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Fixed Income Asset Liability Management

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Summary

This paper presents a risk control model that serves to determine an optimal investment strategy within the following setting. An investor faces a series of known future liabilities. To be able to service all debts, the investor decides to purchase a portfolio of fixed income securities now. Therefore he wants to determine the cheapest possible portfolio, such that the revenues from this portfolio plus revenues from reinvesting excess cash minus costs of future borrowing (to offset temporary cash shortages) suffice to meet all liabilities. The risk control model composes such a portfolio, with the guarantee that all liabilities can be met without ending up with a deficit at the end of the planning period. This guarantee holds under all future interest scenario's that the investor deems relevant. It enables the investor to expresss both his risk attitude and his vision with respect to future interest rate developments.

Résumé

Le Gestion des Engagements à Revenu Fixe

Cet article présente un modèle de contrôle de risque qui sert à déterminer une stratégie d'investissement optimale dans le cadre suivant. Un investisseur est confronté à une série d'engagements futurs connus. Afin de pouvoir servir toutes les dettes, l'investisseur décide d'acheter aujourd'hui un portefeuille de titres à revenu fixe. Il veut donc déterminer le portefeuille le moins cher possible de façon à ce que les produits de ce portefeuille plus les produits du réinvestissement de l'argent excédentaire moins les coûts d'emprunts futurs (pour compenser les manques d'argent temporaires) soient suffisants pour couvrir tous les engagements. Le modèle de contrôle de risque compose un tel portefeuille, avec la garantie que tous les engagements peuvent être honorés sans arriver à un déficit à la fin de la période planifiée. Cette garantie reste valable dans tous les scénarios d'intérêts futurs que l'investisseur juge pertinents. Elle permet à l'investisseur d'exprimer à la fois son attitude vis-à-vis du risque et ses prévisions en ce qui concerne l'évolution des taux d'intérêt.

1. Introduction

This paper presents a linear programming model that serves to determine an optimal investment strategy within the following setting. An investor faces a series of known future liabilities. To be able to service all debts, the investor decides to purchase a portfolio of fixed income securities now. The portfolio must satisfy the following requirements:

- coupon payments and principal payments received from the portfolio plus future reinvestment revenues, minus future costs of borrowing should be sufficient to meet all liabilities;

- at the end of the planning period there should be a nonnegative terminal value.

Well known approaches that aim at addressing the above problem are cash flow matching and immunization. *Cash flow matching* offers the investor maximum security with respect to his ability to meet all liabilities without ending up with a deficit at the planning horizon. The other side of the coin is a high initial investment and a low guaranteed yield over the planning period. Using an immunization model will generally lead to a lower initial investment and a higher expected yield over the planning period. This will be achieved at the cost of accepting a considerable degree of interest rate risk and the necessity of rebalancing the portfolio frequently. Both approaches will be discussed at some more length in the next chapter. The risk control model that will be presented in chapter 3 enables the investor to compose a minimum cost portfolio such that all liabilities can be met with certainty and that no deficit remains at the planning horizon. This guarantee holds under all future interest rate scenario's that the investor deems relevant.

In the fourth chapter it will be shown that the cash flow matching model can be viewed as special case of the risk control model.

2. Cash flow matching and Immunization

The presentation of the cash flow matching models and the immunization models in this paragraph will be aimed at conveying the basic ideas underlying these models. We shall not survey all the relevant literature on this matter. The interested reader is referred to [Dahl, Meeraus and Zenios, 1989], [Granito, 1984], [Elton and Gruber, 1987, 1989].

Cash flow matching strategies adopt the starting point that liquidity shortages are not allowed at any point during the planning period. So at the end of every period, the portfolio revenues up to that point should be greater than or equal to the sum of liabilities due, up to that period.

Cash excesses can be reinvested. However, the yield on these reinvestments is not known yet. When following the cash matching approach it is customary to make a conservative reinvestment assumption, e.g. a reinvestment rate equal to zero. The cash flow matching formulation (I) in this paper assumes a one period reinvestment rate equal to r^c.

We shall use t to indicate both points in time and periods of time. Period t refers to the time span from moment $t-1$ to moment t. It is assumed that all periods are equally long; however, this is not essential to what follows.

In the sequel we shall use the following notation:

$x_i =$	amount of bond i that is purchased at time 0	$i=1,\ldots,n$
$a_{ti} =$	cashflow that is received at time t from 1 unit of bond i	$i=1,\ldots,n$ $t=1,\ldots,T$
$c_i =$	price per unit of bond i	$i=1,\ldots,n$
$b_t =$	liability at time t	$t=1,\ldots,T$

We shall denote the row vector $(a_{t1},\ldots,a_{tn})$ by a_t.

