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Actualization Process and Financial Risk

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Summary

The purpose of this paper is to present a general stochastic model of capitalization, based on the semi-martingale theory.

We study in particular the influence of the financial risk on the actualization process which is currently used in life insurance.

Résumé

Procede d'Actualisation et Risque Financier

L'objectif de cet article est de présenter un modèle stochastique général de capitalisation, basé sur la théorie semi-martingale.

Nous étudions en particulier l'influence du risque financier sur le processus d'actualisation actuellement utilisé dans l'assurance sur la vie.

1. INTRODUCTION.

Actuaries are more and more involved with financial problems, specially with financial risk.
The recent creation of the financial section AFiR of the International Actuarial Association illustrates this increasing interest : this is the time of the actuaries of the third kind !
Indeed, as the actuary is the specialist of the risk, the concept of financial risk is naturally devoted to him.
Nevertheless, if the technical risk in non-life insurance has been widely studied, and stochastic methods have been commonly used by financials (Option theory, portfolio theory, ...), the concept of financial risk in classical life insurance, generated by the actualization by means of a pre-determined but guaranteed technical interest rate, has been almost ignored.
We have tried to build financial stochastic models which allow to define and compute in a stochastic environment the notions of capitalization and actualization and to detect the influence of financial risk on these elements.
In a first time, we had used a brownian model (see Devolder [6]). In this particular model, we had found a result of penalization of the actualization with respect to the capitalization : the mean values of the actualization process must be computed with lower interest rates than the mean values of the capitalization process (concept of rate with and without risk).

The purpose of the paper is to extend this result in a general stochastic model of interest rates.
More precisely, we shall look at the largest class of stochastic processes for which the capitalization equation has always some meaning : these are the semi-martingales, which can be defined as the sum of a process of finite variation (modelization of the trend) and of a local martingale (modelization of the noise).
This powerful tool of "semi-martingale" has been also recently used in other fields, particularly in portfolio theory and risk theory (see Aase [1] and [2]).

In this general model, we show that the property of mean value in the brownian model is but a particular version of a general result of multiplicative decomposition of the capitalization and actualization processes, which allows to define the trends with and without risk.
In the particular case of processes with independent increments which contains the brownian model, this decomposition can be used to obtain again a result of mean values.
All these theoretical results give explicit formulas of what we should call "the price of the financial risk", in a general stochastic model.
Examples of particular stochastic models can be derived from the general approach (discrete or continuous models, gaussian models, Poisson processes, see Devolder [7]).

2. NOTATIONS AND DEFINITIONS.

In this chapter we present the basic concepts of the general theory of stochastic processes.
(e.g. DELLACHERIE-MEYER [5], JACOD [9], MEYER [12]).

2.1. Definitions :

Let $(\Omega, a, \mathbb{P})$ denote a probability space, and $\{F_t, t > 0\}$ a filtration on this space.
The filtered space $(\Omega, a, \mathbb{P}, \{F_t\})$ will fulfil the usual conditions if :

(i) the space $(\Omega, a, \mathbb{P})$ is complete and all $\mathbb{P}$-negligible subsets of F belong to F_o

(ii) the filtration is right-continuous :
$$F_t = F_{t^+} = \bigcap_{s>t} F_s \qquad (t \geqslant o)$$

A stochastic process X is an application :

$$X : [0, \infty[\times \Omega \to \mathbb{R} : (t, \omega) \to X_t(\omega) = X(t, \omega)$$

Let $X(t^-, \omega)$ or $X_{t^-}(\omega)$ denote the left hand limit at $t > 0$, and
$\Delta X(t, \omega) = \Delta X_t(\omega) = X(t, \omega) - X(t^-, \omega)$, the jump of the process at $t > 0$.
All the stochastic processes which will be used, are supposed to be measurable, adapted to the filtration $\{F_t\}$ and right continuous on $\mathbb{R}^+$ with the left hand limits on $\mathbb{R}_o^+$.
Two fundamental σ-fields can be introduced on the space $\mathbb{R}^+ \times \Omega$:

- the optional σ-field, denoted θ, generated by the adapted processes, right continuous with left hand limit
- the predictable σ-field, denoted $\mathcal{P}$, generated by the adapted process, left continuous on $\mathbb{R}_o^+$.

A stochastic process is then optional (resp. predictable) if the application $(t, \omega) \to X(t, \omega)$ is measurable on the space $R^+ \times \Omega$ with the σ-field θ (resp. $\mathcal{P}$).

2.2. Stopping times :

A stopping time T is a random variable $T : \Omega \to \mathbb{R}^+ \cup \{\infty\}$ so that $\{T \leq t\} \in F_t$ for every $t \in \mathbb{R}^+$.
A stopping time T is said to be predictable if there is an increasing sequence of stopping times $\{T_n\}$ so that : $\lim_n T_n = T$ and $T_n < T$ on the set $\{T > 0\}$.
The sequence $\{T_n\}$ is said to announce T.
A stopping time T is said to be totally inaccessible if $\mathbb{P}[S = T < \infty] = 0$ for predictable stopping time S.
A stopping time T is said to be accessible if there is a countable set $\{T_n\}$ of predictable stopping times so that

$$[\![T]\!] \subset \bigcup_n [\![T_n]\!]$$

(where $[\![T]\!] = \{(t, \omega) \in [0, \infty[\times \Omega : T(\omega) = t\}$ is the graph of the stopping time T)

2.3. Classes of processes :

We shall work with two important classes of stochastic processes which together generate the space of semi-martingales: the processes of finite variation and the local martingales.

