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# Quantifying the Risk of Deviation from Experience Assumptions

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## Summary

This paper defines the risk inherent in an insurance portfolio, and segments the risk into two components. These are analysed and quantified separately. Statistical variance is the measure used for the quantification. The discussion then turns to combining the two separate measures into an overall quantification of risk. Sections of the paper are:

* Discussion of Risk
* Analysis of Portfolio Risk
* Analysis of Assumption Risk
* Synthesis: Portfolio/Assumption Risk

## Résumé

## Quantifier le Risque d'Ecart par Rapport aux Suppositions Basées sur l'Experience

Cet article définit le risque inhérent à un portefeuille d'assurance et segmente le risque en deux composants. Ceux-ci sont analysés et quantifiés séparément. L'écart statistique est la mesure utilisée pour la quantification. Puis la discussion se tourne vers la combinaison des deux mesures séparées pour en faire une quantification globale du risque. L'article se compose des sections suivantes:

* Discussion du risque
* Analyse du risque de portefeuille
* Analyse du risque de supposition
* Synthèse: Risque de portefeuille/de supposition

## DISCUSSION OF RISK

What types of risk are inherent in an insurance portfolio? An obvious answer would be that the risk corresponds to the type of coverage: If the policy insures against earthquakes, the risk is that an earthquake will occur. This is only a partial answer.

To continue the earthquake policy example, the salient point is that the insurer's profits, though uncertain, can be statistically analyzed. Suppose the policy runs for five years, and that the actuary has projected probabilities of an earthquake for each of the five years:

| Year | P (Earthquake) |
|---|---|
| 1 | 0.001 |
| 2 | 0.001 |
| 3 | 0.001 |
| 4 | 0.001 |
| 5 | 0.001 |

In making his assumption as to the probability of an earthquake, the actuary has assigned 32 possible outcomes to this policy. That is, each of the five years holds two possibilities: an earthquake either will or will not occur (2 exp 5 = 32). He has also assigned a probability to each of these outcomes, and is therefore in position to calculate a variance.

This actuary can provide a more complete answer to the question regarding the types of risk. He can say that while the risk in a particular year is that an earthquake will occur, the aggregated risk is that the policy has 32 possible outcomes, and he can be confident that his model adequately represents the possibilities. Next, he can assess the risk by calculating the variance of the insurer's profits under the policy.

Let us assume that the claim size, given an earthquake, will be 1 and that the discount rate is zero:

| n | P (n Claims) |
|---|---|
| 0 | 0.995010 |
| 1 | 0.004980 |
| 2 | 0.000010 |
| 3 | 0.000000 |
| 4 | 0.000000 |
| 5 | 0.000000 |

Expected value of claims = .004980 + 2 * .000010 = .005000

Expected value of squared claims = .004980 + 4 * .000010 = .005020

Variance of claims = .005020 - .005000 * .005000 = .004995

Standard Deviation of claims = 0.070675

Now the actuary has described the risk and, by calculating a variance, has assessed it. Yet this answer is still incomplete because the calculations are based on the actuarial assumption as to the probability of an earthquake. Another actuary might make a different assumption. This situation exemplifies the risk that the actuarial assumption is incorrect.

Now we have a complete answer to the question, "What types of risk are inherent in an insurance portfolio?" First, there is the risk of numerous possible outcomes for each policy. The actuary can quantify this risk by calculating an expectation and a variance. But in doing so he must set assumptions, which leads to the second risk: The assumptions used may not accurately predict the future.

The first risk we will call "portfolio risk" and the second "assumption risk". Both arise in connection with the valuation of an insurance portfolio: the first because the constituent policies have various possible outcomes, and the second because the actuary must set assumptions.

Portfolio risk can be reduced by diversification. As more policies are added to the portfolio (assuming independence) the expectation and variance increase proportionately, but the standard deviation increases in proportion to the square root of the number of policies. Assumption risk cannot be reduced in the same fashion.

