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Cashflow Matching Using Modified Linear Programming

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Summary

This paper describes a method of selecting an appropriate mix of the initial assets of a fund so that, at specified future dates, there will be a high probability that sufficient cash will be available to meet the outgo at those dates.

The method generates net accumulated cashflows in a large number of stochastic economic scenarios and uses those cashflows to select optimum distributions of the initial assets to ensure solvency at each test point in all but a small percentage of the scenarios. Optimum results are found by using a modified version of linear programming.

Résumé

Congruence de Cash-Flow Utilisant la Programmation Linéaire Modifiée

Cet article décrit une méthode destinée à sélectionner un éventail approprié d'actifs initiaux pour un fonds de façon à ce que, à des dates futures spécifiées, il y ait une forte probabilité qu'il y ait suffisamment d'argent disponible pour couvrir les dépenses à ces dates.

La méthode génère des cash-flows accumulés nets dans un grand nombre de scénarios économiques stochastiques et utilise ces cash-flows pour sélectionner des distributions optimales des actifs initiaux afin de garantir la solvabilité à chaque point de test dans la plus grande majorité des scénarios. Les résultats optimaux sont calculés en utilisant une version modifiée de la programmation linéaire.

CASHFLOW MATCHING USING MODIFIED LINEAR PROGRAMMING

1 INTRODUCTION

1.1 The actuarial approach to financial risks deals with the art of the possible, seeking to achieve the best practical solution that can be found, whilst keeping within various constraints.

This paper investigates the actuarial problem of identifying, for the investment managers of a life or pension fund, the maximum and minimum percentages of the current assets that can be held in gilt-edged stocks, with the freedom to invest the remainder of the fund in potentially higher-yielding but more volatile assets, in order that there is a high probability of meeting the future liabilities.

The method of solution uses a modified version of Linear Programming. It borrows heavily from the papers on asset selection produced by Wise (1984), and also incorporates the parameters for the Wilkie stochastic investment model included in the paper by Daykin and Hey (1990).

1.2 Linear programming (LP) is a series of techniques for optimising linear functions which are subject to linear inequality constraints. Normally, the variables that form the functions are constrained to be non-negative. A brief description of the process is set out in Appendix 1, although the techniques have been widely researched and published.

The problems in this paper require that not all the variables are necessarily positive. The solutions therefore involve more complicated LP logic and I have called these techniques Modified Linear Programming (MLP). Appendix 2 touches on some of the problems involved.

1.3 The paper is intended to indicate a modular strategy rather than a unique set of solutions. In particular, it is expected that the computer power that will become available over the next decade will enable many of the approximations that have had to be introduced in this paper to be ignored. This rate of computer development has been dramatic in the last twenty years; it has transformed actuarial techniques and seems set to continue this rapid rate of progress. This paper therefore is intended to be a stage on the path of development, rather than an ultimate solution.

2 A SIMPLE EXAMPLE

2.1 Suppose we have a fund with existing assets of A. A premium of 100 will be received at the start of year 2. Payments of 200 are made at the end of years 3 and 5.

2.2 Available investments are dated gilts, equities and a cash fund. For simplicity, we assume that equities will only be sold, and dated gilts only be redeemed, at the end of years 3 and 5. The proceeds from the sale and redemption of these investments, together with interest and dividends received, are all paid into a cash fund, which can also be used for direct investment. The gilts have a coupon of 10%.

2.3 There is no taxation.

2.4 Interest is earned on the cash fund at a variable rate. The cash fund is allowed to go into deficit and the same interest rates are charged on those overdrafts.

2.5 Three possible future economic scenarios A, B and C are assumed. The proceeds (including cash-fund interest) from an investment of 1 in the available investments will, under those three scenarios, be:

Investment Year	Sale/ Redemption Year	Accumulation Year	GILTS A	GILTS B	GILTS C	EQUITIES A	EQUITIES B	EQUITIES C
Outset	3	3	1.3233	1.3246	1.3231	2.2665	1.1164	1.2275
		5	1.5325	1.5280	1.5459	2.6248	1.2877	1.4341
	5	3	0.3233	0.3246	0.3231	0.2080	0.1596	0.1967
		5	1.5823	1.5817	1.5757	3.2293	1.3372	1.9125
2	3	3	1.2001	1.2113	1.2032	1.5957	1.0110	1.2328
		5	1.3898	1.3972	1.4058	1.8479	1.1661	1.4404
	5	3	0.2054	0.2092	0.2062	0.1054	0.1021	0.1454
		5	1.4316	1.4541	1.4410	2.2855	1.2131	1.9451

It can be seen that, where the accumulation year is prior to the sale/redemption year, then the proceeds consist only of the rolled-up dividends or interest payments from the relevant investment.