- An adapted process X is said to be of finite variation if every sample path is a function of finite variation on every compact set and right continuous.
  A process of finite variation can be described as the difference of two increasing processes.

- An adapted process M is said to be a martingale with respect to the filtration $\{F_t,\ t > 0\}$ if :

  (i) $M(t, \omega)$ is integrable and $\mathbb{E}\, M(t, \omega) < \infty$

  (ii) $\mathbb{E}\,(M(s, \omega)\ |F_t) = M(t, \omega) \qquad \forall\ o \leqslant t \leqslant s$

- An adapted process M is said to be a local martingale if there is an increasing sequence of stopping times $\{T_n\}$ such that $\lim_n T_n = \infty$ a.s., and the stopped processes $M(t \wedge T_n, \omega) \cdot 1\!\!1_{\{T_n < \infty\}}$ are uniformly integrable martingales.

- An adapted process Z is said to be a semi-martingale if there is a proc of finite variation X and a local martingale M such that $Z = X + M$ on $[0, \infty] \times \Omega$.

- A semi-martingale Z is said to be special if there exists a decompositi $Z = X + M$ where X is a process of finite variation which is predictable and of local integrable variation and M is a local martingale. This decomposition is unique up to indistinguishable processes.

2.4. Stochastic integration :

We shall use the general theory of stochastic integration with respect to semi-martingale (as DELLACHERIE-MEYER [5]).

The two following concepts will also be useful (DELLACHERIE-MEYER [5]) :

- dual predictable projection of a process :
  if A is an increasing locally integrable process, there exists a single increasing locally integrable and predictable process denoted A^P and called dual predictable projection of A, so that :

$$\mathbb{E} \int_0^\infty X(s, \omega)\, d A^P(s, \omega) = \mathbb{E} \int_0^\infty X(s, \omega)\, d A(s, \omega)$$

for predictable processes X.

This definition can be extended to the locally integrable processes of finite variation (difference of two increasing processes).

- predictable quadratic variation :
  if M is a locally square integrable local martingale, there exists a single predictable and increasing process, denoted $\langle M,M\rangle$ and called predictable quadratic variation of M, so that :

  $$M^2 - \langle M,M\rangle \text{ is a local martingale}$$

  In particular, a continuous local martingale is locally square integrable and has a predictable quadratic variation.

3. SEMI-MARTINGALE OF CAPITALIZATION AND ACTUALIZATION.

3.1. Introduction :

The stochastic model of capitalization presented in DEVOLDER [6], was based on the stochastic differential equation :

$$C\,(t,\,\omega) = C\,(o) + \int_0^t C\,(s,\,\omega)\; d\,I\,(s,\,\omega) \tag{3.1}$$

where :

a) $I\,(t,\,\omega) = \int_0^t \delta\,(s)\, ds + \int_0^t \sigma\,(s)\; d\,w\,(s,\,\omega)$ (3.2)

,called flux of interest, is a stochastic process generated by a brownian motion w (Wiener model)

b) C is the process of capitalization induced by the flux I.

It was interesting to generalize this model and to extend its results in a general stochastic context.
More precisely, the following problems appear :

1) the trend of interest rates, given in the Wiener model by a deterministic function,

   $I_d\,(t) = \int_0^t \delta\,(s)\, ds$, could also be stochastic and become a process of finite variation

2) the noise, which is given in the Wiener model by the stochastic integral

   $I_a\,(t,\,\omega) = \int_0^t \sigma\,(s)\; dw\,(s,\,\omega)$ could become a more general local martingale

3) the Wiener model is continuous; we could also introduce jumps in the trend or in the noise. These jumps will be interpreted as stochastic gains in value.

The general theory of stochastic processes shows precisely that the semi-martingales are the class of processes for which a "reasonable" notion of integration can be developed. Furthermore, the additive decomposition of the semi-martingale in a part of finite variation and a local martingale generalizes naturally the notions of trend and noise introduced in the Wiener model.

3.2. Flux of interest and semi-martingale :

The basic notion is given by the following definition :

DEFINITION 3.1.

A flux of interest I on the filtered space $(\Omega, a, \mathbb{P}, \{F_t\})$ is a special semi-martingale on this space with the following properties :

(i) the jumps of I belong a.s to a compact subset of $(-1, 1)$

(ii) the jumps of I are in finite number on every compact of $\mathbb{R}^+$

(iii) the continuous part of finite variation of I is increasing

(iv) $I(o) = 0$.

So the special semi-martingale I has a unique decomposition in :

$$I(t, \omega) = V(t, \omega) + M(t, \omega) \tag{3.3}$$

where $V(t, \omega)$ is a predictable process of locally integrable variation

$M(t, \omega)$ is a local martingale such that $M(o) = 0$.

Each of the processes V and M can also be decomposed.

For the process V, we can apply for each sample path the Lebesgue decomposition

$$V(t, \omega) = V^c(t, \omega) + V^d(t, \omega)$$

where V^c is the continuous part, supposed here absolutely continuous
V^d is the discrete part with jumps.