In at least one sense, reducing portfolio risk can increase assumption risk. Using the earthquake policy example, if the insurer increases the coverage period to eight years, we have:

| n | P (n Claims) |
|---|---|
| 0 | 0.992028 |
| 1 | 9.007944 |
| 2 | 0.000028 |
| 3 | 0.000000 |
| 4 | 0.000000 |
| 5 | 0.000000 |
| 6 | 0.000000 |
| 7 | 0.000000 |
| 8 | 0.000000 |

Expected value of claims = .007944 + 2 * .000028 = .008000

Expected value of squared claims = .007944 + 4 * .000028 = .008056

Variance of claims = .008056 - .008000 * .008000 = .007992

Standard Deviation of claims = 0.089398

The ratio of standard deviation of claims to expected value of claims is lower with the policy extended to eight years, because this extension is a form of diversification (assuming the years are independent of each other). Portfolio risk has been reduced.

Assumption risk, on the other hand, has probably increased. The further into the future the actuary projects, the less sure he becomes of his assumption. Adding years to the end of a policy lowers his overall confidence in the assumption, increasing assumption risk.

We have classified the risk in valuing an increase portfolio into two types: portfolio risk and assumption risk. Portfolio risk can be described in terms of a calculated expected value, variance and standard deviation. It appears that assumption risk may increase as portfolio risk decreases.

We have defined portfolio risk as the risk of numerous possible outcomes for each policy in a portfolio, where the possible outcomes are encompassed in the assumptions the actuary has made. This risk can be analyzed with rather elementary statistical methods. The name "portfolio risk" is apropos because this risk can be reduced by diversification. An introduction to this risk is contained in the book, Actuarial Mathematics, by Bowers, Gerber, Hickman, Jones, and Nesbitt. In chapters 4 and 5 the authors calculate variances of the basic life contingent present value random variables.

To illustrate, consider a policy that pays one unit for each day of hospital stay, if the hospital stay commences within one year from the issue date. Let us assume that the probability of a hospital stay within one year is .05, and that the continuance table for a hospital stay is:

Number of Hospital Stays 100

| Days Since Hospital Stay Began | Number Remaining Hospital Stays |
|---|---|
| 1 | 100 |
| 2 | 99 |
| 3 | 96 |
| 4 | 91 |
| 5 | 84 |
| 6 | 75 |
| 7 | 64 |
| 8 | 51 |
| 9 | 36 |
| 10 | 19 |
| 11 | 0 |

If we ignore interest discounting and the possibility of more than one hospital stay within the year, we are in a position to analyze variance. For each insured, eleven possible outcomes exist: no hospital stay,hospital stay for a total of one day, hospital stay for a total of two days, and sò on. Given a hospital stay, the expected duration is 7.15 days, and the expected squared duration is 56.65. The variance of the claim cost, given a claim, is 5.53. The unconditional variance can be obtained using the general formula relating the variance to conditional expectations:

$$\begin{aligned} Var\,[W] &= Var\,[E[W|V]] + E[Var[W|V]] \\ &= 7.15 * 7.15 * .05 * .95 + 56.65 * .05 \\ &= 5.26 \end{aligned}$$

If the actuary is so bold as to assume that all hospital stays run for exactly 7.15 days, a variance can still be calculated:

$$\begin{aligned} Var\,[W] &= Var\,[E[W|V]] + E[Var[W|V]] \\ &= 7.15 * 7.15 * .05 * .95 + 0 * .05 \\ &= 2.43 \end{aligned}$$

The variance cannot be calculated only when all contingencies are removed from the assumption. Here that is the case if the assumption is simply that claims per policy will be .36.

This simple example demonstrates that when an actuarial assumption allows for more than one possibility, a variance corresponding to that assumption can be calculated. Examples are a mortality table, a lapse table, a continuance table of any type (length of disability, length of hospital stay, length of nursing home stay), a table of probabilities of premium suspension,a table of probabilities of claim, and a table of claim amounts given that a claim occurs. In other words, almost any assumption that an actuary makes.

This may seem rather obvious, but in practice actuaries rarely,if ever, calculate a variance. One reason may be that, with long duration coverages, the number of possible outcomes can become quite large. That would seem a meager excuse in these times of great computing capabilities. But consider a long duration disability income coverage. The actuary might make assumptions as to:

- Probabilities of claim in each year
- Continuance table
- Interest rates (for discounting)
- Probabilities of lapse in each year
- Expenses
- Premium rates
- Amount of coverage in units

Of these assumptions, three involve multiple outcomes (probabilities of claim, continuance, and probabilities of lapse). Assume that the coverage runs for 20 years and that the continuance table has monthly entries for 120 months. For a given policy, there are 20 possible years of termination. For a policy that terminates in year n, there are $2^n$ possibilities for claim incurrals (either a claim or no claim for each of the n years). So far we have $2 + 2^2 + 2^3 + \ldots + 2^{20} = 2^{21} - 2 = 2{,}097{,}150$ possible outcomes, and that ignores the 120 possible durations for each claim.

