q-t_s)] \qquad [6]$$

$$\ln M = \ln f(t_o,t_n) = \Theta (t_o,t_n) * (t_n - t_o)$$

For unit time intervals:

$$M = \exp ( \Sigma \Theta(j-1,j) ) \qquad j = 1,..,n$$

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[4] Gil,L. op. cit. page 75.

[5] Used by Bierwag,G.O. (1977), Meneu,V., Navarro, E. (1988 and 1990)

We can write:

$$\tau\ (t,t+h) = \frac{r\ (t,\ t+h)}{h}$$

$$\Theta\ (t,t+h) = \frac{\Phi\ (t,\ t+h)}{h}$$

$$\lim_{h \to o+} \tau\ (t,t+h) = \lim_{h \to o+} \frac{r\ (t,\ t+h)}{h} = \tau\ (t) \qquad [7]$$

$$\lim_{h \to o+} \Theta\ (t,t+h) = \lim_{h \to o+} \frac{\Phi\ (t,\ t+h)}{h} = \Theta\ (t) \qquad [8]$$

The variable $\tau\ (t)$ is known as force of interest [6]. It is verified that $\tau\ (t) = \Theta\ (t)$[7].

It is easy to demonstrate, under integrability conditions, that:

$$f\ (t_s,t_q) = \exp\left(\int_{t_s}^{t_q} \tau(t)\ dt\right) \qquad [9]$$

$$\Theta\ (ts,tq) * (tq-ts) = \int_{t_s}^{t_q} \tau\ (t)\ dt \qquad [10]$$

and:

$$M = \exp\left(\int_{t_o}^{t_n} \tau(t)\ dt\right) = \exp\left(\int_{t_o}^{t_n} \Theta(t)\ dt\right)$$

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[6] Used by Fisher and Weil (1971) and Vasiceck and Fong (1982 as "forward instantaneous rate of interest compounded continuosly".

[7] Gil, L. op. cit. page 76.

These equations establish the relationship between the discrete and continuos fields.

We can obtain the following mean values:

$$\bar{f}(t_o,t_1) = [\ f(t_o,t_1)\ ]$$

$$\bar{f}(t_o,t_2) = [\ f(t_o,t_1) * f(t_1,t_2)\ ]^{1/(t_2-t_o)}$$

$$\ldots \qquad \ldots\ldots$$

$$\bar{f}(t_o,t_n) = [f(t_o,t_1) * f(t_1,t_2)*\ldots*f(t_{n-1},t_n)]^{1/(t_n-t_o)}$$

known as mean accumulation factors.

With respect to the variable r(.), we have [8]:

$$[1 + \bar{r}(t_o,t_1)] = [1 + r(t_o,t_1)]$$

$$[1+\bar{r}(t_o,t_2)] = [(1+r(t_o,t_1)) * (1 + r(t_1,t_2))]^{1/(t_2-t_o)}$$

$$\ldots \qquad \ldots\ldots$$

$$[1+\bar{r}(t_o,t_n)]=[(1+r(t_o,t_1))*\ldots*(1+r(t_{n-1},t_n))]^{1/(t_n-t_o)}$$

For unit time intervals:

$$\bar{r}(.) = \bar{\tau}\ (.)$$

So far, the TSIR can be characterized with any of the following variables:

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[8] See:
- Bierwag, Kaufman, Schweitzer and Toevs, 1983, pages 341-343
- Bierwag, Kaufman and Toevs, 1983, pages 145-147
- Bierwag, 1987, page 220
- Babbel, D. 1983, page 242
- Hopewell and Kaufman, 1973,

The variable $\bar{r}(.)$ is called by these authors as:
- h(0,t) "zero coupon yield equivalent for period spanning 0 to t"
- $R_t$ "spot rate of interest for t years..."
- $r_t$ "interest rate aplicable for..."

* accumulation factor $f(t_o,t_1)$, .... , $f(t_{n-1},t_n)$
* log. accumul. factor $\Phi(t_o,t_1)$, .... , $\Phi(t_{n-1},t_n)$
* log. density $\Theta(t_o,t_1)$, .... , $\Theta(t_{n-1},t_n)$
* force of interest $\tau(t)$
* inst. density $\Theta(t)$
* mean "interest rate" $r(t_o,\bar{t}_1)$, .... , $r(t_o,\bar{t}_n)$

If the TSIR is FLAT:

$$f(t_o,t_1) = f(t_1,t_2) = \ldots = f(t_{n-1},t_n) = f$$

and, consecuently (unit time intervals):

$$\Phi = \ln f$$

$$r = f - 1$$

$$\tau(t) = \text{constant}$$

$$\Theta(t) = \text{constant}$$

$$\tau = r = \Phi = \Theta$$

$$f = \exp \left[ \int_o^1 \tau * dt \right] = \exp(\tau)$$

The TSIR is also said to be flat if:[9]

$$[1+ \bar{r}(t_o,t_1)] = [1+ \bar{r}(t_o,t_2)] = \ldots\ldots = [1+ \bar{r}(t_o,t_n)]$$

