

An Actuarial Approach to Option Pricing

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Summary

This paper proposes a direct, actuarial, approach to the question of option pricing.

With reference to a European call option, under normal conditions, the behaviour of a writer is like that of an insurer making probabilistic estimates of the value of a share at expiration and deducing the price at which the option should be offered. This paper presents a method of constructing the random share value variable based on a three-state Markov process and numerical aspects of the option price from the point of view both of "equity" and "expected utility".

The aim of our work is to formulate a model which will enable the writer or the buyer to decide whether the market price of the option corresponds to his expectations about the share's performance.

Résumé

Une Approche Actuarielle au Prix des Options

Cet article propose une approche actuarielle directe à la question du prix des options.

En ce qui concerne une option d'achat européenne, dans des conditions normales, l'attitude d'un souscripteur est la même que celle d'un assureur qui fait des estimations probabilistes de la valeur d'une action à l'échéance et déduit le prix auquel l'option devrait être offerte. Cet article présente une méthode pour construire la variable aléatoire d'une valeur d'action basée sur un procédé de Markov sur trois Etats. Cet article présente aussi par des aspects numériques le prix des options du point de vue à la fois des "fonds propres" (equity) et de l'utilité attendue.

L'objectif de notre travail est de formuler un modèle qui permettra au souscripteur ou à l'acheteur de décider si le prix du marché de l'option correspond à ses attentes sur la performance de l'action.

1. Option pricing by means of theoretical models is of great practical importance.

Estimates provided by models have the advantage of indicating a rational basis for the construction of the actual price of the option. But the actual price has to take into account a large number of relevant factors which theoretical models purposely leave out for the sake of formal simplicity.

A theoretical model cannot, by its very nature, provide immediate practical solutions, but it is indispensable for their construction.

To introduce the considerations with which this paper deals, we shall illustrate a typical approach to option pricing, the well-known one proposed by J. Cox and M. Rubinstein (1).

The authors simplify the problem as much as possible in order to make the negotiation straightforward and clear. The option in question is a European call with a price, to be established, of C (2). The underlying stock has an initial value of S and a striking price of K . The expiration date t is just one period away. At the end of this period, on expiration, the share can take on only two values (this is a fundamental condition): dS and uS , where d and u stand in the following relation to the capitalisation factor of the operating period t , $r_t = 1+i_t$:

$$d < r_t < u .$$

At expiration the value of the call (that is to say, the buyer's economic result) may therefore be of two types only:

- if the share has fallen: $C_d = \max(0; dS - K)$

- if the share has risen: $C_u = \max(0; uS - K)$.

Now consider the case of an investor who buys a portfolio consisting of n shares (of the type subject to call) and riskless bonds (with an interest rate of i_t over the period t) for the sum of N .

This portfolio has an initial value of P given by

$$P = nS + N$$

while the expiration value t may be

- in the event of a fall: $P_d = n d S + N r_t$

(1) The model was first proposed by the authors in 1979. In this paper we refer to the version presented in their more recent work "Options Markets", published by Prentice-Hall in 1985.

(2) The usual conditions apply: the stock pays no dividends; there are no taxes or transaction costs; the market is competitive; the interest rate is constant.

- in the event of a rise: $P_u = n u S + N r_t$.

The portfolio will be "equivalent" to the call, that is to say it will have exactly the same values as the call in the event both of a fall and a rise, as soon as n and N are such that

$$\begin{aligned} P_d &= C_d \\ P_u &= C_u \end{aligned}$$

which implies that

$$\begin{aligned} n &= (C_u - C_d) / (S(u-d)) \\ N &= (uC_d - dC_u) / ((u-d)r_t) \end{aligned}$$

where, as may easily be demonstrated, n proves to be non-negative and N non-positive.

Therefore the initial value P of the equivalent portfolio, achieved through the purchase of n shares and the sale of an amount -N in riskless bonds is given by

$$P = nS + N = \frac{1}{r_t} \left[\left(\frac{u-r_t}{u-d} \right) C_d + \left(\frac{r_t-d}{u-d} \right) C_u \right]$$

and, from $(u-r_t)/(u-d) = \alpha (>0)$ and $(r_t-d)/(u-d) = \beta (>0)$, the result is

$$P = (\alpha C_d + \beta C_u) / r_t$$

As in the definition of the equivalence between portfolio and call the two prices must coincide, the desired call price C is expressed by

$$C = (\alpha C_d + \beta C_u) / r_t$$

We have thus obtained the call price over a single period.

Dividing the period t into two or more periods gives the well-known Cox and Rubinstein formula, which is similar in form to Bernoulli's binomial distribution formula.