For the Cash Fund, accumulated proceeds are:

Investment Year	Accumulation Year	A	B	C
Outset	3	1.2423	1.2515	1.2419
	5	1.4387	1.4437	1.4509
2	3	1.1557	1.1642	1.1552
	5	1.3383	1.3429	1.3497
3	5	1.1580	1.1535	1.1684

2.6 We define the fund as being solvent if we have sufficient assets in the cash fund at the end of years 3 and 5 to meet the outgo of 200 at those test-points.

We are allowed a 33 1/3% ruin probability so that, at each of the two test-points, the solvency test can fail under not more than one of the three scenarios A, B and C. The scenario causing a deficit in year 3 need not be the same one as that causing a deficit in year 5.

2.7 We do not know at the start of the problem how the premium of 100 in year 2 will be invested. This will be found as an output of the main solution.

2.8 The problem to be solved is:

What is the smallest holding of initial assets that we need to meet the required solvency constraints, and what are the investments required for the initial assets and for the renewal premium?

The answers are:

a) Minimum initial assets 151.806
b) Asset-mix

Initial	88.775%	in a 3 year gilt
	2.424%	in equities, to be sold in year 3
	8.801%	in equities, to be sold in year 5
Renewal	100%	in equities, to be sold in year 5

The net cash funds after deducting the outgo are, for the three scenarios at test-points 3 and 5:

Scenario	A	B	C
Test-point 3	0	-5.0	0
Test-point 5	56.3	-80.9	0

For example, the accumulated cash-fund income available at test point 3 under scenario A is made up of

Proceeds of initial assets in 3 year gilt 151.806 x .88775 x 1.3233	=	178.34
+ proceeds of initial assets in equities sold in year 3 151.806 x .02424 x 2.2665	=	8.34
+ income from initial assets in equities due to be sold in year 5 151.806 x .08801 x .208	=	2.78
+ income from renewal premium in equities due to be sold in year 5 100 x .1054	=	10.54
		200.00

which is just sufficient to meet the outgo of 200 in year 3.

We are not able to reduce the size of the initial assets any further by reducing the proportions in equities. The higher income from the equities is needed in certain scenarios to provide the required payments out.

It can be seen that the two in three solvency requirement is met at each test-point. The one allowable insolvent scenario happens to be B at both test-points, but this is not a necessity.

2.9 If we hold more assets at outset, then we will not need to invest so great a proportion in gilts. The extra "free reserves" allow some mismatching into higher yielding but more volatile equities. If assets of more than about 160 are held at outset, then 100% can be invested in equities if required. Some results are:

Initial assets	Minimum % in gilts
151.806	88.8
155	60.5
160	18.5

2.10 If initially we hold less than 151.806, then it will not be possible to find any mix of initial and future investments that will meet the required 2 in 3 solvency standard.

It is fairly straightforward, using the table of proceeds in 2.5, to solve for this limit by trial and error. However, in practice we can be dealing with up to 50 years of future cash flows and say 1000 possible future economic scenarios (which would be the minimum needed for any credible stochastic ruin probabilities).

Also, we need to allow investment in irredeemable gilts, and to include the facility of selling equities and irredeemable gilts (and choosing redemption years for dated gilts) in non-test years.

With all these requirements, the problem becomes much more complex. The detailed problem is described in the next section, with a broad outline of the method of solution in section 4.

3 THE DETAILED PROBLEM

3.1 For the purposes of this solution, asset classes are limited to dated and irredeemable gilt-edged stocks, and equity shares. The dated gilt stocks are assumed to be held until redemption; the irredeemable gilt stocks and equity shares are divided into predetermined groups which will be sold at future dates determined at the time of purchase, at the prices ruling at the time of sale.

The dividends from equities and gilts are assumed to be reinvested in a cash fund with varying future interest rates.

3.2 In order to test for solvency with a high probability of security, future equities and gilt prices and expense inflation rates are projected using the stochastic model devised by Wilkie (1986) with parameters updated by Daykin and Hey (1990). A description of those parameters is set out in Appendix 1 of the latter paper.

Due to computer space problems, 1000 stochastic runs have been used, although a minimum of 5000 runs would be preferable.

3.3 Future premiums or pension contributions are constrained to be used firstly to meet outgoing liability payments of the same type of inflationary currency. Any residue is free to be invested each year, plus the existing assets at outset, in a notional series of dated gilts maturing in each of the next 50 years, together with sets of irredeemable gilts and equities which are assumed to be sold in each of those future years.

Dated gilts maturing, or irredeemables or equities sold, ahead of the year in which they are needed to pay for liability outgo are assumed to be held in the cash fund accumulating at the stochastic rates of deposit interest in force over the period of withdrawal. These rates are normally taken as 75% of the full gilt-edged yields at the time, and apply also to overdraft facilities in years where strict solvency is not required and the liability outgo exceeds the then available cash fund.

3.4 The investment pattern for net future income is found as part of the solution to the overall problem.

3.5 Solvency is defined as having sufficient residual cash funds, at the end of specified future test years, to pay out the liability outgo at each of those test points in all but x% of the stochastic runs investigated.