We shall denote :

a) $V^c(t, \omega) = \int_0^t \delta(s, \omega)\, ds$

The derivative process δ, almost surely positive and defined almost everywhere on each sample path, extended by right continuity, is called <u>instantaneous stochastic interest rate</u>

b) $V^d(t, \omega) = \sum_{k \mid T_k \leqslant t} s_k(\omega)$

where (i) $0 = T_0 < T_1 \ldots < T_n < \ldots$ is a sequence of predictable stopping times with disjoined graphs.

(ii) the jumps of the process, denoted $s_k(\omega)$, are called <u>stochastic impulses</u>.

The sum $V(t, \omega) = \int_0^t \delta(s, \omega)\, ds + \sum_{k \mid T_k \leqslant t} s_k(\omega)$ represents the trend of the interest rates.

For the process M, we can apply the additive decomposition of the martingale:

$$M(t, \omega) = M^c(t, \omega) + M^d(t, \omega)$$

where a) $M^c(t, \omega)$ is the continuous local martingale part of the flux and is called <u>continuous noise</u>

b) $M^d(t, \omega)$ is a compensated sum of jumps and is called <u>discontinuous noise</u>

This jumps are bounded and in finite number on every compact, so we can write :

$$M^d(t, \omega) = A(t, \omega) - A^P(t, \omega)$$

where (1) $A(t, \omega) = \sum_{j \mid S_j \leqslant t} m_j(\omega)$

- $0 = S_0 < S_1 \dots < S_n < \dots$ is a sequence of totally inaccessible stopping times.

- the jumps m_j are called <u>impulsed noises</u>

(2) A^P is the dual predictable projection of A and is a continuous process of finite variation, called <u>compensator of the noise</u>.

A general flux of interest admits so the following representation

$$I(t, \omega) = \int_0^t \delta(s, \omega)\, ds + \sum_{j \mid T_j \leqslant t} s_j(\omega) + M^c(t, \omega) + \left(\sum_{j \mid S_j \leqslant t} m_j(\omega) - A^P(t, \omega) \right) \quad (3.4)$$

The four parts can be financially interpreted :

a) $\int \delta(s, \omega)\, ds$ represents the stochastic interest rate

b) M^c represents the noise on this rates

c) $\sum_j s_j$ represents a sequence of predictable gains in value

d) $\sum_j m_j$ represents a sequence of totally inaccessible gains in value, compensated by the continuous process A^P

This general model can be used therefore for continuous yield or for "surprising" gains in value.

3.3. Capitalization process :

The capitalization process can be defined as the future value of a unitary deposit at time $t = 0$.
This process, denoted C, is associated to a given flux of interest I and will be solution of the stochastic differential equation :

$$C(t, \omega) = 1 + \int_0^t C(s^-, \omega)\, dI(s, \omega) \tag{3.5}$$

where I has the form (3.4).
The first term is the initial deposit, the second term which is expressed as a stochastic integral, modelizes the accumulation of interest on this deposit.
The solution of this equation is given by :

THEOREM 3.1. :

There exists a unique semi-martingale, solution of the capitalization (3.5) associated to the flux (3.4), given by :

$$C(t, \omega) = \exp\left(\int_0^t \delta(s, \omega)\, ds + M^c(t, \omega) - \frac{1}{2} \langle M^c, M^c \rangle_t - A^P(t, \omega)\right).$$

$$\cdot \prod_{j \mid T_j \leqslant t} (1 + s_j(\omega)) \cdot \prod_{j \mid S_j \leqslant t} (1 + m_j(\omega)) \tag{3.6}$$

Proof :

This theorem is a simple application of the concept of exponential of a semi-martingale (see JACOD [9]) : if X is a semi-martingale and z a finite random variable F_o-measurable, then the equation

$$Z(t, \omega) = z + \int_0^t Z(s^-, \omega)\, dX(s, \omega)$$

has a unique solution in the space of semi-martingale, given by

$$\mathcal{E}(t, \omega) = z \cdot \exp\left[X(t, \omega) - X(0, \omega) - \frac{1}{2} < X^c, X^c >_t\right]$$

$$\cdot \prod_{0<s\leq t} \left[1 + \Delta X_s \, e^{-\Delta X_s}\right]$$

By using the particular form of the flux I, one obtains directly (3.6)

REMARK 3.1. :

The presence in the capitalization equation, of the left hand limit $C(s^-, \omega)$ instead of the process $C(s, \omega)$ can be justified from a financial and a technical point of view :

a) financially :

When we use the left hand limit, the jump of the capitalization process, at time t_o of discontinuity of the flux of size s_o, is given by

$$C(t_o) = C(t_o^-) \cdot (1 + s_o)$$

The size S_o can be interpreted as a stochastic impulse which is applied on the capital before the jump.

b) technically :

In a deterministic model, it is possible to consider the integral equation without left hand limit

$$C(t) = 1 + \int_o^t C(s) \, dI(s)$$

The solution is then :

$$C(t) = \exp(I^c(t)) \cdot \prod_{k \mid T_R \leq t} \left(1 + \frac{s_k}{1 - s_k}\right)$$

Where - $I(t) = I^c(t) + \sum_{k \mid T_k \leq t} s_k$

- $I^c(t)$ is the continuous part of $I(t)$

But in the semi-martingale model, the equation cannot be because the process C is not necessarily predictable, and so the stochastic integral $\int C(s, \omega)\, dI(s, \omega)$ does not exist.
The capitalization process (3.6) has the following properties :

PROPOSITION 3.1. :

(i) property of the sample paths :

The capitalization process admits sample paths which are right continuous, with left hand limits, and strictly positive.