A cash flow matching model minimizing the initial cash outlay can now be formulated as:

$$\text{(I)} \quad \min_{x} \; cx$$

$$\text{s.t.} \quad \sum_{\tau=1}^{t} (1+r^{c})^{t-\tau} (a_{\tau}x - b_{\tau}) \geq 0 \qquad t=1,\ldots,T$$

$$x \geq 0$$

If r^{c} is chosen sufficiently conservative, then the cash flow matching approach has some attractive properties:

- all liabilities can be met with certainty;
- it is not necessary to rebalance the portfolio purchased at time 0.

On the other hand, the conditions imposed upon the portfolio by the cash flow matching constraints are so restrictive that the initial investment will be quite high in comparison with the other methods that will be discussed here.

Constructing an immunized portfolio can be viewed as composing a cash flow matching portfolio where the investor has an increased degree of freedom: the option, at any time t $(t=1,..,T-1)$ to lend excess cash flow and to borrow shortcomings until $t+1$. It is assumed that borrowing as well as lending can be done against an interest rate r, which is known in advance. Denote the amount borrowed at time t by f_t ($f_t<0$ means lending). Consider the following investment strategy:

1. buy a portfolio at time 0.
2. At time t borrow an amount f_t from time t to $t+1$ with
   $f_t = (1+r)f_{t-1}+b_t-a_t x \qquad t = 1,\ldots, T-1.$

Then, what are the constraints that should be imposed upon the portfolio at time 0, to guarantee that all liabilities b_t can be met, and that no debt from one period borrowing remains? Clearly there is no need to formulate any constraints concerning time $1,\ldots,T-1$ since all liabilities can be met by practizing the investor's strategy specified above. Only a constraint with respect to the end of the planning period is required:

$$-(1+r)f_{T-1}-b_T+a_Tx \;\geq\; 0$$

The investor's strategy combined with the horizon constraint and the objective of minimizing the initial investment can be described by means of a linear program:

$$\begin{array}{lll}
\min\limits_{x,f} \;\; cx & & \\
\text{s.t.} & f_t \;=\; b_t-a_tx & \\
 & f_t \;=\; b_t-a_tx+(1+r)f_{t-1} & t \;=\; 2,\ldots T-1 \\
 & -(1+r)f_{T-1}+a_T-b_T \;\geq\; 0 & \\
 & x \;\geq\; 0 &
\end{array}$$

Notice that $f_t=\sum\limits_{\tau=1}^{t}(1+r)^{t-\tau}(a_\tau x-b_\tau)$, for $t=1,..,\ T$. Using this observation one can reformulate the above program to obtain a much simpler, equivalent program:

$$\begin{array}{ll}
\min\limits_{x} \;\; cx & \\
\text{s.t.} & \sum\limits_{t=1}^{T}(1+r)^{-t}(a_tx-b_t) \;\geq\; 0 \\
 & x\geq 0
\end{array}$$

Thus formulated, the remaining constraint is readily recognized as the requirement that the Net Present Value of the cash flows, computed with a discount rate r, should be positive.

To simplify the notation in the remainder of this paper we define

$$NPV(r,x)=\sum_{t=1}^{T}(1+r)^{-t}(a_tx-b_t)$$

Now suppose that the one period interest rate r shows some fluctuations over time. Then the constraint $NPV(r,x)\;\geq\;0$ does not guarantee a positive terminal value any more. Costs of borrowing (revenues from lending) may turn out higher (lower) than expected. The approach to this problem is to add the constraint

$$\frac{\partial NPV(r,x)}{\partial r} = 0$$

Thereby forcing the portfolio to be at a stationary point of the *NPV* as a function of the discount rate r. However, what would happen to the *NPV* if this stationary point happens to be a maximizer of *NPV* as a function of r? Then there exists a neighbourhood $(r-\varepsilon_0, r+\varepsilon_0)$ of r such that for any $\varepsilon \in(-\varepsilon_0, \varepsilon_0)$ $NPV(r+\varepsilon, x) < NPV(r,x)$, in other words any small change in r will
result in a decrease of the *NPV*. This undesirable situation can be avoided by including a second order condition:

$$\frac{\partial^2 NPV(r,x)}{\partial r^2} > 0$$

Now x has to be chosen so that the stationary point will be a minimizer, implying that a change of r within the neighbourhood of r will result in an increase of the NPV. It is the following model that is often referred to as an immunization model:

$$\begin{array}{lll}
(II) & \min\limits_{x} & cx \\
\text{s.t.} & & NPV(r,x) \geq 0 \\
& & \dfrac{\partial NPV(r,x)}{\partial r} = 0 \\
& & \dfrac{\partial^2 NPV(r,x)}{\partial r^2} > 0 \\
& & x \geq 0
\end{array}$$

It can be verified easily that all constraints are linear in the decision variables x.