It may also be observed that if the division is intensified to the extent of making the individual operating periods tend to zero, there ensues a formulation of the problem in continuous time which leads - under appropriate conditions which render the model a highly particular one - to F. Black and M. Scholes' well-known formula (3).

- (3) The proof is contained in the worked mentioned in note (1). The procedure with which Black and Scholes obtained their formula was completely different and much more complex. The convergence between the two is in accordance with the observation the two models yield very similar numerical values and also makes it clear that they have similar conceptual bases.

What interests us here, however, is the interpretation of the elementary call formula just obtained. The first thing to observe is that the following applies:

$$\alpha + \beta = 1$$

It follows from this that C is obtained as the present value of a weighted average, with weights α and β , of the two values C_d and C_u .

This weighted average coincides with the expectation of the random variable "value of the call on its expiration date" if α and β is considered as the probability of fall and rise of the share respectively.

Another relationship also applies and may be immediately demonstrated:

$$\alpha d + \beta u = r_t$$

which tells us that, once factors d and u have been fixed, the evaluation of the probability of the alternative events α and β is precisely such as to establish "on average" parity with the rate of return of the riskless bond. The condition of no arbitrage therefore translates into the hypothesis that the share yields on average as much as the riskless bond.

This, then, is a very particular situation. In this case the interpretation of α and β as probabilities is perfectly correct. It is not correct in all the other cases in which α and β do not correspond to the probability values given to the fall and rise of the share.

2. Let us consider the position of the who sells a call, that is to say, the writer. When he establishes the price of a call the writer thinks like an insurer, he thinks in actuarial terms, that is, he uses probabilistic methods.

Let us return to the system with two alternatives elaborated above. Once d and u are fixed, the (subjective) probabilities of fall and rise are evaluated by the writer - on the basis of his knowledge and experience - as q and p (obviously $p+q=1$). The present value of the expectation of the final value of the call is

$$C' = (q C_d + p C_u) / r_t .$$

If the buyer were not prepared to pay more than the price C (equal to the value P of the equivalent portfolio) and this price were lower than C' , there would be no agreement between buyer and writer.

These considerations based on an extremely simplified system highlight the conceptual difference between the purely economic approach described above and the actuarial approach. From a practical point of view, however, the simplified system represents a pure abstraction. Even in

the case of very short periods, it is realistic to postulate a wide variation of values (not only two) that the share may take on at expiration. In this case the construction of the equivalent portfolio becomes virtually impossible and therefore the same is true of this method of calculating the option price.

With the actuarial method the price can always be quantified by means of the consideration of a random variable (continuous or discrete) designed to represent the value of the share on expiration. It is to this question that we turn our attention in the following section.

3. If we knew the future value of the share we would already have solved the problem of establishing the price of the option. In the case of a call it would be sufficient to calculate the present value of the maximum between 0 and the difference between the value of the share and striking price of the option. As said above, we designate the future price of the share as a random variable. The random variable "share value after m periods of time" is thus indicated by $x^{(m)}$ and its distribution of probability by $F_{x^{(m)}}$

4. We now turn to the random variable "performance of the writer of a call after m periods", indicated by $y^{(m)}$. Bearing in mind that the buyer has no reason to exercise his right if the share price on expiration is lower than the striking price (K), $y^{(m)}$ is defined as follows:

$$y^{(m)} = \begin{cases} 0 & \text{if } x^{(m)} < K \\ x^{(m)} - K & \text{if } x^{(m)} \geq K \end{cases}$$

The distribution function of the random variable $y^{(m)}$ is therefore

$$F_{y^{(m)}}(x) = \begin{cases} 0 & \text{if } x < 0 \\ F_{x^{(m)}}(K+x) & \text{if } x \geq 0 \end{cases}$$

It is now possible to construct the expectation and the variance of $y^{(m)}$. The expectation will in general be given by:

$$E(y^{(m)}) = \int_0^{+\infty} s \, d F_{y^{(m)}}(s) = \int_0^{+\infty} s \, d F_{x^{(m)}}(s+K)$$

proceeding by substitution, and considering $z=s+K$ it is possible to rewrite $E(Y^{(m)})$ as

$$E(Y^{(m)}) = \int_K^{+\infty} z \, dF_{X^{(m)}}(z) - K + K F_{X^{(m)}}(K)$$

Turning now to the calculation of the variance given $E(Y^{(m)})=M$, the result is:

$$\begin{aligned} \text{var } Y^{(m)} &= \int_0^{+\infty} (s-M)^2 \, dF_{Y^{(m)}}(s) \\ &= \int_0^{+\infty} (s-M)^2 \, dF_{X^{(m)}}(s+K) \end{aligned}$$

proceeding in a way similar to that followed in the case of the expectation, as $z=s+K$ the following final relationship is obtained:

$$\text{var } Y^{(m)} = M^2 F_{X^{(m)}}(K) + \int_K^{+\infty} (z-(K+M))^2 \, dF_{X^{(m)}}(z)$$

Considering the position of the writer of a put we can repeat for the random variable "performance of the writer of a put on expiration m " similar considerations to those laid out above.