When the TSIR has a positive slope:

$$f(t_o,t_1) < f(t_1,t_2) < \ldots\ldots < f(t_{n-1},t_n)$$

$$\Phi(t_o,t_1) < \Phi(t_1,t_2) < \ldots\ldots < \Phi(t_{n-1},t_n)$$

$$\Theta(t_o,t_1) < \Theta(t_1,t_2) < \ldots\ldots < \Theta(t_{n-1},t_n)$$

$\tau(t) = \Theta(t)$ strictly increasing function with respect to t

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[9] This assumption is the most usual hipothesis in the financial literature, despite it seldom holds in reality.

The TSIR is also said to have positive slope when:

$$[1+\bar{r}(t_o,t_1)] < [1+\bar{r}(t_o,t_2)] < \ldots\ldots < [1+\bar{r}(t_o,t_n)]$$

However, the latter condition is not equivalent to the former ones.

When the TSIR has a negative slope:

$$f(t_o,t_1) > f(t_1,t_2) > \ldots\ldots > f(t_{n-1},t_n)$$

$$\Phi(t_o,t_1) > \Phi(t_1,t_2) > \ldots\ldots > \Phi(t_{n-1},t_n)$$

$$\Theta(t_o,t_1) > \Theta(t_1,t_2) > \ldots\ldots > \Theta(t_{n-1},t_n)$$

$\tau(t) = \Theta(t)$ strictly decreasing function with respect to t.

The TSIR is also said to have negative slope when:

$$[1+\bar{r}(t_o,t_1)] > [1+\bar{r}(t_o,t_2)] > \ldots\ldots > [1+\bar{r}(t_o,t_n)]$$

As before, the latter condition is not equivalent to the former ones.[10]

## 2.- SHIFTS OF THE TSIR

Now, we are going to analyse the relationships among different sort of shocks or shifts of the TSIR, pointing out the need of clearly stating what variable we are working with.

### A.- ADDITIVE SHOCK

By an additive shock we mean a parallel shift of the variable we are using to describe the TSIR. We can distinguish then:

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[10] The TSIR may have other shapes, but we are not going to deal with them in this paper.

A.1 Additive shock on the force of interest:

$$\tau^{\circ}(t) = \tau(t) + h$$

i.e. the value of the force of interest suffers an increment (decrement) of amount h for all t.

* effects on the accumulation factor:

$$f^{\circ}(t_m, t_q) = \exp\left[ \int_{t_m}^{t_q} (\tau(t) + h)\, dt \right]$$

$$f^{\circ}(t_m, t_q) = f(t_m, t_q) * \exp[h * (t_q - t_m)] =$$

$$= f(t_m, t_q) * \delta$$

this implies that an additive shock on the force of interest is equivalent to a multiplicative shock $\delta$ on the accumulation factor.

* effects on the mean accumulation factor:

$$[\bar{f}^{\circ}(t_o, t_n)] = [\bar{f}(t_o, t_n) * \delta]$$

or,

$$(1 + \bar{r}^{\circ}) = (1 + \bar{r}) * \delta$$

* effects on the logarithmic density

$$\Theta^{\circ}(t_m, t_q) * (t_q - t_m) = \int_{t_m}^{t_q} (\tau(t) + h)\, dt \qquad t_m < t_q$$

$$\Theta^{\circ}(t_m, t_q) * (t_q - t_m) = \ln f(t_m, t_q) + h * (t_q - t_m)$$

$$\Theta^{\circ}(t_m, t_q) = \frac{\ln f(t_m, t_q)}{t_q - t_m} + h$$

$$\Theta^{\circ}(t_m, t_q) = \Theta(t_m, t_q) + h$$

The resulting effect is then additive.

A.2 Additive shock on the logarithmic accumulation factor

$$\Phi^{\circ} (t_{j-1},t_j) = \Phi (t_{j-1},t_j) + h \qquad j= 1,2,..,n$$

* effects on the accumulation factor:

$$f^{\circ} (t_{j-1},t_j) = \exp [ \Phi (t_{j-1},t_j) + h ]$$

$$f^{\circ} (t_{j-1},t_j) = f (t_{j-1},t_j) * \exp (h)$$

The effect is a multiplicative shock.