Choosing the portfolio which results from solving the immunization model offers the investor the certainty that:

- all debts can be met timely if interest rates do not change during the planning period;

- if r changes into $r+\varepsilon$, before any payment has been made, then all debts can still be met if $r+\varepsilon$ is in the neighbourhood of the (local) minimizer r of $NPV(r,x)$ as a function of r.

The immunization model described above is sometimes called a duration matching model, because the model forces the duration of the assets to be equal to the duration of the liabilities.

Duration is measure of interest rate sensitivity for a future cashflow. This concept was introduced by Macaulay [Macaulay 1938]. Based on his definition, the duration of a stream of cash flows can be formulated as the average time to maturity of the cash flows, weighted by the present values of the cash flows:

$$D = \frac{\sum_{t=1}^{T} t(1+r)^{-t} CF_t}{\sum_{t=1}^{T} (1+r)^{-t} CF_t}$$

Notice that the definition of the present value of a cash flow CF_t at time t is $(1+r)^{-t}CF_t$, implying a flat yield curve. Since Macaulay's publication several authors have proposed different duration measures (see e.g. [Fisher and Weil 1971] and [Khang 1979]). The main differences between duration measures are usually due to the underlying assumptions with respect to the shape of the yield curve and the structure of future changes in the yield curve.

In the context of immunization models one is frequently confronted with "the duration constraint". This duration constraint is motivated by the argument that and investor can reduce his interest rate risk by composing a portfolio which exhibits the same sensitivity to interest rate changes as the liabilities that it should fund. Therefore one requires the duration of the portfolio to be equal to the duration of the stream of liabilities. It is verified easily that this constraint is equivalent to the first order condition of model (II).

When interest rates change, however, they may do so in a way that the duration constraint does not offer much consolation. That is, in spite of "equating the interest rate sensitivity of the present value of assets and liabilities", real world interest rate changes typically do not have an identical impact on the value of cash inflows and cash outflows because they do not cohere with the assumptions underlying the duration measure. To model interest rate changes in a more realistic manner, the **convexity** constraint was introduced. Loosely speaking, the idea behind convexity boils down to: "If we cannot guarantee identical behaviour of asset value and liability value when interest rates change, then it would be nice if the present value of the assets would increase more (or decrease less) than the present value of the liabilities." In order to achieve this portfolio property one requires the convexity of the assets to be greater than the convexity of the liabilities. Convexity can be quantified as the second derivative of the present value with respect to the discount factor:

$$C = \sum_{t=1}^{T} t(t+1)(1+r)^{-t-2}CF_t$$

As with the duration measure, different definitions for convexity have been proposed. If we assume that the net present value constraint is binding($NPV(r,x) = 0$), then the convexity requirement mentioned above in conjunction with the convexity definition given earlier, leads to the second order constraint of the immunization model.

The major problem with the use of the immunization concept stems from the strict assumptions that have to made about the magnitude (infinitesimal) and structure (e.g. parallel shifts) of changes in the yield curve. If changes in interest rates occur that do not cohere with these assumptions, then no guarantee can be given that the investor will be able. to make all liability payments timely. In addition one should rebalance the portfolio after each cashflow (portfolio revenue or liability payment) in order to maintain an immunized portfolio. This will generally lead to a high level of transaction costs. On the other hand, the initial investment to purchase an immunized portfolio will be substantially smaller than the costs of buying a cash flow matching portfolio.

3. A risk control model

This chapter presents a model, such that a portfolio constructed by this model has the property that all liabilities can be met if the one period interest rate r stays between a prespecified upper and lower bound. The upper and lower bound can be specified arbitrarily. However, as always, there is a trade off involved: the greater the possible fluctuations of r against which the portfolio should be immunized, the smaller the set of feasible portfolios and the higher the initial investment will be.

The risk control model generates a portfolio which is immunized against any series of shifts of the yield curve as long as:

$$r_t^{min} \leq r_t \leq r_t^{max} \qquad t=1,\ldots,T-1$$

where r_t = one period interest rate at time t

r_t^{min} = a prespecified lower bound on r_t

r_t^{max} = a prespecified upper bound on r_t

For notational convenience we shall assume that borrowing and lending at time t can be done against the same interest rate r_t. This assumption will be dropped later on.