3.6 Future cash-flow in or out can be partially inflation-linked, enabling a final salary pension scheme to be investigated.

4 METHOD OF SOLUTION

4.1 We generate a large number of future economic scenarios, using the stochastic model of 3.2. If we are testing say s points for cash flow solvency, then each stochastic run will generate s lines in the Simplex matrix described in Appendix 1.

4.2 Slack variables $V_{s,k}$ (for test year s in stochastic run k) must be positive or zero if the cash flow at the test point is to be non-negative for the stochastic run.

We use the modified LP logic of Appendix 2 to find the required minimum gilt, equity or total initial assets subject to these constraints.

4.3 Define $g_{i,j}$ as the proportion of gross positive cash-flow invested in year i in a dated gilt-stock maturing in year j, with $h_{i,j}$ and $e_{i,j}$ similarly the proportions of irredeemable gilt stock and equity shares bought in year i and sold in year j.

Then if we are investigating liability outgo for the next N years, with M years of premium income, we have

$$\sum_{j=i}^{N} \left[g_{i,j} + h_{i,j} + e_{i,j}\right] = 1 \text{ for } i = 0 \text{ to } M, \text{ where } N \geqslant M.$$

$g_{i,i}$ represents cash and $h_{i,i} = 0$ and $e_{i,i} = 0$ for all i.

If we introduce artificial variables U_i, representing the uninvested proportion of the year's premium income, we get the set of equations

$$U_i + \sum_{j=i}^{N} \left[g_{i,j} + h_{i,j} + e_{i,j}\right] = 1 \text{ for } i = 0 \text{ to } M.$$

4.4 The redemption or sale proceeds of stocks and shares are paid in to the cash fund, from which the liability outgo is paid out.

Then the residual accumulated cash fund for each specified test year must be sufficient to cover the liability outgo in that year.

Therefore, for any test year s, for all stochastic runs k except a limited number which the ruin probability allows to be insolvent, we need

$$\left[\sum_{i=0}^{M'} \left[\sum_{j=i}^{s} \left\{g_{i,j}.GP(i,j,s,k) + h_{i,j}.GIP(i,j,s,k) + e_{i,j}.EP(i,j,s,k)\right\}\right].CP(i,k)\right] \geqslant \sum_{i=1}^{s} CN(i,k).WP(i,s,k)$$

where g, h and e are defined in 4.3 and $M' = \min\{M\ ;\ s\}$

GP(i,j,s,k) is the accumulated value to the end of year s, for stochastic run k, of an investment of 1 at the end of year i in a gilt stock redeemed at the end of year j and thereafter rolled-up in the cash fund. Income, reinvested in the cash fund, is included in this value.

GIP and EP are similar to GP, for respectively an irredeemable gilt or an equity sold at the end of year j. WP is the cash fund accumulation from year i to year s.

The net cash flow, eg premium income or pension scheme contributions receivable less liability outgo payable, at the end of year i in the stochastic conditions of run k is shown as CP(i,k) if positive or CN(i,k) if negative.

4.5 If we are solving for the minimum proportion of the fixed total current assets A that can be held in gilt-edged, we need to minimise the Objective Function

$$Z = A.\sum_{j=0}^{N}\left[g_{o,j} + h_{o,j}\right]$$

and where we are minimising the proportion of the fixed total current assets that can be held in equities,

$$Z = A.\sum_{j=0}^{N} e_{o,j}$$

If we wish to find the smallest amount of current assets, however invested, the Objective Function is taken as

$$Z = A.\sum_{j=0}^{N}\left[g_{o,j} + h_{o,j} + e_{o,j}\right]$$

and we replace the constraint $\sum_{j=0}^{N}\left[g_{o,j} + h_{o,j} + e_{o,j}\right] = 1$

by $$\sum_{j=0}^{N}\left[g_{o,j} + h_{o,j} + e_{o,j}\right] + U_o = 1$$

where $0 \leqslant U_o \leqslant 1$

That is, we define the variable U as slack instead of artificial, so that it may remain $\geqslant 0$ in a feasible solution.

4.6 The detailed logic of the solutions is described in an unpublished working paper by the author (1990).

5 SOME PRACTICAL EXAMPLES

Immediate Annuity certain

5.1 As a first practical example, consider an Immediate Annuity certain, with 1000 payable at the end of each of years one to five.

If we had, to take an extreme example, current assets of 1 million then, even if they were all invested in equities, the probability of failure to have sufficient cash to pay any of the five annuity payments would be virtually zero.

Equally, if we had no initial assets, the risk of insolvency would be certain.

Assume that at outset the gilts and irredeemable stocks yield 10% and are standing at par; ie they have coupons of 10%.

Taking the Wilkie stochastic model, as amended by Daykin and Hey, we get the following ranges of initial investment, using 1000 stochastic runs and a ruin probability of 0.5%; ie, 5 of the runs, not necessarily the same ones, can be in deficit at each of the five test points.