The random variable in question, indicated by $Z^{(m)}$, is defined by

$$Z^{(m)} = \begin{cases} K - X^{(m)} & \text{if } X^{(m)} \leq K \\ 0 & \text{if } X^{(m)} > K \end{cases}$$

The distribution function thus turns out to be

$$F_{Z^{(m)}}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - F_{X^{(m)}}(K-x) & \text{if } x \geq 0 \end{cases}$$

5. It may be observed at this point how the fair value of the option call (P_C) is given by the present value of the expectation of the random variable $Y^{(m)}$, which, it should be remembered, represents the random performance of the writer on expiration. Bearing in mind that t represents the time remaining before the expiration of the option the following is obtained:

$$\begin{aligned} P_c &= E(e^{-\delta t} Y^{(m)}) \\ &= e^{-\delta t} \left[\int_K^{+\infty} z \, dF_{X^{(m)}}(z) - K + K F_{X^{(m)}}(K) \right] \end{aligned}$$

It is interesting to observe the perfect similarity between fair option price and fair insurance premium. From this point of view writer and buyer are exactly the same as insurer and insured, and the risk which passes from the latter to the former is the risk inherent in the fluctuations of the stock market.

The put price may now be immediately formulated

$$P_p = E(e^{-\delta t} z^{(m)}) \\ = e^{-\delta t} \left[K F_{X^{(m)}}(K) - \int_0^K h dF_{X^{(m)}}(h) \right]$$

and

$$P_e - P_p = p_A^{(m)} - e^{-\delta t} K$$

which, given

$$p_A^{(m)} = e^{-\delta t} \int_0^{\infty} x dF_{X^{(m)}}(x)$$

where $p_A^{(0)}$ represents the present value of the share price, reproduces the put-call parity relationship.

6. Nothing has yet been said about the distribution of the random variable $X^{(m)}$ (from which $Y^{(m)}$ is derived). We now propose to formulate a model which will enable the writer or the buyer to decide whether the market price of the option corresponds to his expectations about the share's performance.

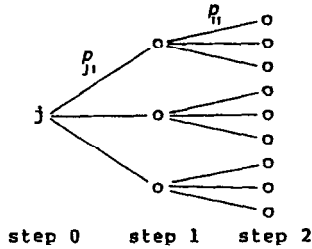
The stochastic process chosen to obtain the random variable $X^{(m)}$ "value of the share after m periods" is a particular discrete first order three-state Markov process. It was chosen in order to verify the hypothesis that the price of share is not independent from its history - this hypothesis appeared more realistic than the widely accepted one of stochastic independence.

The three states of the process represent a fall (state 1), parity (state 2) and rise (state 3) in the price of the share. The process will be known once the initial distribution vector \underline{a}_0 and the matrix of transition probabilities have been determined

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix}$$

the meanings of the symbols are self explanatory.

The process may be graphically represented by a three-way tree:



$$j = \{ 1; 2; 3 \}$$

The probability of the entire walk leading to each node is given by the product of the probabilities on each segment of the walk. The determination of the random variable $x^{(m)}$ are assumed to be equal to those of the previous node multiplied by factors R_1, R_2 or R_3 according to the state of arrival. R_1, R_2 and R_3 ($R_1 < R_2 < R_3$ and $R_2 \sim 1$) stand for the variations of the share in one step. It can be seen that the initial distribution is degenerate in that the state is known at step one, as is the value of the share (S) at the corresponding node (the root of the tree).

Using m_1, m_2 and m_3 to indicate the number of times each state 1, 2 and 3 ($m = m_1 + m_2 + m_3$) has been passed through the general determination will be

$$S R_1^{m_1} R_2^{m_2} R_3^{m_3}$$

associated with the probability:

$$\sum_{h_1} \sum_{h_2} \dots \sum_{h_m} P_{j, h_1} P_{h_1, h_2} \dots P_{h_{m-1}, h_m}$$

$$j, h_1, \dots, h_m = \begin{cases} 1 \\ 2 \\ 3 \end{cases}$$

$h_k = 1$ m_1 times
 $h_k = 2$ m_2 times $\forall k$
 $h_k = 3$ m_3 times

Note that the number of distinct walks is 3^m , but

$$\frac{m!}{m_1! m_2! m_3!}$$

lead to the same determination

$$S R_1^{m_1} R_2^{m_2} R_3^{m_3}$$

and the number of distinct determinations is $3+(m-1)+(m+(m-1)+\dots 2)$.