Let $V(t,r,x)$ be the cumulative cash position resulting from portfolio revenues, liability payments and one period borrowing/lending at time t, given some interest rate scenario r, an initial portfolio x and the investor's strategy specified above:

$$V(t,r,x) = a_t x - b_t + \sum_{j=1}^{t-1} (a_j x - b_j) \prod_{\tau=j}^{t-1} (1+r_\tau)$$

Then the investor will have met all liabilities, without ending up with any debts if $V(T,r,x) \geq 0$.

Now construct a portfolio that satisfies the following linear constraints:

(III) $$V(T,r,x) \geq 0 \ \forall \ r \in S^*$$
$$x \geq 0$$
$$\text{with } S^* = \{r \in \mathbb{R}^{T-1} \mid r_t \in \{r_t^{min}, r_t^{max}\}, t = 1,\ldots,T-1\}$$

Then $V(T,r,x) \geq 0 \ \forall \ r \in S$ with $S = \{r \in \mathbb{R}^T \mid r_t^{min} \leq r_t \leq r_t^{max}, t = 1,\ldots,T-1\}$

Proof.

From the definition of $V(t,r,x)$ it follows that

$$V(t,r,x) = (1+r_{t-1})\ V(t-1,r,x) + a_t x - b_t \quad t = 2,\ldots,T$$
$$V(1,r,x) = a_1 x - b_1$$

And hence for $r_t^{min} \leq r_t \leq r_t^{max}$

$$V(t,r,x) \geq \begin{cases} (1+r_{t-1}^{max})V(t-1,r,x) + a_t x - b_t \text{ if } V(t-1,r,x) < 0 \\ \\ (1+r_{t-1}^{min})V(t-1,r,x) + a_t x - b_t \text{ if } V(t-1,r,x) \geq 0 \end{cases} \qquad t = 2,\ldots,T$$

So there exists an interest rate scenario $r^* \in S^*$ such that $V(T,r^*,x) \leq V(T,r,x) \ \forall \ r \in S$, and since x was chosen such that $V(T,r,x) \geq 0 \ \forall \ r \in S^*$ it follows that $V(T,r,x) \geq V(T,r^*,x) \geq 0 \ \forall \ r \in S$.

□

Now let r_t^{min} and r_t^{max} be specified such that r_t^{min} and r_t^{max} can be percepted as a lower bound on the reinvestment rate and an upper bound on the borrowing rate. Then conditions (III) combined with the objective of minimizing the initial investment can be formulated as a linear program:

(IV) $\min_x cx$

s.t. $V(T,r,x) \geq 0 \quad \forall\, r \in S^*$

$x \geq 0$

with $S^* = \{r \in \mathbb{R}^{T-1} \mid r_t \in \{r_t^{min}, r_t^{max}\},\ t = 1,\ldots,T-1\}$[1]

If the upper and lower bounds on the future interest rates were specified correctly, then the investor's strategy combined with the portfolio x, guarantees that all liabilities will be met and that a nonnegative net value will remain at the planning horizon.

So, if the upper and lower bounds are specified such that the probability that interest rates violate the upper or lower bound equals α in each period, then a portfolio x, constructed by the program presented earlier, will have the property that there exists a strategy such that the probability of ending up with a deficit at the planning horizon is less than or equal to α.

How to specify the upper and lower bounds such that the probability of ending up with a deficit equals some desired probability that is implied by the risk/return attitude of the investor? The answer to this question is dependent upon one's belief in methods to model and forecast future interest rates.

One could provide the decision maker with a series of optimal portfolios corresponding to different upper/lower bound interest rate scenarios. The portfolios will be priced differently and the investor can determine which of the (price, interest rate scenario) combinations he likes best. Thus, the investor specifies for what additional certainty (in his perception) he still wants to pay and at what point he does not value additional guarantees any

[1] Model (IV) is less attractive from a computational point of view: the set S^* consist of 2^{T-1} potential worst case scenario's. This implies that the number of constraints of model (IV) will rise exponentially fast with T, the number time periods. The authors have a paper in preparation in which it will be shown that that an equivalent model can be formulated, such that the number of constraints and the number of variables are linear T.

more.

Another approach to this question would be to model the interest rate movements by a stochastic process, estimate the process parameters and compute the upper and lower bounds such that, given the stochastic process, α takes on the desired value. One could for instance use the term structure models proposed in [Vasicek 1977], [Cox, Ingersoll and Ross 1985] or [Ho and Lee 1986] to determine the upper and lower bounds of future interest rates in a systematic and consistent manner.