	Minimum % in gilts				Maximum % in gilts			
Current assets	3791	3800	4000	4500	3791	3800	4000	4500
Cash	-	-	-	-	-	10.5	-	100.0
Gilt redemption								
Yr 1	16.4	16.0	9.1	-	16.4	6.0	76.9	-
Yr 2	18.0	18.0	17.3	4.8	18.0	18.0	-	-
Yr 3	19.8	19.3	9.3	10.2	19.8	19.8	-	-
Yr 4	21.8	21.8	21.0	18.7	21.8	21.7	-	-
Yr 5	24.0	23.7	17.8	9.9	24.0	24.0	-	-
Irredeemable sold								
Yr 1	-	-	-	-	-	-	23.1	-
Yr 2	-	-	-	7.2	-	-	-	-
Yr 3	-	0.4	8.7	-	-	-	-	-
Yr 4	-	-	-	-	-	-	-	-
Yr 5	-	-	-	-	-	-	-	-
Total Gilts	100.0	99.2	83.2	50.8	100.0	100.0	100.0	100.0
Total Equities	-	0.8	16.8	49.2	-	-	-	-

It will be seen, as one would expect, that as the total initial asset holding reduces, the gap between maximum and minimum equity holdings reduces until, at a critical size, we require all the stocks to be in matching gilts. At this stage we have an absolute matching position similar to that in the example in Wise's 1984 paper (para 5.5).

Deferred Annuity Certain

5.2 As a second example, we take a five year annuity certain, with payments deferred for five years. This is identical to the example in para 5.6 of Wise (1984), but with the same stochastic model, initial gilt yields and ruin probability as in 5.1 above.

The results are shown in Appendix 3. It will be seen that, because of the uncertainty of the roll-up of future interest over the deferred five year period, absolute matching is not possible. (This is comparable to Wise (1984), para 5.6).

Groupings

5.3 So far the examples in 5.1 and 5.2 have examined each individual year of the future cashflow outgo for solvency. Where however there are many future test points, the available computer power may not be sufficient to handle the resulting simplex matrices.

In this case, certain specific test points only will be examined for solvency, with the remaining years' accumulated cashflows not necessarily being sufficient to cover the outgo in those years. Also, cashflows in and out are grouped over a period of several years, with stochastic interest accumulation to the end of each period as in 3.3.

An example of this is shown in the next paragraph, which takes an existing (notional) pension scheme with liability outgo over each of the next 25 years. In this case, every fifth year only is taken as a test point. Net cashflows are separately accumulated over quinquennial intervals, for each stochastic run, at the then relevant stochastic rate of cashflow accumulation.

Provided that the test points are spread reasonably over the period, a satisfactory result should be achieved, even if not an optimum one.

Pension Scheme

5.4 Cashflows representing an established final salary pension scheme are shown below, grouped quinquennially for simplicity. Contribution income and pension entitlement up to retirement are linked to salary and hence to inflation. Pensions in payment remain static. The figures are given in current units of salary.

Cashflow Year	Inflation to Year	Contribution income	Pension outgo
5	0		50
	5	80	55
10	5		55
	10	60	60
15	10		60
	15	40	60
20	15		60
	20	20	50
25	20		50
	25		40

Because of the more flexible nature of the future funding rate, a weaker ruin probability of 5.0% is used at quinquennial intervals.

The results for different total current assets, using again the Daykin/Hey variation of the Wilkie model, with 1000 runs are:

Current assets	Minimum % in equities	Maximum % in equities
364	50	50
380	22	82
400	7	100

Because the liabilities are to a great extent linked to inflation, the equity proportions in the initial assets are more evident than in the two previous examples. The maximum % in equities increases as the ruin probability allowed is increased. For example, with 10% ruin probability, minimum assets reduce from 364 to 329 and the % in equities increases from 50% to 66%.

Fixed initial gilt assets

5.5 As an alternative use of the cashflow matching model, we can specify the current holding in dated gilt stocks, by redemption year, and the current total of irredeemable stocks. The model is then used to test whether a particular level of ruin probability is achievable; ie whether a feasible solution can be found.

5.6 We tackle this problem by introducing further equations of the form $g_{o,i} = A_i$ for each of the specified redemption years i, plus

$$\sum_{i=1}^{N} h_{o,i} = H$$

for a specified total of current irredeemable stocks.
These equations are added to the simplex matrix by means of further rows of the form

$$T_i + g_{o,i} = A_i$$

where T_i are artificial variables, taken as the Basic Variables for the initial simplex matrix, with value A_i , and the $g_{o,i}$ are taken as non-basic variables with initial value zero. Similarly the irredeemable equation can be introduced as

$$T_j + \sum_{i=1}^{N} h_{o,i} = H$$

with T_j another artificial variable set initially to H. As with the U_i artificial variables, until pivoting has removed all the T_i a feasible solution has not been reached.