Fortunately the calculations can be simplified almost as effectively as the possible outcomes mount. The first step is to calculate the first and second moments of the present value of a claim, given that a claim has occurred. Suppose these are 7.15 and 56.65 respectively (as in the previous example). Then, each year that the policy is in force, the variance of the insurer's experience is:

$$7.15 * 7.15 * q * (1-q) + 56.65 * q$$

where q is the probability of a claim in that year. (This variance does not involve the premium rate or expense levels because these assumptions are not stochastic.) Now with the assumption that the claim experience in each year is independent of other years, and the formula relating variance to conditional expectations, the problem can be solved. The preliminary calculations are:

| | Probability of | | Variances of Gain | | Expected Gain | |
|---|---|---|---|---|---|---|
| Year | Claim | Termin-ation | for the yr | sum to date | for the yr | sum to date |
| 1 | 0.010 | 0.060 | 1.07 | 1.07 | 0.43 | 0.43 |
| 2 | 0.015 | 0.058 | 1.61 | 2.68 | 0.39 | 0.82 |
| 3 | 0.020 | 0.056 | 2.14 | 4.81 | 0.36 | 1.18 |
| 4 | 0.025 | 0.054 | 2.66 | 7.48 | 0.32 | 1.50 |
| 5 | 0.030 | 0.052 | 3.19 | 10.66 | 0.29 | 1.79 |
| 6 | 0.035 | 0.050 | 3.71 | 14.37 | 0.25 | 2.03 |
| 7 | 0.040 | 0.048 | 4.23 | 18.60 | 0.21 | 2.25 |
| 8 | 0.045 | 0.046 | 4.75 | 23.35 | 0.18 | 2.43 |
| 9 | 0.050 | 0.044 | 5.26 | 28.61 | 0.14 | 2.57 |
| 10 | 0.055 | 0.042 | 5.77 | 34.38 | 0.11 | 2.68 |
| 11 | 0.060 | 0.040 | 6.28 | 40.66 | 0.07 | 2.75 |
| 12 | 0.065 | 0.038 | 6.79 | 47.45 | 0.04 | 2.78 |
| 13 | 0.070 | 0.036 | 7.29 | 54.75 | -0.00 | 2.78 |
| 14 | 0.075 | 0.034 | 7.80 | 62.54 | -0.04 | 2.75 |
| 15 | 0.080 | 0.032 | 8.29 | 70.84 | -0.07 | 2.67 |
| 16 | 0.085 | 0.030 | 8.79 | 79.63 | -0.11 | 2.57 |
| 17 | 0.090 | 0.028 | 9.29 | 88.91 | -0.14 | 2.42 |
| 18 | 0.095 | 0.026 | 9.78 | 98.69 | -0.18 | 2.24 |
| 19 | 0.100 | 0.024 | 10.27 | 108.96 | -0.22 | 2.03 |
| 20 | 0.105 | 0.202 | 10.75 | 119.71 | -0.25 | 1.78 |

The probabilities of claim and termination in a given year are assumptions made by the appraisal actuary. The variance of the experience for the year is the result of applying the formula relating variance to conditional expectations. Assuming the years are independent, the sum of variances to date is the variance of the experience to date given that the policy terminates this year. The insurer's expected gain for the year equals premium less expenses (here assumed to be a level .50), less q times the expected claim size of 7.15. Interest discounting has been ignored for simplicity; to include it the actuary would discount yearly variances at twice the force of interest, and yearly expectations at the force of interest in developing the summations to date.