(ii) probabilistic property :

If the flux of interest is a process of finite variation (resp. a local martingale), the capitalization process is also a process of finite variation (resp. a local martingale).
In particular, if M is a local martingale,

the process $M^*(t, \omega) = \exp\left(M^c(t, \omega) - \frac{1}{2}\langle M^c, M^c\rangle_t\right)$ (3.7)

is a continuous local martingale.

and the process $A^*(t, \omega) = \prod_{j \mid S_j \leq t} (1 + m_j) \exp - A^p(t, \omega)$ (3.8)

is a compensated sum of jumps.

(iii) property of decomposition :

The additive decomposition (3.3) of the flux generates a multiplicative decomposition of the associate capitalization process (3.5) :

$$C\ (t,\ \omega) = C_1\ (t,\ \omega)\ .\ C_2\ (t,\ \omega) \tag{3.9}$$

Where C_1 is a process of finite variation
C_2 is a local martingale.

Proof :

(i) The elementary properties result directly (the form of the solution and the sizes of the jumps, included in the interval (-1, 1)).

(ii) The stochastic integral with respect to a process of finite variation (resp. a local martingale) generates a process of finite variation (resp. a local martingale) (see JACOD [9]). Then (ii) is a consequence of the equation of the capitalization process.

(iii) The following decomposition is possible :

$$C\ (t,\ \omega) = C_1\ (t,\ \omega)\ .\ C_2\ (t,\ \omega)$$

Where
$$C_1\ (t,\ \omega) = \exp \int_0^t \delta\ (s,\ \omega)\ ds \prod_{j \mid T_j \leqslant t} (1 + s_j\ (\omega)) \tag{3.10}$$

$$C_2\ (t,\ \omega) = \exp \{M^c\ (t,\ \omega) - \frac{1}{2} \langle M^c, M^c \rangle_t - A^p\ (t,\ \omega)\}$$

$$.\ \prod_{j \mid S_j \leqslant t} (1 + m_j\ (\omega)) \tag{3.11}$$

The process C_1 is generated by the flux of finite variation V; the process C_2 is generated by the local martingale M. Apply then (ii).

REMARK 3.2. :

In the multiplicative decomposition, it is interesting to observe that the part of finite variation of C is independent of the part local martingale of the flux I.

3.4. Actualization process :

The actualization process can be defined as the present value at time $t = 0$ of a future unitary amount.
Explicitly, it is the inverse of the capitalization process.

DEFINITION 3.2. :

The actualization process ℓ, associated to a given flux of interest (3.4) is given by :

$$\ell(t, \omega) = \frac{1}{C(t, \omega)} = \exp\{\frac{1}{2} \langle M^c, M^c\rangle_t + A^p(t, \omega) - \int_0^t \delta(s, \omega)\, ds - M^c(t, \omega)\} . \prod_{j \mid T_j \leqslant t} \frac{1}{(1 + s_j(\omega))} . \prod_{j \mid S_j \leqslant t} \frac{1}{(1 + m_j(\omega))} \tag{3.12}$$

This process exists a.s. because the capitalization process is always strictly positive (property 3.1.i).
The properties of this actualization process will be compared with those of the capitalization process. We shall obtain a principle of irreversibility in a stochastic environment.
Let us study first the continuous model, and then the model with jumps.

3.4.1. Continuous model :

In this case, the flux of interest has the form :

$$I(t, \omega) = V(t, \omega) + M^c(t, \omega)$$

$$= \int_0^t \delta(s, \omega)\, ds + M^c(t, \omega) \tag{3.13}$$

In a continuous deterministic model (i.e. $I(t) = \int_0^t \delta(s)\, ds$), the inversion of time induces a simple inversion of sign in the differential equation :

$$d\, C(t) = C(t)\, dI(t)$$

$$\text{and} \quad d\, \ell(t) = -\, \ell(t)\, dI(t)$$

The reversibility disappears in a general continuous stochastic model as :

PROPOSITION 3.2. :

When the flux of interest is a continuous semi-martingale (3.13), the actualization process is solution of the stochastic differential equation :

$$d\, \ell(t, \omega) = -\, \ell(t, \omega)\, dI(t, \omega) + \ell(t, \omega)\, d\langle M^c, M^c\rangle_t \tag{3.14}$$

(in comparison with $dC(t, \omega) = C(t, \omega)\, dI(t, \omega)$).