Missing assets

5.7 Sections 5.19 to 5.25 of Wise (1984) investigated matching portfolios when certain assets were not available for purchase. In particular, the matching portfolio for a ten-year annuity certain was examined, in 5.19 of that paper, where the only assets available were gilts maturing after 1, 3, 5, 7 and 9 years. The minimum asset holding, allowing for a 2.5% risk of deficit at maturity was shown to be 6.226 for an annual liability outgo of 1, ie 6226 per mille.

It is interesting to apply the current model to the same situation. There will not be exact agreement because of the different stochastic models used in the two investigations, but the comparison highlights the fundamental differences in the two approaches. As in Wise's investigation, 2.5% ruin probabilities are used, and the following tests are made:

Run	Test years	Cash fund interest as % of gilts rates
a	1, 3, 5, 7, 9 and 10	75
b		100
c		110
d	10	75
e		100
f		110

The resulting minimum initial assets and their distribution is shown in the following table:

	% of initial assets						
	Wise	Current paper run					
		a	b	c	d	e	f
Gilt Redemption							
Year 1	10.2*	6.7	11.6	100.0	-	11.6	100.0
3	14.8	15.2	14.2	-	-	13.7	-
5	17.8	18.1	19.6	-	2.6	19.5	-
7	18.7	21.3	17.6	-	66.0	19.1	-
9	38.5	38.7	37.0	-	31.4	36.1	-
	100.0	100.0	100.0	100.0	100.0	100.0	100.0
Minimum assets	6226	5966	5978	5927	5918	5978	5927

* The holding of 8.9% in Wise's distribution has been increased for the extra assets required to give the 2½% confidence level of surplus.

It can be seen that, as we reduce the number of test points, thereby allowing overdraft facilities to finance more of the annuity payments, the matching asset-distribution becomes more sensitive to the interest rates in the cash fund (ie the overdraft rates). However, since the process requires the same interest rates to be used for cash fund deposits and overdrafts, it is important in practice to keep the test points spread evenly over the full term. Otherwise, the program will seek to optimise the asset-mix by investing short if cash fund rates are a high proportion of gilts, or alternatively by investing long (in higher-yielding equities if available) and running a long-term overdraft if borrowing rates are too generous. The spread of test points will counter this latter tactic.

Where the only test point is that at the end of the term, which is similar to the situation used by Wise, this sensitivity is most acute. The reason for this is that, whereas his approach seeks to minimise the ultimate surplus, and hence, as he points out in para 5.20 of his paper, aims to minimise the exposure to future changes in the interest rate, the present approach is only concerned with keeping the chance of a deficit at each test point to below 2½%; it puts no constraint on the probability that large surpluses may occur at those test points. Therefore the closest comparison with Wise's figures occur not in runs (d)-(f), as one might at first expect, but in runs (a)-(c), with run (b) being the closest.

Stochastic feed-back

5.8 Since each line of the stochastic part of the simplex matrix represents an independent equation, it is possible to introduce cashflow elements linked to the stochastic variables. For instance, the salary-linked elements of contribution income and pension-outgo in the pension scheme in para 5.4 vary in absolute amounts depending on the rates of inflation generated in each stochastic run.

It is therefore possible to introduce more complex 'black-box' feedback into the equations, without prejudicing the generality of the method.

5.9 For instance, rates of future bonus in a life fund could be linked to the past n years investment results or rates of inflation. Withdrawal rates for guaranteed surrender values or retirement rates could be linked to investment conditions at the time (see Ford and Masters, 1978).

The strategy in the black-box itself could be tested for robustness against required levels of ruin and varied until a satisfactory strategy was achieved.

6 CONCLUSION

6.1 The approach to solving many actuarial problems consists of the art of the possible. Inequality constraints, feasible areas of possible solutions and finding the optimum solution subject to those constraints fit in with this philosophy, and linear programming methods are a natural consequence.

6.2 The complexities of the practical problems on which actuaries are asked to advise means that the straightforward LP methods can often not be applied. It is hoped that the example of Modified Linear Programming in this paper may show how the normal LP methods can be adapted to tackle some of these difficult practical problems.

6.3 The constraints of available computer power have led to various approximations in this paper, and it is expected that more general methods will become available as computer power increases.

For example, an obvious extension would be to assume higher interest rates for the cash fund in those years when it was in overdraft, an approach originally discussed by Benjamin (1959). Solutions can be investigated by including in the simplex formulae one of the constraining algorithms described in the paper by Kocherlakota, Rosenblum and Shiu (1988). These additions would however substantially increase the computer power required.