Now the formula relating variance to conditional expectations is applied once again, this time with conditioning on thė year of termination:

| Year | (1) P[termination] | (2) Variance to date | (3) Gain to date | (4) Squared Gain to date | (1)*(2) | (1)*(3) | (1)*(4) |
|---|---|---|---|---|---|---|---|
| 1 | 0.060 | 1.07 | 0.43 | 0.18 | 0.064 | 0.026 | 0.011 |
| 2 | 0.058 | 2.68 | 0.82 | 0.67 | 0.155 | 0.048 | 0.039 |
| 3 | 0.056 | 4.81 | 1.18 | 1.39 | 0.270 | 0.066 | 0.078 |
| 4 | 0.054 | 7.48 | 1.50 | 2.25 | 0.404 | 0.081 | 0.121 |
| 5 | 0.052 | 10.66 | 1.79 | 3.19 | 0.554 | 0.093 | 0.166 |
| 6 | 0.050 | 14.37 | 2.03 | 4.14 | 0.719 | 0.102 | 0.207 |
| 7 | 0.048 | 18.60 | 2.25 | 5.06 | 0.893 | 0.108 | 0.243 |
| 8 | 0.046 | 23.35 | 2.43 | 5.89 | 1.074 | 0.112 | 0.271 |
| 9 | 0.044 | 28.61 | 2.57 | 6.60 | 1.259 | 0.113 | 0.291 |
| 10 | 0.042 | 34.38 | 2.68 | 7.16 | 1.444 | 0.112 | 0.301 |
| 11 | 0.040 | 40.66 | 2.75 | 7.55 | 1.627 | 0.110 | 0.302 |
| 12 | 0.038 | 47.45 | 2.78 | 7.74 | 1.803 | 0.106 | 0.294 |
| 13 | 0.036 | 54.75 | 2.78 | 7.74 | 1.971 | 0.100 | 0.279 |
| 14 | 0.034 | 62.54 | 2.75 | 7.54 | 2.126 | 0.093 | 0.256 |
| 15 | 0.032 | 70.84 | 2.67 | 7.15 | 2.267 | 0.086 | 0.229 |
| 16 | 0.030 | 79.63 | 2.57 | 6.58 | 2.389 | 0.077 | 0.198 |
| 17 | 0.028 | 88.91 | 2.42 | 5.87 | 2.490 | 0.068 | 0.164 |
| 18 | 0.026 | 98.69 | 2.24 | 5.03 | 2.566 | 0.058 | 0.131 |
| 19 | 0.024 | 108.96 | 2.03 | 4.11 | 2.615 | 0.049 | 0.099 |
| 20 | 0.202 | 119.71 | 1.78 | 3.16 | 24.181 | 0.359 | 0.638 |
| Total | | | | | 50.869 | 1.965 | 4.316 |

$$\begin{aligned}\text{Var}[W] &= \text{Var}[E[W|V]] + E[\text{Var}[W|V]] \\ &= 4.316 - 1.965 * 1.965 + 50.869 \\ &= 51.324\end{aligned}$$

The repeated use of the conditional formula for variance is noteworthy.

While such calculations are straightforward and practical enough, the actuary would not do them for every policy in a large portfolio. Modelling techniques that are already well established in practice could be employed with very little loss of accuracy. Also, different techniques and approximations would be developed for different lines of business. The example studied here is intended to be complex enough to convince the reader that these calculations can and should be done, but is not intended to illuminate detailed procedures for various lines of business.

The point is that the tools for analyzing portfolio risk already exist. They include elementary statistics, assumption setting,modeling, and computational ability. Actuaries could begin making these types of calculations in connection with the valuation of insurance portfolios almost immediately.

What is the nature of portfolio risk? Our numerical example can be used to start a discussion of this question. In the example the variance is 51.324 compared to an expected gain for the insurer of 1.965. These figures are for one policy. Imagine two separate portfolios made up of policies similar to this, one (portfolio A) containing ten policies and the other (B) containing 10,000. We would have:

| | Expected | | Standard |
|---|---|---|---|
| Portfolio | Gain | Variance | Deviation |
| A | 19.65 | 513.24 | 22.65 |
| B | 19650 | 513240 | 716.41 |

The insurer interested in taking on portfolio A might pay only a small percentage of the expected gain to the transferor, because of the relatively high standard deviation. With only ten policies, the normal approximation is not appropriate, but the insurer might guess that there is a 20% chance that the present value of cash flows will be negative. The insurer interested in portfolio B can use the Normal Distribution to closely approximate the distribution of the present value of cash flows, and can pay very close to the expected present value, even if highly risk averse. This illustrates the diversify of portfolio risk.