Proof :

Let us apply the ITO formula (cf. DELLACHERIE-MEYER [5]) to the function $F(x) = \frac{1}{x} \in C^2(\mathbb{R}_o^+)$.

$$\ell(t, \omega) = F(C(t, \omega)) = 1 + \int_{(o, t]} -\frac{1}{C^2(s, \omega)}\, d\, C(s, \omega)$$

$$+ \frac{1}{2} \int_{(o, t]} \frac{2}{C^3(s, \omega)}\, d\, [C^c, C^c]_s$$

But a) $dC(s, \omega) = C(s, \omega)\, dI(s, \omega)$

b) $d\,[C^c, C^c]_s = C^2(s, \omega)\, d\,[I^c, I^c]_s$

$$= C^2(s, \omega)\, d\,\langle M^c, M^c\rangle_s$$

So :

$$\ell(t, \omega) = 1 + \int_{(o,\, t]} - \frac{1}{C(s, \omega)}\, dI(s, \omega) + \int_{(o,\, t]} \frac{1}{C(s, \omega)}\, d\,\langle M^c, M^c\rangle_s$$

or :

$$\ell(t, \omega) = 1 + \int_{(o,\, t]} - \ell(s, \omega)\, dI(s, \omega) + \int_{(o,\, t]} \ell(s, \omega)\, d\,\langle M^c, M^c\rangle_s$$

i.e.

$$d\,\ell(t, \omega) = -\,\ell(t, \omega)\, dI(t, \omega) + \ell(t, \omega)\, d\,\langle M^c, M^c\rangle_t$$

II

REMARK 3.3. :

The continuous local martingale part generates a complementary term in the stochastic equation of actualization.

The equation can be written :

$$d\,\ell(t, \omega) = -\,\ell(t, \omega)\, d\,(V(t, \omega) - \langle M^c, M^c\rangle_t) - \ell(t, \omega)\, d\,M^c(t, \omega)$$

The process denoted $U(t, \omega) = V(t, \omega) - \langle M^c, M^c\rangle_t$ (3.15)
will be called trend without risk.
This process is also relevant in the multiplicative decomposition of the actualization process.

PROPOSITION 3.3. :

When the flux of interest is a continuous semi-martingale (3.13), the actualization process admits the following multiplicative decomposition

$$\ell\ (t, \omega) = \ell_1\ (t, \omega)\ .\ \ell_2\ (t, \omega)$$

Where $\ell_1\ (t, \omega) = \exp\ (-\ U\ (t, \omega)) = \exp\ (-\int_o^t \delta\ (s, \omega)\ ds + \langle M^c, M^c\rangle_t)$ (3.16)

is a continuous process of finite variation

$$\ell_2\ (t, \omega) = \exp\ (-\ M^c\ (t, \omega) - \frac{1}{2} \langle M^c, M^c\rangle_t) \qquad (3.17)$$

is a continuous local martingale.

Proof :

In the continuous model, the actualization process has the form

$$\ell\ (t, \omega) = \exp\ (\frac{1}{2}\ \langle M^c, M^c\rangle_t - M^c\ (t, \omega) - \int_o^t \delta\ (s, \omega)\ ds)$$

1. Let us try the following decomposition which is a natural extension of the decomposition of the capitalization process :

$$\ell\ (t, \omega) = \ell_1\ (t, \omega)\ .\ \ell_2\ (t, \omega)$$

Where

$$\ell_1\ (t, \omega) = \frac{1}{C_1\ (t, \omega)} = \exp\ (-\int_o^t \delta\ (s, \omega)\ ds)$$

$$\ell_2\ (t, \omega) = \frac{1}{C_2\ (t, \omega)} = \exp\ (\frac{1}{2}\ \langle M^c, M^c\rangle_t - M^c\ (t, \omega))$$

In fact, the process ℓ_2 is not a local martingale.

For this, let us look at the stochastic equation :

$$Z(t, \omega) = 1 + \int_0^t Z(s, \omega)\, d(-M^c(s, \omega))$$

The solution process Z is a local martingale because the process $-M^c$ is a local martingale (proposition 3.1.(ii)).
The theorem (3.1) gives the explicit solution of this equation :

$$Z(t, \omega) = \exp\left(-M^c(t, \omega) - \frac{1}{2} <-M^c, -M^c>_t\right)$$

$$= \exp\left(-M^c(t, \omega) - \frac{1}{2} <M^c, M^c>_t\right)$$

But $\ell_2(t, \omega) = Z(t, \omega) \cdot \exp(<M^c, M^c>_t)$.

This means that the process ℓ_2 is the product of a local martingale and of an increasing process, and is not a local martingale.

(ii) The process Z gives naturally the decomposition :

$$\ell(t, \omega) = Z(t, \omega) \cdot \exp\left(-\int_0^t \delta(s, \omega)\, ds + <M^c, M^c>_t\right)$$

$$= \ell_2(t, \omega) \cdot \ell_1(t, \omega)$$

The process $\ell_2(t, \omega) = Z(t, \omega)$ is a local martingale (cf. (i)).
The process $\ell_1(t, \omega) = \exp\left(-\int_0^t \delta(s, \omega)\, ds + <M^c, M^c>_t\right)$ is a process of finite variation.

CONCLUSION 3.1. :

The structure of the stochastic equation (proposition 3.2) and the multiplicative decomposition (proposition 3.3) of the actualization process illustrate the importance of the "trend without risk", equal to the trend minus the quadratic variation of the noise.

This quadratic variation, $\langle M^c, M^c \rangle$ can be interpreted as the price of the risk in an actualization process.

3.4.2. Model with jump :

We generalize here the results of the last section, in the general model of semi-martingale.