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STANDARD LINEAR PROGRAMMING - THE SIMPLEX SOLUTION

A1.1 Linear programming looks to minimise $Z = \sum_{j=1}^{n} B_j . x_j + B_o$ (known as the Objective Function) subject to the constraints

$$\sum_{j=1}^{n} a_{i,j} . x_j \leqslant K_i \quad (i = 1, \ldots, m),$$

where B_j and a_{ij} are constants or known data and x_j are the variables to be found.

If the x_j are all constrained to be non-negative, the problem is said to be in standard equality form. Where a certain number (possibly unnamed) of the x_j are unrestricted, the LP solutions become more complex. These unrestricted x_j are called free variables and I have called their solutions modified linear programming (MLP).

A1.2 The following description is intended to focus on the main aspects of the simplex solution which are necessary for an understanding of the logic of the modified linear programming algorithms in Appendix 2. Inevitably it will appear trivial to those readers who already have an understanding of LP, whilst giving insufficient detail for those who have not yet dealt in practice with these techniques.

However, for these latter readers, the books by Vajda, Beale and Hartley mentioned in the references provide an excellent insight into the subject and I give no apologies for failing to reproduce their detailed explanations. The following paragraphs hopefully set the main framework for the understanding of the subsequent sections, leaving the reader who wishes to follow the proposed methods in practice to fill in the missing gaps from the textbooks.

A1.3 The inequalities of A1.1 can be converted into equalities by introducing slack variables v_i, to give:

$$V_1 + a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots a_{1n}x_n = K_1$$
$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots$$
$$V_m + a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots a_{mn}x_n = K_m$$

These can be set out in a tableau form (T1), known as the simplex method of solution, as follows:

T1	x_1	x_2	....	x_n	Constraint
v_1	a_{11}	a_{12}		a_{1n}	K_1
⋮					⋮
v_m	a_{m1}	a_{m2}		a_{mn}	K_m
Z	$-B_1$	$-B_2$	....	$-B_n$	B_o

where the bottom row is the Objective Function equation for Z.

The variables v_i in the left-hand column are called Basic Variables (BV) and the variables x_i at the top of the columns are called Non-Basic Variables (NBV).

In the standard LP algorithm, all v_i and x_i must be kept $\geqslant 0$.

A1.4 By multiplying rows by non-zero constants and adding or subtracting them from each other, we can effect a transformation that exchanges one BV with one NBV; this is called pivoting.

Suppose for instance that we wish to exchange the NBV x_c with the BV V_r. To achieve this we need to remove x_c from all terms except the r^{th} row.

This is done by dividing the rth row by a_{rc} (which must therefore be non-zero). We then multiply this new row r by a_{ic} and subtract the result from row i, for each row i except r.

A1.5 The purpose of the above pivot is to reduce the Objective Function $Z' = B_o - \frac{K_r . B_c}{a_{rc}}$ as fast as possible.

A1.6 The element a_{rc} of the pivotal row r and pivotal column c in Tableau 1 is called the pivot. If we put all the NBV equal to zero, then each BV will equal the value in the constant column for its particular row. The Objective Function Z equals the constant value for its row.

A1.7 Pivoting continues until no transformation can be found that will make Z any smaller. At this stage we have found an Optimum Basic Feasible Solution for the problem, consisting of the variables representing at that stage the Basic Variables in the left hand column. The sets of Basic Variables in the earlier tableau stages are called a Basic Feasible Solution (BFS) providing they meet all the required constraints, which at present require all the variables to be non-negative.

A1.8 We choose the first tableau in a form that makes all the BV positive. This will clearly be the case where all the K_i are > 0 and for the moment we assume that this is true.

Then if, after the first pivot, all the BV are to remain positive, we need to find a pivotal row and column such that

a) $a_{rc} > 0$. This will make $K'_r \; (= \frac{K_r}{a_{rc}}) > 0$ since we know that K_r is already > 0

and b) $K'_i = K_i - K_r \left(\frac{a_{ic}}{a_{rc}} \right) > 0$ for all $i \neq r$.

If we choose the row r which has the smallest positive value of K_i/a_{ic} (where c is the pivot column) then $K' = (a_{ic})\left[\frac{K_i}{a_{ic}} - \frac{K_r}{a_{rc}}\right] > 0$ since we have chosen $0 < \frac{K_r}{a_{rc}} < \frac{K_i}{a_{ic}}$ and, since K_i is already > 0, then if a_{ic} is > 0, K' = (+) [+] > 0; if $a_{ic} < 0$, then

$K' = [K_i - a_{ic}.\frac{K_r}{a_{rc}}] = [(+) - (-)(+)] > 0$ again.

A1.9 The algebraic logic for the pivotal row selection has been set out in some detail because it is fundamental to the amended logic of MLP in Appendix 2.

A1.10 The choice of pivotal column is easier. We are seeking to reduce the value $Z' = B_o - \frac{K_r.B_c}{a_{rc}}$ after pivoting.

We have already determined that K_r/a_{rc} must be > 0 so, if we choose a column with $B_c > 0$, we will have

$Z' < B_o = Z$.