With large enough portfolios of independent policies, portfolio risk becomes negligible. This is true even if the policies are not at all similar; the only requirement is that they are independent. Does this mean that very large insurers can take on additional insurance portfolios without reservation, paying very close to the expected present value of cash flows as determined by the actuary, regardless of the line of business involved? What if the actuary's expected present value is clearly wrong?

These questions imply that portfolio risk is not a complete measure of inherent risk. Portfolio risk is determined completely by the actuary's set of assumptions. If those assumptions are inadequate, the resulting expected present value will be unreliable, as will the standard deviation used to measure the reliability! This is the insidious nature of portfolio risk.

Can any progress be made toward measuring the rest of the risk? The rest of the risk is assumption risk. To it we turn next.

We have defined assumption risk as the risk that the actuarial assumptions are incorrect. This risk can be difficult even to conceptualize; a discussion of it can have philosophical implications. We proceed to discuss it.

Actuarial assumptions can be incorrect, as actuaries freely admit. Recall the policy insuring against earthquakes, where the actuary had estimated the probabilities of an earthquake to be:

| Year | P (Earthquake) |
|---|---|
| 1 | 0.0010 |
| 2 | 0.0010 |
| 3 | 0.0010 |
| 4 | 0.0010 |
| 5 | 0.0010 |

Suppose another actuary, working with the same coverage, estimated the probabilities to be:

| Year | P (Earthquake) |
|---|---|
| 1 | 0.0005 |
| 2 | 0.0005 |
| 3 | 0.0005 |
| 4 | 0.0007 |
| 5 | 0.0010 |

Can we say that both actuaries cannot be right, and so at least one must be wrong? The implication of such a statement is that a correct assumption does exist. But what is the correct assumption?

One answer is that the correct assumption is the one that is borne out by experience. Suppose then that no earthquake occurs within the next five years (this would most likely be the case). Both actuaries were wrong; the correct assumption was:

| Year | P (Earthquake) |
|---|---|
| 1 | 0.0000 |
| 2 | 0.0000 |
| 3 | 0.0000 |
| 4 | 0.0000 |
| 5 | 0.0000 |

In other words, the earthquake policies should have been given away. All insurance company presidents and most actuaries would interject here, saying that the absence of an actual earthquake does not imply the absence of a probability of an earthquake. We return to the question: What is the correct assumption?

To get a better answer, the real world must be viewed as a stochastic process. The five year period of coverage, in which no earthquake actually occurred, was merely five trials of the process. This process is controlled by a particular set of probabilities which actuaries can estimate but can never know with certainty. The correct assumption corresponds to the probabilities that describe the real world stochastic process.

The idea that the real world is the result of stochastic processes has great philosophical implications. Quantum physicists endorse the idea; Albert Einstein refused to believe it. Philosophers have argued at length about it. Actuaries who wish to understand the concept of assumption risk are forced to adopt it as a model, if not as a philosophy.

To complete this discussion, consider the expense assumption. Our model of the real world as a stochastic process does apply to expenses, although the determination of actual expense levels involves a combination of many related processes. Some of the processes involved in life insurance would be requests for policy loans, computer system crashes, and the setting of public utility rates.

Assumption risk is the risk that the actuarial assumptions are incorrect; that is, they do not correspond exactly to the stochastic processes that define the real world. This risk applies to every assumption that the actuary makes. In this way assumption risk contrasts with portfolio risk. The latter relates only to actuarial assumptions that allow for more than one possibility. For example, expense assumptions are usually set in a deterministic fashion, such as $40 per policy, and would involve assumption risk but not portfolio risk.

Having satisfactorily characterized assumption risk, we next attempt its quantification. The first stumbling block is the cynical argument:

- The actuary is asked to measure the risk that his assumptions are incorrect
- To measure this risk, the actuary must estimate how far wrong he may be
- This estimate is an assumption in its own right, and is also subject to assumption risk
- Therefore, attempting to quantify assumption risk introduces further assumption risk

This argument is sound, and its point is that assumption risk cannot be eliminated. Still, the actuary does have varying levels of confidence in his assumptions; some he will feel quite sure of, and others he will say are very tentative. Thinking about the assumptions in this way, the actuary begins to gauge his susceptibility to being wrong, and takes the first step toward quantifying assumption risk. Were it not possible for him to do this, contingency reserves and margins for adverse deviation (established concepts for the valuation actuary) would be meaningless.