The flux of interest has then the form :

$$I(t, \omega) = \int_0^t \delta(s, \omega)\, ds + M^c(t, \omega) - A^p(t, \omega) + \sum_{j \mid T_j \leqslant t} S_j(\omega)$$

$$+ \sum_{j \mid S_j < t} m_j(\omega) \qquad (3.18)$$

In a deterministic model with jump, (i.e. $I(t) = \int_0^t \delta(s)\, ds + \sum_{j \mid T_j \leqslant t} S_j$)

it can be easily shown that, if the capitalisation process is solution of the differential equation :

$$d\, C(t) = C(t^-)\, dI(t)$$

then the actualization process is solution of

$$d\, \ell(t) = -\, \ell(t^-)\, dI^*(t)$$

Where I* is a corrected flux defined by its continuous and discrete parts :

$$I^{*c}(t) = I^c(t) = \int_0^t \delta(s)\, ds$$

$$I^{*d}(t) = \sum_{j \mid T_j \leqslant t} \frac{s_j}{1 + s_j}$$

The reversibility of the equation appears again thanks to an adaptation of the jumps, in the deterministic model.
To extend this in a stochastic model, let us consider the corrected stochastic flux associated to (3.18).

$$I^* (t, \omega) = \int_0^t \delta (s, \omega)\, ds + M^c (t, \omega) - A^p (t, \omega) + \sum_{j \mid T_j \leq t} \frac{s_j (\omega)}{1 + s_j (\omega)}$$

$$+ \sum_{j \mid S_j \leq t} \frac{m_j (\omega)}{1 + m_j (\omega)} \tag{3.19}$$

We have then the following extension of the proposition 3.2 :

PROPOSITION 3.4. :

In the general model (3.18), the actualization process is solution of the stochastic differential equation :

$$d\, \ell (t, \omega) = - \ell (t^-, \omega)\, d\, I^* (t, \omega) + \ell (t^-, \omega)\, d \langle M^c, M^c \rangle_t \tag{3.20}$$

Proof :

Let us apply the ITO formula with discontinuities (cf. DELLACHERIE-MEYER [5]) to the function $F (x) = \frac{1}{x} \in C^2 (\mathbb{R}_0^+)$

$$\ell\ (t, \omega) = F\ (C\ (t, \omega)) = 1 + \int_{(o,\ t]} - \frac{1}{C^2\ (s^-, \omega)}\ dC\ (s, \omega)$$

$$+ \frac{1}{2} \int_{(o,\ t]} \frac{2}{C^3\ (s^-, \omega)}\ d\ [C^c, C^c]$$

$$+ \sum_{0 \leqslant s \leqslant t} (\ell\ (s, \omega) - \ell\ (s^-, \omega) + \frac{1}{C^2\ (s^-, \omega)}\ (C\ (s, \omega) - C\ (s^-, \omega)))$$

or :

$$\ell\ (t, \omega) = 1 + \int_{(o,\ t]} - \ell\ (s^-, \omega)\ dI\ (s, \omega) + \int_{(o,\ t]} \ell\ (s^-, \omega)\ d < M^c, M^c >_s$$

$$+ \sum_{0 \leqslant s \leqslant t} (\ell\ (s, \omega) - \ell\ (s^-, \omega) + \frac{1}{C^2\ (s^-, \omega)}\ (C\ (s, \omega) - C\ (s^-, \omega))$$

The last term, at a stopping time t_k of discontinuity of the flux of size $\Delta I_k = I\ (t_k, \omega) - I\ (t_k^-, \omega)$, becomes :

$$\ell\ (t_k, \omega) - \ell\ (t_k^-, \omega) + \frac{1}{C^2\ (t_k^-, \omega)}\ \Delta C\ (t_k, \omega)$$

$$= \ell\ (t_k^-, \omega)\ (1 - \frac{\Delta I_k}{1 + \Delta I_k}) - \ell\ (t_k^-, \omega) + \ell\ (t_k^-, \omega)\ .\ \Delta I_k$$

$$= \ell\ (t_k^-, \omega)\ .\ \Delta I_k - \ell\ (t_k^-, \omega)\ .\ \frac{\Delta I_k}{1 + \Delta I_k}$$

Now the jump of the stochastic integral $\int_{(o,\ t]} - \ell\ (s^-, \omega)\ dI\ (s, \omega)$

at $t = t_k$ is given by

$$- \ell\ (t_k^-, \omega)\ \Delta I_k$$

So the jump of ℓ at $t = t_k$ is

$$\Delta\ell\ (t_k,\ \omega) = - \frac{\Delta I_k}{1 + \Delta I_k}\ \ell\ (t_k^-,\ \omega)$$

and we can write

$$\ell\ (t,\ \omega) = 1 - \int_{(o,\ t]} \ell\ (s^-,\ \omega)\ dI^*\ (s,\ \omega) + \int_{(o,\ t]} \ell\ (s^-,\ \omega)\ d <M^c,\ M^c>_s$$

II

A multiplicative decomposition can also be considered in the general model: (extension of proposition 3.3).