In order to reduce the Objective Function Z as fast as possible, it would seem sensible to choose the column with the largest positive value of B and therefore we do so (although this is not mandatory).

If no row has a pivotal element $a_{rc} > 0$, then we will need to look for the column with the next largest value of B and see if a suitable pivot can be found in that column.

If no suitable column can be found, then we cannot find a pivot to reduce Z any further, and the Optimum Basic Feasible Solution has been reached.

<u>Artificial variables</u>

A1.11 In A1.8 we made the assumption that all the K_i are > 0. For any row i where $K_i < 0$, we cannot initially put the slack variable v_i as the Basic Variable since we would have $V_i + \sum_{j=1}^{n} a_{ij} . x_j = K_i$

and, putting all x_i to zero as NBVs, we would have $v_i = K_i < 0$ which is inadmissible.

For any such runs, therefore, we introduce an additional <u>artificial variable</u> D_i so that

$$D_i - \sum_{j=1}^{n} a_{ij} . x_j - v_i = -K_i$$

and make that artificial variable D_i the Basic Variable.
Then the slack variable v_i becomes another NBV and the simplex tableau row is of the form

	x_1 .... x_n	v_i	Constant
D_i	$-a_{i1}$ .... $-a_{in}$	-1	$-K_i$

Since $D_i = -K_i > 0$, the constraint that D_i must be > 0 is satisfied and the solution is feasible. This solution contains an additional variable which does not interest us. However, if we can pivot until D_i becomes non-basic, and thereafter ignore that column for future pivoting, we will have driven D_i to zero and it will therefore not affect the optimum solution.

In order to drive out any live D_i that remain as BV, we form an additional Objective Function row DZ, which represents the totals of the elements in the rows which still have a D_i as their BV. Pivot selection operates on this row DZ, rather than on the normal Z row, until all the artificial variables have become non-basic (at which stage the DZ row will be all zero elements). Thereafter normal pivoting selection using the Z bottom row continues as before.

APPENDIX 2

MODIFIED LINEAR PROGRAMMING

Free variables

A2.1 The LP algorithm in Appendix 1 required the slack variables v_i all to be non-negative. However in practice certain problems may need to be solved where some of the v_i are free; ie they are unconstrained and may be < 0.

Where this situation occurs, the normal LP logic breaks down and modifications are required. As mentioned earlier, solutions of this sort are called Modified Linear Programming (MLP).

Two types of MLP are considered in this paper, firstly where specified slack variables are free and secondly where a maximum number of unspecified slack variables are allowed to be negative at any one time.

Local minima

A2.2 Define the set of all possible feasible solutions of an LP problem as the feasible region R of those solutions.

A region R is defined to be convex if the point

$$(1-Q)\ x_1 + Qx_2 \qquad (0 < Q < 1)$$

is always in the region, provided the points x_1 and x_2 also belong to it. A convex function (x) is one such that, if

$$x = (1-Q)\ x_1 + Q\ x_2 \qquad (0 < Q < 1)$$

then $f(x) < (1-Q) \cdot f(x_1) + Q \cdot f(x_2)$

Then it can be shown (see Beale (1988) pages 4/5) that, if the region R is convex and f(x) is a convex function in R, then any local minimiser x^* of f(x) in R is also a global minimiser.

A2.3 The LP method in Appendix 1 ensures that, at all stages, the solution lies within the feasible area R. However, where the form of the inequalities does not ensure that that region is necessarily convex, we cannot be certain, when an optimum solution is reached, that we have found a global rather than a local minimum solution.

One can take the analogy of rolling a ball down into a bowl. If a local minimum is reached, the ball will have lodged in a small pit in the side of the bowl but will not have reached the global minimum at the bottom of the bowl.

A2.4 The methods of MLP accentuate this problem and care is needed to avoid a local minimum solution which differs widely from the required global minimum solution. The MLP logic used in this paper appears to avoid local minima in the situations examined. However, it has not been proved that the method works in all cases and some inspection of the results is necessary.

Specified free variables

A2.5 In this case, when choosing the pivotal row as in A1.8, we can ignore all rows where the BV is free in choosing the row with smallest k/a value, since we are not concerned whether or not that BV becomes negative after pivoting. We will therefore only pivot on that row if no other positive k/a row exists.

Also, where a pivotal column has as its NBV a free variable, we can pivot on a row which will have $k/a < 0$, since its new BV will be the free (NBV) variable as a result of the pivoting.

Unspecified free variables

A2.6 The logic here is similar to that in A2.5, but somewhat more laboured, since, after carrying out a pivot, we cannot assume without further tests that the solution remains feasible and so R would not be convex.

In order to retain solutions that are still feasible, we need to test before carrying out a pivot that the total number of variables that would be negative would not exceed the allowed maxima at specified points. This adds significantly to the calculation time and needs to be avoided as far as possible. It is, however, necessary for the methods of solution used below in A2.7 and A2.8.