From an operational point of view the problem is that of assigning numerical values both to the transition matrix and to factors R_1, R_2 and R_3 . This is solved by calculating a succession of quotients on past weeks' closures of the Stock Exchange:

$$q_h = \frac{a_h}{a_{h-1}} \quad h=1, \dots, n$$

a_h = week' closure

n = number of weeks' closures observed

These quotients are divided into three classes according to the rule:

$$q_h < \frac{1}{1+\epsilon} \quad \text{I class}$$

$$\frac{1}{1+\epsilon} \leq q_h \leq 1+\epsilon \quad \text{II class}$$

$$1+\epsilon < q_h \quad \text{III class}$$

It is therefore the choice of ϵ that will determine the fluctuation period within which the variation of the share is considered so weak as to be insignificant (parity).

Given n_1, n_2 and n_3 = number of q_h of classes 1, 2 and 3 and q_h^1, q_h^2 and q_h^3 general quotients of classes 1, 2 and 3 it is possible to calculate factors R_1, R_2 and R_3 as a geometrical average as follows:

$$R_1 = \sqrt[n_1]{\prod_{h=1}^{n_1} q_h^1}$$

$$R_2 = \sqrt[n_2]{\prod_{h=1}^{n_2} q_h^2}$$

$$R_3 = \sqrt[n_3]{\prod_{h=1}^{n_3} q_h^3}$$

Having thus determined the historical walk and knowing the percentages of the various classes it is possible to assign values to the transition matrix P.

It should be remembered that in our subjective formulation (according to de Finetti's view) it is always the person deciding who fixes the probabilities. In this case, therefore, matrix P obtained as above may be adjusted by the writer who decides another fundamental element: the length of the period of observation.

The idea of the whole model is to give an evaluation of the option price taking account of information acquired: it must be possible to adjust continuously our valuation immediately after a new information (for example a change of the share's price).

7. In order to give concrete form to this model we have written a series of programmes which are run on an I.B.M. 4381 at LLOYD ADRIATICO S.p.A., using APL language. The structure of the programmes will be described in another publication - the main problems derived from the large number of determinations of the random variable $X^{(m)}$. Some numerical results, however, are presented here in tables 1 and 2.

It may again be observed that the distributions obtained with the share examined prove to be fairly close to normal distributions.

It should be pointed out that at present there is not an Option market in Italy (although it soon will be); nevertheless we have calculated the call prices for some shares using our model.

As starting data for the calculation of factors R_1 , R_2 and R_3 and matrix P we used the figures of the weekly closures of the first seven months of 1989 on the Milan Stock Exchange. Below are the calculations carried out on the shares of three important companies on the Italian market.

GENERALI

-Price on	18.8.89	45.800
-Striking price		45.800
- Fair price and variance of the three-month call		
	1.462	3.961.149

FIAT

-Price on	18.8.89	11.450
-Striking price		11.450
- Fair price and variance of the three-month call		
	874	586.504

FERFIN

-Price on	18.8.89	3.370
-Striking price		3.370
- Fair price and variance of the three-month call		
	176	29.819

Tab. 1

Up to now we have always spoken in terms of the fair value of the call, calculated with the criterion of expected value. But remembering once more the similarity between the writer and the insurer it is natural to introduce a safety loading.

Making reference to utility theory, in particular assuming a preferibility function $E(X) - a\text{Var}(X)$, the call price (P_C) is determined as follows:

$$\tilde{P}_c = e^{-\delta t} E(y^{(m)}) + a e^{-2\delta t} \text{var } y^{(m)}$$

in which case the resulting values for the risk-aversion parameter $a = 0,0001$ are:

GENERALI

-Price on 18.8.89	45.800
-Striking price	45.800
-Risk-aversion a	0,0001
-Price (P_C) of the three-month call	1.800

FIAT

-Price on 18.8.89	11.450
-Striking price	11.450
-Risk-aversion a	0,0001
-Price (P_C) of the three-month call	930

FERFIN

-Price on 18.8.89	3.370
-Striking price	3.370
-Risk-aversion a	0,0001
-Price (P_C) of the three-month call	178

Tab. 2

Now we are working on the Milan Stock Exchange to estimate the parameters both to obtain a good approximation of the future values of the share and to examine the difference between the results of the classical formulae of Option Pricing Theory and our model.

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