One approach that has been used sparingly in practice is sensitivity analysis. This involves adjusting a particular assumption, such as lapse, on a purely judgmental basis, and redoing all calculations. The inadequacy of this approach is that the actuary does not (indeed cannot) assign a probability weight to either assumption set. The user gains very little information as to the level of risk involved.

To quantify the assumption risk in determining the value of an insurance portfolio, the actuary must replace the static projection developed by current practice with a number of projections, each with an equal probability of occurring. In a numerical example considered previously, the actuary had projected cash flows from a portfolio to be:

| Year | Projected Net Cash Flows |
|---|---|
| 1 | -3141 |
| 2 | -3420 |
| 3 | -3706 |
| 4 | -4007 |
| 5 | -4326 |
| 6 | -4674 |
| 7 | -5058 |
| 8 | -5485 |
| 9 | -5962 |
| 10 | -283351 |

This is a static projection; it is based on one particular assumption set. If the actuary were to include random fluctuations in the generation of the assumption set, he would be able to make projection after projection, and would soon be able to provide information as to the susceptibility of the cash flows to changes in the assumptions. Such methods have been called Monte Carlo methods.

To continue with the numerical example, it is necessary to review exactly how it was developed. The portfolio consists of 600 paid-up life insurance policies that will endow at the end of ten years:

| Year | Number of Policies Remaining | Mortality Rate per Thousand | Claims |
|---|---|---|---|
| 1 | 600.00 | 5.235 | 3141 |
| 2 | 596.86 | 5.730 | 3420 |
| 3 | 593.44 | 6.245 | 3706 |
| 4 | 589.73 | 6.795 | 4007 |
| 5 | 585.73 | 7.385 | 4326 |
| 6 | 581.40 | 8.040 | 4674 |
| 7 | 576.73 | 8.770 | 5058 |
| 8 | 571.67 | 9.595 | 5485 |
| 9 | 566.18 | 10.530 | 5962 |
| 10 | 560.22 | 11.570 | 283351 |

Each death claim is for 1,000; each endowment is for 500. The year 10 claims equal 560.22 * .011570 * 1000 + 560.22 * (1 - .011570) * 500. Reserve effects are ignored since they provide no information useful to the analysis of risk. Portfolio cash flows, if certain, should be discounted at the risk-free rate. If the cash flows are uncertain, but result from a stochastic process that allows for risk, then they also should be discounted at the risk-free rate. If the cash flows are uncertain, and result from a static projection, it would be foolish to try to allow for the risk simply by adjusting the discount rate.

Here we will use Monte Carlo methods to allow for assumption risk, and are fully justified in discounting at the risk-free rate, which is assumed to be 10% in all years. This gives a present value of claims of 133,455.

In our simplified numerical example, the actuary is making two assumptions: mortality and risk-free rate of interest. The task is to introduce a random element into the generation of these two assumptions, with the amount of randomness being reflective of the actuary's confidence in each. The element of randomness should be kept as simple as possible for two reasons: computational ease, and to avoid asking the actuary to do more than he is capable of. The simplest approach, and it does have some theoretical basis, is to use a random walk. The actuary need set only the maximum step size, a larger step size being used when he is less confident in the assumption. To take interest first, suppose the actuary sets a step size of .005:

| | | Range Under Random Walk | |
|---|---|---|---|
| Year | Static Assumption | Low | High |
| 1 | 0.100 | 0.095 | 0.105 |
| 2 | 0.100 | 0.090 | 0.110 |
| 3 | 0.100 | 0.085 | 0.115 |
| 4 | 0.100 | 0.080 | 0.120 |
| 5 | 0.100 | 0.075 | 0.125 |
| 6 | 0.100 | 0.070 | 0.130 |
| 7 | 0.100 | 0.065 | 0.135 |
| 8 | 0.100 | 0.060 | 0.140 |
| 9 | 0.100 | 0.055 | 0.145 |
| 10 | 0.100 | 0.050 | 0.150 |