PROPOSITION 3.5. :

In the general model (3.18), the actualization process admits the following multiplicative decomposition :

$$\ell\ (t,\ \omega) = \ell_1\ (t,\ \omega)\ .\ \ell_2\ (t,\ \omega)$$

Where $\ell_1\ (t,\ \omega) = \exp\ (- \int_o^t \delta\ (s,\ \omega)\ ds + <M^c,\ M^c>_t + A^p\ (t,\ \omega) + B^p\ (t,\ \omega))$

$$\prod_{j | T_j \leqslant t} \frac{1}{1 + s_j} \tag{3.21}$$

is a process of finite variation

$$\ell_2\ (t,\ \omega) = \exp\ (- M^c\ (t,\ \omega) - \frac{1}{2} <M^c,\ M^c>_t - B^p\ (t,\ \omega))\ .\ \prod_{j | S_j \leqslant t} \frac{1}{1 + m_j}$$

is a local martingale (3.22)

and with B^p, the dual predictable projection of the process of finite variation B defined by :

$$B(t, \omega) = \sum_{j \mid S_j \leq t} - \frac{m_j(\omega)}{1 + m_j(\omega)}$$

Proof :

We can write :

$$\ell(t, \omega) = (\exp(-\int_o^t \delta(s, \omega)\, ds + \langle M^c, M^c \rangle_t \cdot \prod_{j \mid T_j \leq t} \frac{1}{1 + s_j})$$

$$\cdot (\exp(-\frac{1}{2} \langle M^c, M^c \rangle_t - M^c(t, \omega))$$

$$\cdot (\prod_{j \mid S_j \leq t} \frac{1}{1 + m_j} \exp A^p(t, \omega))$$

We must study the third term

$$\ell_3(t, \omega) = \prod_{j \mid S_j \leq t} \frac{1}{1 + m_j} \exp A^p(t, \omega)$$

For this, let us denote

$$B(t, \omega) = \sum_{j \mid S_j \leq t} - \frac{m_j(\omega)}{1 + m_j(\omega)}$$

which is of finite variation

and $\qquad N(t, \omega) = B(t, \omega) - B^p(t, \omega)$

which is a compensated local martingale

(B^p is the dual predictable projection of the process B).

The capitalization equation associated to the flux N

$$W(t, \omega) = 1 + \int_o^t W(s^-, \omega)\, dN(s, \omega)$$

has the following solution (cf. (3.6)) :

$$W(t, \omega) = \prod_{j \mid S_j \leqslant t} \left(1 - \frac{m_j(\omega)}{1 + m_j(\omega)}\right) \exp - B^p(t, \omega)$$

This process is also a local martingale (proposition 3.1.(ii)), then we can write :

$$\ell_3(t, \omega) = W(t, \omega) \cdot \exp\left(A^p(t, \omega) + B^p(t, \omega)\right)$$

where W is a local martingale

$\exp(A^p + B^p)$ is a process of finite variation.

In fact, this last process is increasing.
Indeed, $A^p + B^p$ is the dual predictable projection of the process :

$$(A + B)(t, \omega) = \sum_{j \mid S_j \leqslant t} m_j(\omega) - \sum_{j \mid S_j \leqslant t} \frac{m_j(\omega)}{1 + m_j(\omega)}$$

$$= \sum_{j \mid S_j \leqslant t} \frac{m_j^2(\omega)}{1 + m_j(\omega)}$$

The sizes of the jumps, $\frac{m_j^2}{1 + m_j}$ are positive, because $m_j \in (-1, 1)$

So the process A + B is increasing, and also $A^p + B^p$.
We obtain the decomposition

$$\ell(t, \omega) = \exp\left(- \int_0^t \delta(s, \omega)\, ds + \langle M^c, M^c\rangle_t\right) \cdot \prod_{j \mid T_j \leqslant t} \frac{1}{1 + s_j(\omega)}$$

$$\cdot \exp\left(- \frac{1}{2} \langle M^c, M^c\rangle_t - M^c(t, \omega)\right) \cdot W(t, \omega) \cdot \exp\left(A^p(t, \omega) + B^p(t, \omega)\right)$$

The two processes, $\exp(-\frac{1}{2}\langle M^c, M^c\rangle_t - M^c(t, \omega))$ and $W(t, \omega)$, are two local martingales without the same discontinuities.
So their product is also a local martingale, which is equal to $\ell_2(t, \omega)$ (cf. (3.22)).
The process $\ell_1(t, \omega)$ (cf. (3.21)) is the product of two processes of finite variation and is also a process of finite variation.

CONCLUSION 3.2. :

The multiplicative decomposition of the actualization process illustrates the importance of the continuous trend without risk equal to the continuous trend minus an increasing process :

$$U(t, \omega) = \int_0^t \delta(s, \omega)\, ds - \langle M^c, M^c\rangle_t - (A^p(t, \omega) + B^p(t, \omega)) \quad (3.23)$$

The penalization, $\langle M^c, M^c\rangle_t + (A^p(t, \omega) + B^p(t, \omega)$, which is an increasing process can be interpreted as the price of the risk in this general model.

4. EXAMPLE : PROCESSES WITH INDEPENDENT INCREMENTS.

In the particular case where the flux (3.4) is a process with independent increments, we can obtain a stronger version of the multiplicative decomposition : an explicit computation of the mean values.