3.3 The risk control strategy in relation to cash flow matching

The basic formulation of a cash flow matching model has been presented in the previous chapter. Assuming a return on one period reinvestment equal to r^c, a cash flow matching model can be formulated as:

$$\text{(V)} \quad \min \; cx$$

$$\text{s.t.} \qquad \sum_{t=1}^{T} (1+r^c)^{t-\tau} (a_t x - b_t) \geq 0 \qquad\qquad t=1,\ldots,T$$

$$x \geq 0$$

The risk control model minimizes the same objective function as the cash flow matching model. By setting r_t^{max}, the worst case borrowing rate at time t, equal to infinity, the optimal solution will not allow for cash shortages at any moment: costs of borrowing would be infinite. Now choose r_t^{min} equal to r^c for all t. Then any feasible solution to the cash matching problem is also a feasible solution to risk control model. Furthermore, it can be proven that the above choice of r_t^{min} and r_t^{max} forces optimal solutions to the risk control model to be feasible to model (V). Consequently, if x^* is an optimal solution to either the cash flow matching model or the risk control model, then x^* is an optimal solution to the other model as well. The following theorem states the relation between the cash flow matching model and the risk control model in a more rigorous way.

Theorem. Let $r_t^{max} = \infty$ and $r_t^{min} = r^c$. If there exists a feasible solution to (V), then x^* is an optimal solution to (V) if and only if x^* is an optimal solution to (IV).

Proof.
Let x^c be a feasible solution to (V), then $\sum_{\tau=1}^{t}(1+r^c)^{t-\tau}(a_t x - b_t) \geq 0,\ t = 1,\ldots,T \Rightarrow a_t x^c - b_t + \sum_{\tau=1}^{t-1}(a_\tau x^c - b_\tau)(1+r^c)^{t-\tau} = V(t,r^c,x^c) \geq 0, t = 1,\ldots,T$.
Since $r_t^{min} = r_t^c$ and $V(t,r^c,x^c) \geq 0\ \forall\ t$, it follows that $V(t,r,x^c) \geq V(t,r^{min},x^c)\ \forall\ r \in S^* \Rightarrow V(T,r,x^c) \geq 0\ \forall\ r \in S^* \Rightarrow x^c$ is a feasible solution to (IV).

Let x be an optimal solution to (IV) that is infeasible to (V). Then for some index $t \in 1,.., T-1$, $V(t,r^c,x) < 0$ and $r^c \in S^* \Rightarrow V(t,r^*,x) < 0$. Since x is a feasible solution to (IV), $V(T,r^*,x) \geq 0 \Rightarrow \exists$ an index $s \leq T-1$ such that $V(s,r^*,x) < 0$ and $V(t,r^*,x) \geq 0\ \forall\ t \geq s+1$. This implies:

$$V(s+1,r^*,x) = V(s,r^*,x)(1+r_s^{max}) + a_{s+1}x - b_{s+1} \geq 0 \Rightarrow$$

$$a_{s+1}x \geq b_{s+1} - (1+r_s^{max})V(s,r^*,x) \Rightarrow$$

$$\sum_{i=1}^{n} x_i \geq \frac{b_{s+1} - (1+r_s^{max})V(s,r^*,x)}{\max_i\{a_{i+1}\}} \Rightarrow$$

$$cx \geq \frac{\min_i\{c_i\}}{\max_i\{a_{s+1,i}\}}\left[b_{s+1} - (1+r_s^{max})V(s,r^*,x)\right] = \infty > cx^c \Rightarrow$$

x cannot be an optimal solution to (IV).

Thus, any feasible solution to (V) is feasible to (IV) and any optimal solution to (IV) is feasible to (V). Hence x^* is an optimal solution to (IV) if and only if x^* is also optimal to (V).

□

4 Concluding remarks

The risk control approach enables the investor to determine a minimum cost portfolio of fixed income securities. The portfolio has the property that revenues from coupon payments and redemptions, plus revenues from future reinvestments minus costs of borrowing will suffice to meet a stream of known liabilities. This can be guaranteed for any series of future changes of the term structure, as long as there remains an opportunity to borrow and lend, during the planning period, against an interest rate that fluctuates between upper and lower bounds which may be chosen freely by the investor. In contrast with the cash flow matching approach and the immunization approach, the risk control model offers the investor the possibility to specify against which set of future interest rate developments he wants to be insured.

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