A.3 Additive shock on the variable r(.)

$$r^{\circ} (t_{j-1},t_j) = r (t_{j-1},t_j) + h$$

then

$$[1 + r^{\circ} (t_{j-1},t_j)] = [1 + r (t_{j-1},t_j) + h]$$

* effects on the accumulation factor:

$$\begin{aligned} f^{\circ} (t_{j-1},t_j) &= [1 + r^{\circ} (t_{j-1},t_j)] = \\ &= [1 + r (t_{j-1},t_j) + h] = \\ &= f (t_{j-1},t_j) + h \end{aligned}$$

B.- MULTIPLICATIVE SHOCK

By a multiplicative shock we mean that the variable we are using to describe the TSIR is multiplied by a parameter of value g.

B.1 Multiplicative shock on the force of interest:

$$\tau^{\circ} (t) = \tau (t) * g$$

* effect on the accumulation factor

$$f^{\circ} (t_m,t_q) = \exp [ \int_{t_m}^{t_q} ( \tau (t) * g ) dt ]$$

$$f^{o}\ (t_m, t_q) = [\ f\ (t_m, t_q)\ ]\ \hat{}\ [\ g\ ]$$

Thus, the result is an exponencial shock.

* effects on the logaritmic density

$$\Theta^{o}\ (t_m, t_q) * (t_q - t_m) = \int_{t_m}^{t_q} (\ \tau\ (t) * g\ )\ dt \qquad t_m < t_q$$

$$\Theta^{o}\ (t_m, t_q) * (t_q - t_m) = g * \ln f\ (t_m, t_q)$$

$$\Theta^{o}\ (t_m, t_q) = g * \frac{\ln f\ (t_m, t_q)}{t_q - t_m}$$

$$\Theta^{o}\ (t_m, t_q) = g * \Theta\ (t_m, t_q)$$

Now the effect is multiplicative too.

B.2.- Multiplitative shock on the accumulation factor

From A.1 follows that a multiplicative shock on the accumulation factor is equivalent to an additive shock on the force of interest.

Many other shocks could be considered and they can be analysed in a similar way.[11]

Sometimes we can be interested in studing the effects on the TSIR caused by economic events as inflation, considering a risk premiun, etc. In such cases, we can analyse these topics in a analogous way.

For instance, the force of interest which may be considered as an instantaneous price per money unit, can be split up into two parts: one of them reflecting inflation and the other the "real" instantaneous price. Thus:

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[11] For instance, see Khang (1979) and Babbel (1983).

$$\tau(t) = \alpha(t) + \beta(t)$$

where: $\tau$ is the nominal force of interest

$\alpha$ is the instantaneous inflation rate

$\beta$ is the "real" force of interest

The accumulation factor would be then separated multiplicatively into two parts:

$$f(t_m, t_q) = \exp \left[ \int_{t_m}^{t_q} \tau(t)\, dt \right] =$$

$$= \exp \left[ \int_{t_m}^{t_q} \alpha(t)\, dt \right] * \exp \left[ \int_{t_m}^{t_q} \beta(t)\, dt \right]$$

$$= f_\alpha(t_m, t_q) * f_\beta(t_m, t_q)$$

## 3.- FINAL REMARKS

We have studied the actual relationship among different variables that can be used to define the TSIR, from the viewpoint of the mathematics of finance. This paper also includes the equivalence between discrete and continuous variables.

This allows the treatment of the movements of the TSIR due to changes in the variables used to describe it:

- an additive shock on the force of interest is equivalent to a multiplicative shock on the accumulation factor
- an additive shock on the force of interest is equivalent to an additive shock on the logarithmic density
- an additive shock on the logarithmic accumulation factor is equivalent to a multiplicative shock on the accumulation factor

- an additive shock on the variable r(.) is equivalent to an additive shock on the accumulation factor
- a multiplicative shock on the force of interest is equivalent to an exponential shock on the accumulation factor

- a multiplicative shock on the force of interest is equivalent to a multiplicative shock on the logarithmic density

- a multiplicative shock on the accumulation factor is equivalent to an additive shock on the force of interest

We have specially pointed out the use of mean variables.

When the effects of changes of the TSIR are analysed in financial topics (such as duration, immunization, etc.), we can find papers where different sort of shocks are used over different variables obtaining equivalent results and, just the opposite, the same shock on different variables may lead to different results. For instance:

FISHER, WEIL (1.971), use the force of interest $\tau(t)$ and additive shocks

BIERWAG (1.977), use the logarithmic density and additive and multiplicative shocks

BIERWAG, KAUFMAN, SCHEITZER, TOEVS ( 1983), use the mean accumulation factor and additive and multiplicative shifts

BIERWAG (1987): additive shifts and multiplicative shifts over the mean value of the variable $\tau$ (rate of interest)

MENEU, NAVARRO (1988,1990): arbitrary shocks over the logarithmic density

KHANG (1979): use a logarithmic shifts depending on term to maturity

Finally, a brief comment about other applications is made concerning economic topics as inflation.

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