DEFINITION 4.1. (see GIMMAN-SKOROKOD [8]) :

A flux (3.4) is a process with independent increments if :

$$I(t, \omega) = \int_0^t \delta(s)\, ds + \int_0^t \sigma(s)\, dw(s, \omega) + \sum_{n \mid T_n \leqslant t} j_n(\omega) + \sum_{n \mid S_n \leqslant t} i_{S_n}(\omega)$$

$$- \int_0^t \int_{-1}^1 x\, d\, G_s(x)\, d\, a(s) \tag{4.1}$$

where :

1) $V^c(t, \omega) = \int_0^t \delta(s)\, ds$ is a deterministic function (deterministic interest rates)

2) $M^c(t, \omega) = \int_0^t \sigma(s)\, dw(s, \omega)$ is a stochastic integral with respect to a brownian motion

3) $V^d(t, \omega) = \sum_{n \mid T_n \leqslant T} j_n(\omega)$

   with :
   - $\{T_n\}$ are deterministic times
   - The $\{j_n\}$ are independent random variables in a compact of $(-1, 1)$

4) $M^d(t, \omega) = \sum_{n \mid S_n \leqslant t} i_n(\omega) - \int_0^t \int_{-1}^1 x\, dG_s(x)\, da(s)$

   with :
   - The $\{i_n\}$ are independent random variables in a compact of $(-1, 1)$, of distribution denoted $G_s(.)$ where s is the time of jump

- The counting process of the jumps denoted N, is a Poisson process with mean value

$$E\,N(t, \omega) = a(t)$$

where a is a continuous non decreasing function.

In this model, the local martingale M^d is in fact a real martingale (see GIMMAM-SKOROKOD [8]).

The process M^c is also a martingale if $\int_o^t \sigma^2(s)\,ds$ is finite for all $t > 0$. In this case, we can compute the mean value of the flux

$$\mathbb{E}\, I(t, \omega) = \int_o^t \delta(s)\,ds + \sum_{n \mid T_n \leqslant t} E\, j_n(\omega) \tag{4.2}$$

The capitalization process is given by (cf. theorem 3.1) :

$$C(t, \omega) = (\exp \int_o^t \delta(s)\,ds \prod_{n \mid T_n \leqslant t} (1 + j_n) \,.\, \exp \int_o^t \sigma(s)\,dw(s, \omega)$$

$$.\ \exp - \frac{1}{2} \int_o^t \sigma(s)\,ds$$

$$.\ \prod_{n \mid S_n \leqslant t} (1 + i_{S_n}) \,.\, \exp (- \int_o^t \int_{-1}^1 x\, dG_s(x)\, da(s))$$

$$= C_1(t, \omega) \quad . \quad C_2(t, \omega)$$

The local martingale C_2 becomes a real martingale if the function σ is bounded (cf. LEPINGLE-MENIN [10]).
In this case, we can write (independence of the processes) :

$$E\, C(t, \omega) = \exp \int_o^t \delta(s)\,ds \,.\, \prod_{n \mid T_n \leqslant t} (1 + E\, j_n) \tag{4.3}$$

For the actualization process, we have

$$\ell\,(t,\omega) = \ell_1\,(t,\omega)\,.\,\ell_2\,(t,\omega)$$

with $\ell_1\,(t,\omega) = \exp \int_o^t -(\delta\,(s) - \sigma^2\,(s))\,ds\,.\,\exp \int_o^t \int_{-1}^{1} \frac{x^2}{x+1}\,d\,G_s\,(x)\,d\,a\,(s)$

$$\prod_{n \mid T_n \leqslant t} \frac{1}{1+j_n}$$

$$\ell_2\,(t,\omega) = \exp \int_o^t -\frac{1}{2}\,\sigma^2\,(s)\,ds - \int_o^t \sigma(s)\,dw\,(s,\omega) \prod_{n \mid S_n \leqslant t} \frac{1}{1+i_{S_n}}$$

$$\exp \int_o^t \int_{-1}^{1} \frac{x}{1+x}\,d\,G_s\,(x)\,da\,(s)$$

The local martingale ℓ_2 becomes again a real martingale if the function σ is bounded, and we can write :

$$\mathbb{E}\;\ell\,(t,\omega) = \exp \int_o^t -(\delta\,(s) - \sigma^2\,(s))\,ds\,.\,\exp \int_o^t \int_{-1}^{1} \frac{x^2}{x+1}\,d\,G_s\,(x)\,da(s)$$

$$E \prod_{n \mid T_n \leqslant t} \frac{1}{1+j_n}$$

Let us denote

$1 + r_n = E\,\left(\frac{1}{1+j_n}\right)(1 + E\,j_n)$ which is $\geqslant 1$ by the JENSEN inequality.

Then (4.4) becomes

$$\mathbb{E}\ \ell\ (t,\omega) = \frac{1}{E\ C\ (t,\omega)}\ .\ \left(\exp \int_0^t \sigma^2\ (s)\ ds\right)\ .\ \exp \int_0^t \int_{-1}^1 \frac{x^2}{x+1}\ d\ G_s(x)\ da(s)$$

$$\prod_{n \mid T_n \leqslant t} (1 + r_n) \tag{4.5}$$

The three terms of penalization are non decreasing functions.

REMARK 4.1. :

As fundamental examples of flux with independent increments, we can consider the Wiener processes, or the Poisson processes.
The particular formula for these cases results directly from (4.3) and (4.4) and have been developed together with other examples in DEVOLDER |7|.

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