The stochastic generation of the assumption would be:

$$d(0) = 0$$
$$d(n) = d(n-1) + rand(.01) - .005$$
$$sto(n) = sta(n) + d(n)$$

Here "rand(.01)" means a randomly generated number between 0 and .01, "sto(n)" means the stochastic assumption in year n, and "sta(n)" means the static assumption in year n. Turning to mortality, suppose the actuary sets a step size of one death per thousand:

| Year | Static Assumption | Range Under Random Walk Low | Range Under Random Walk High |
|---|---|---|---|
| 1 | 5.235 | 4.235 | 6.235 |
| 2 | 5.730 | 3.730 | 7.730 |
| 3 | 6.245 | 3.245 | 9.245 |
| 4 | 6.795 | 2.795 | 10.795 |
| 5 | 7.385 | 2.385 | 12.385 |
| 6 | 8.040 | 2.040 | 14.040 |
| 7 | 8.770 | 1.770 | 15.770 |
| 8 | 9.595 | 1.595 | 17.595 |
| 9 | 10.530 | 1.530 | 19.530 |
| 10 | 11.570 | 1.570 | 21.570 |

The stochastic generation of the assumption would be:

$$d(0) = 0$$
$$d(n) = d(n\text{-}1) + \text{rand}(2) - 1$$
$$\text{sto}(n) = \text{sta}(n) + d(n)$$

The generalized formulas, using "ss" for step size, are:

$$d(0) = 0$$
$$d(n) = d(n\text{-}1) + \text{rand}(2{*}\text{ss}) - \text{ss}$$
$$\text{sto}(n) = \text{sta}(n) + d(n)$$

A theoretical justification of these formulas follows. The actuary can review the current state of the world and recent experience and set a static assumption for the next ten years. He is less confident in his assumption as he moves further out into the future, because he is getting further away from his point of full information (year 0), and the real world stochastic processes can change. But if the year 1 assumption turns out to be exactly right, it means the real world stochastic processes match those that existed in year 0, and the year 2 assumption is more likely to be accurate. To the extent the year 1 assumption is inaccurate, the real world stochastic processes have changed, and that change will carry into year 2. This procedure was applied to the numerical example using step sizes of .005 for interest and 1 death per thousand for mortality. The results were:

| Trial # | Present Value of Net Cash Flow |
|---|---|
| 1 | -132946 |
| 2 | -134566 |
| 3 | -133903 |
| 4 | -134252 |
| 5 | -134190 |
| 6 | -130372 |
| 7 | -136517 |
| 8 | -134533 |
| Mean | -133910 |
| Estimated Standard Deviation of the Present Value | 1743 |

This procedure provides sample values of the random variable. These values are used to estimate the population variance. This variance (or standard deviation) measures assumption risk.

The random walk procedure for stochastic generation of an assumption is not the only possibility. For example, a "random hit" procedure might be used to allow for the chance of an external shock, such as an influenza epidemic. Such an event would temporarily alter the real world stochastic process. The actuary could estimate the probability of a shock. Then if a shock occurs, the assumption he is generating could become a random variable uniformly distributed across a specified range. Formulas to accomplish this could be:

```
d(0) = 0
d1(n) = d1(n-1) + rand(2*ss) - ss
d2(n) = rand(range) + LB
sto(n) = sta(n) + d1(n) if rand(1) > P[shock]
sto(n) d2(n) if rand(1) < P[shock]
```

Here P[shock] is the actuary's estimate of the probability of a shock in a given year, "LB" is the lower bound on the assumption given that a shock occurs, and "range" is the upper bound less the lower bound. After the shock, the random walk generation is resumed.

We have introduced a technique for estimating assumption risk that results in an estimated standard deviation of the present value of portfolio cash flows. The technique requires the actuary to set one parameter for each assumption, the size of the parameter reflecting the degree of confidence in the assumption. It is true that the actuary is estimating his assumption risk, which leads to another type of assumption risk, but it is quite plausible that the actuary can develop a useful estimate.

## SYNTHESIS: PORTFOLIO/ASSUMPTION RISK

Portfolio risk and assumption risk have been separately analyzed and quantified in connection with valuing an insurance portfolio. To make his analysis more useful to management, the actuary will need to combine the two measures of variance.

Since portfolio risk is diversifiable, with very large portfolios it will become insignificant. In such a case the synthesis is accomplished by looking only at the variance on account of assumption risk.

For smaller portfolios, where both risks are significant, the actuary could proceed as follows:

1) Measure the variance due to portfolio risk based on a static best-estimate assumption set

2) Measure the variance due to assumption risk by running a number of Monte Carlo trials with a stochastic assumption set

3) Assume that the variance due to portfolio risk is constant across the various stochastic assumption sets that are developed for the Monte Carlo trials

4) Apply the general formula relating variance to conditional expectations.