Pivoting Logic to Avoid Local Minima

A2.7 Two possible approaches for solving this problem are:

A2.7.1 Introduce pairs of slack and artificial variables $V_{s,k}$ and $D_{s,k}$ in the equations in 4.4 to give equations of the form

$$D_{s,k} - V_{s,k} + \sum_{i=0}^{n'} \Big[\sum_{j=i}^{\omega} g_{i,j}.GP(i,j,s,k) + h_{i,j}.GIP(i,j,s,k) + e_{i,j}.EP(i,j,s,k) \Big] CP(i,k) = \sum_{i=1}^{\omega} CN(i,k).WP(i,s,k)$$

for test year s and stochastic run k.

D and V are both non-negative variables.

This approach can result in very large simplex matrices, since we have $V_{s,k}$ initially as a Non-Basic Variable in order to put $D_{s,k}$ as the (positive) Basic Variable. Therefore we will need a separate column for each value of s and each value of k.

The logic needs to allow a number, R say, out of the k runs at each test point s, to have positive values of D-V (ie net deficits) with (R/k)% being the probability of ruin. The logic therefore for the simplex pivoting is not entirely straightforward.

A simplex with ten test points s and 1,000 runs k would have more than 10,000 rows and 10,000 columns. This could result in an 800 megabyte simplex matrix which would be very time-consuming to solve.

I have not therefore pursued this approach, although it is the obvious one to be used if the algorithm referenced in 6.3 is to be included, but have used the approach in A2.7.2 below.

A2.7.2 Introduce only slack variables $V_{s,k}$, but allow them to start with negative values and constrain them so that ultimately only R of the variables at each test point s can be negative at any one time.

This approach requires the same number of simplex rows, ie equations, as in A2.7.1, but, since we now start by putting

$$V_{s,k} = -\sum_{i=1}^{A} CN(i,k).WP(i,s,k)$$

as the first feasible solution, we have each $V_{s,k}$ as the Basic Variable and therefore only need columns for the non-basic variables g, h and e, resulting in a much smaller matrix than in A2.7.1.

The logic for constraining the number of $V_{s,k}$ that can be negative for each s is much more complex than that in A2.7.1, where we had the artificial variables $D_{s,k}$ as well.

A2.8 Since the slack variables $V_{s,k}$ that are negative may alter after each simplex pivot, it is not obvious that the feasible area of solution is convex. Therefore we may well reach a solution which is only a local minimum, possibly a long way from the true global minimum result that, with a different choice of initial assets, could have been achieved.

In order to lessen the chance of reaching such a local minimum, an artificial and somewhat more lengthy approach is used instead of the straightforward LP logic. A detailed description is given in the working paper (1990).

	Minimum % in gilts				Maximum % in gilts			
Current assets	2446	2500	2750	3000	2446	2540	2750	3000
Cash								
Gilt redemption								
Year 1	-	-	-	-	-	-	-	-
Year 2	-	-	-	-	-	-	-	-
Year 3	-	-	-	-	-	-	-	-
Year 4	-	-	-	-	-	-	-	-
Year 5	-	-	-	-	-	-	-	-
Year 6	-	-	-	-	-	-	100.0	100.0
Year 7	-	-	-	-	-	-	-	-
Year 8	18.6	11.5	3.6	-	18.6	27.8	-	-
Year 9	33.8	31.9	12.3	-	33.8	32.8	-	-
Year 10	36.6	36.4	19.8	-	36.6	39.4	-	-
Irredeemables sold								
Year 1	-	-	-		-	-	-	-
Year 2	-	-	-		-	-	-	-
Year 3	-	-	-		-	-	-	-
Year 4	2.4	-	-		2.4	-	-	-
Year 5	2.4	-	-		2.4	-	-	-
Year 6	5.2	8.2	-		5.2	-	-	-
Year 7	0.6	2.7	-		0.6	-	-	-
Year 8	-	-	10.0	8.6	-	-	-	-
Year 9	-	1.1	6.4	6.2	-	-	-	-
Year 10	0.4	-	11.5	26.0	0.4	-	-	-
Equities sold								
Year 1	-	5.2	23.3	40.4	-	-	-	-
Year 2	-	-	1.2	0.8	-	-	-	-
Year 3	-	-	-	-	-	-	-	-
Year 4	-	0.6	-	-	-	-	-	-
Year 5	-	-	-	-	-	-	-	-
Year 6	-	0.9	-	-	-	-	-	-
Year 7	-	-	-	-	-	-	-	-
Year 8	-	1.5	1.8	3.9	-	-	-	-
Year 9	-	-	-	13.7	-	-	-	-
Year 10	-	-	10.1	0.4	-	-	-	-
Total Gilts	100.0	91.8	63.6	40.8	100.0	100.0	100.0	100.0
Total Equities	0	8.2	36.4	59.2	0	0	0	0