To illustrate, imagine that the actuary has done the computational work and developed the following results:

| Assumption Set | Present Value of Net Cash Flows | Variance |
|---|---|---|
| Static | -133455 | 1569204 (portfolio risk) |
| Stochastic-1 | -135409 | |
| Stochastic-2 | -140926 | |
| Stochastic-3 | -132757 | |
| Stochastic-4 | -133954 | |
| Stochastic-5 | -134439 | |
| Stochastic-6 | -131075 | |
| Stochastic-7 | -134141 | |
| Stochastic-8 | -131894 | 9122492 (assumption risk) |

The eight stochastic projections have equal probability weight since they are Monte Carlo trials. In the general formula,

$$\mathrm{Var}\,[W] = \mathrm{Var}\,[E[W|V]] + E\,[\mathrm{Var}[W|V]]$$

The random variable W is the present value of net cash flows, and V is the stochastic assumption set number. Under the assumption that the variance due to portfolio risk is constant, the application of the general formula is:

$$\mathrm{Var}\,[W] = 9122492 + 1569204 = 10691696$$

That is, the two variances simply add.

If the assumption of constant variance due to portfolio risk is relaxed, the actuary must calculate this variance for each stochastic assumption set. The general formula would still be used to combine the measures of variance.

The actuary has arrived at a single measure of expectation and variance for the present value of net cash flow random variable. By invoking the central limit theorem he can state that this random variable is approximately normally distributed, if the portfolio consists of a sufficient number of identical policies (many elementary statistics textbooks recommend that this number be at least 30 for the approximations to be reasonable). The theoretical justification for this is:

- By definition, the Monte Carlo technique used to analyze assumption risk by definition generates repeated trials from the same statistical distribution.

- Consider one particular trial. A single resent value will be generated. Because of portfolio risk, this single present value is actually the expectation of random present values. But if there are a sufficient number of identical independent policies in the portfolio, this expectation is normally distributed, under the central limit theorem.

- Consider sets of n trials, generating n present values of cash flows. The mean of these n present values will be normally distributed, because each present value is normally distributed.

- The expected value of the mean of these n present values equals the present value using the actuary's static best-estimate assumption set.

So the actuary can make an initial static projection, obtaining his expectation of the present value of cash flows. He can then proceed to quantify portfolio and assumption risk, and to combine the quantifications into a single measure of variance. Finally, if the portfolio is large enough, he can state that the present value random variable is normally distributed. He has completely specified its distribution.

A problem is that insurance portfolios do not consist of completely identical policies. This means that the usual requirement of 30 or more must be increased before the actuary can fairly state that the present value random variable is normally distributed. With small portfolios, this cannot be stated, but the measurement of the variance would retain some usefulness. Chebysheff's inequality, for example, would retain applicability.

# ANNOTATED BIBLIOGRAPHY

Symposium on Monte Carlo Methods
Herbert A. Meyer, ed.
New York: John Wiley & Sons, Inc, 1956

Discusses the theory of Monte Carlo techniques and illustrates various practical applications in a variety of fields

The Commodity Futures Game
Richard J. Teweles
New York: McGraw-Hill, Inc., 1974

Contains a description of utility theory and discusses in plain terms how a utility curve might be constructed

Actuarial Mathematics
Newton L. Bowers, Jr.; Hans U. Gerber; James C. Hickman;
Donald A. Jones; and Cecil J. Nesbitt
Chicago: The Society of Actuaries, 1986

Introduces the concept of a variance in connection with life insurance and annuity present values (the variance corresponding to portfolio risk)

Introduction to Mathematical Statistics
Robert V. Hogg and Allen T. Craig
New York: Macmillan Publishing Co., Inc., 1970

Deals with elementary statistical methods; includes discussions of the central limit theorem and Chebyshev's inequality (one of many excellent textbooks)

Actuarial Standard of Practice Exposure Draft: Actuarial Appraisals of Insurance Companies, Segments of Insurance Companies, and/or Blocks of Insurance Contracts
Actuarial Standards Board (U.S.)
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Would prescribe the techniques to be used in valuing insurance portfolios; summarizes current practice