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Consequences of the Variations in the Rate of Return on the Financial Equilibrium of a Pension Fund

S. M. Coppini

Via Flaminia 21, 00196 Roma, Italy

Summary

The use of an approximate formula is suggested in order to estimate the ratio between funds and actuarial reserves of a pension fund varying the expected investment yield. It is shown that the same formula allows the inverse problem to be solved, that is to calculate the variation in the yield on assets, having set up the ratio between funds and reserves which one intends to achieve. The purpose is to integrate the technical management of the pension fund and the financial management of the reserves, providing certain parameters within which the optimal investment policy may be sought.

Résumé

Conséquences des Variations du Taux de Rendement sur l'Equilibre Financier d'un Fond de Retraite

On suggère l'utilisation d'une formule approximative pour estimer le rapport entre les fonds et les réserves actuarielles d'un fonds de retraite en variant le rendement d'investissement attendu. On démontre que la même formule permet de résoudre le problème inverse, c'est-à-dire de calculer la variation du rendement des actifs, ayant fixé le rapport entre les fonds et les réserves que l'on se propose d'atteindre. L'objectif est d'intégrer la gestion technique du fonds de retraite et la gestion financière des réserves, et de fournir quelques paramètres dans lesquels la politique optimale des investissements peut être recherchée.

1. The investiments' management of a pension fund is somewhat complex and it may deal with various ways. There are some specific features of the investments such as safety, duration that often impose different choices from those oriented to achieve maximun investment income.
Much attention has been payed to this topic in the british literature with reference to a general short-term investment model: see for example references (1).
An analysis of the consequences on the technical management of a pension fund of possible variations of the actual investment experience from that assumed, is the goal of this paper.
As it is well known to determine premium rates which are function of the average yield expected over a medium-long term, it is necessary to adopt a conservative basis for the rate of interest which almost always will not occur in reality. When the actual rates are known it is necessary review the progress of a pension fund, to adjust the contributions and the benefit. If it is not allowed for changes in the rate of interest the pension fund, in relation to the financing system adopted, is not in equilibrium. In this case, if V (t) denotes the reserve at time t calculated on the basis of the expected interest rate (or rates) and V'(t) denotes the funds that would be accumulated with the actual rates of return, the ratio

$$\gamma(t) = \frac{V'(t)}{V(t)}$$

gives an index of the increase or decrease of the "guarantee" provided by the financing system.

In general the actuarial techniques allow to simulate, time by time, the value of $\gamma(t)$ in function of a change in the rate of interest. However the use of simulation techniques requires a lot of calculations and in any case it doesn't provide, unless various attempts are made, the solution of the inverse problem, that is to estimate the variation in the interest rate in function of a variation in $\gamma(t)$.
Following we suggest some simple formulae providing a solution of the two above mentioned problems and also some arguments to achieve further understanding of the effect of a change in the rate of interest.

2. As it is known the recurrence relation of the pension fund reserves may be written as follows:

$$(1) \qquad \frac{d\,V(t)}{dt} = \delta(t)\,V(t) + (\pi(t) - p(t))\,S(t)$$

where:

$p(t)$ denotes the rate of the benefits at time t per salary unit; (salary unit = 1 Lira)

$\pi(t)$ denotes the rate of the contributions at time t per salary unit;

$S(t)$ are total salaries at time t;

$\delta(t)$ is the force of interest at time t.

Formula (1) is a differential equation of the function V(t) which, for the initial value V(o), has the solution

$$(2) \qquad V(t) = \int_0^t (\pi(s) - p(s))\,S(s)\,e^{\int_s^t \delta(u)\,du}\,ds + V(0)\,e^{\int_0^t \delta(u)\,du}$$

3. We now shall try to solve the two problems in section 1, by introducing some hypothesis which, although simple, are more than acceptable taking in account of the wide margin of approximation within which the variation in the interest rate is expected.

We shall assume that:

- the flat force of interest used in the beginning in the valuations is δ ;
- the effective force of interest (or the expected one) is varying in arithmetic progression, that is, it can be expressed in the following form:

$$(3) \qquad \delta(t) = \delta - \alpha t \qquad \alpha \neq 0$$

We shall use the known formulae with which the definite integral of the product of two functions is calculated throughout the approximate relation:

$$(4) \qquad \int_a^b f(s)\, \phi(s)\, ds = G \int_a^b \phi(s)\, ds$$

where G is calculated through determided values of $f(s)$ and moments of $\phi(s)$.
There are several examples of these formulae (see references (2), (3)), we shall apply one that is simple to use.
At this stage it is worth considering how the hypothesis (3) are not strictly compulsory because the procedure, that follows, is readily extended to more complex assumptions.

4. By hypothesis (3), given:

$$K(s) = (\pi(s) - p(s))\, S(s) \qquad (5) \qquad (*)$$

and hence the (2) becomes:

$$V'(t) = \int_0^t K(s)\, e^{\delta(t-s)}\, e^{-(1/2)\alpha(t^2-s^2)}\, ds + V(0)\, e^{\delta t}\, e^{-(1/2)\alpha t^2} \qquad (6)$$

Let:

$$\phi(s) = K(s)\, e^{\alpha(t-s)}$$

(7)

$$f(s) = e^{-(1/2)\alpha(t^2-s^2)}$$

it follows that for $V(0) = 0$

$$\gamma(t) = \frac{V'(t)}{V(t)} = G \qquad (8)$$

(*) We shall now suppose that the values of $K(s)$ have all the same sign; in case of a variation in the sign, the procedure may be used considering separately the positive values and the negative ones.

In a analogous way, if $V(0) \neq 0$

given: $${}^{0}V(t) = V(0)\, e^{\delta t}$$

we obtain:

$$(9) \qquad \gamma(t) = \frac{G\left(V(t) - {}^{0}V(t)\right) + {}^{0}V(t)\, e^{-(1/2)\alpha t^2}}{V(t)} =$$

$$= G + \left(e^{-(1/2)\alpha t^2} - G\right){}^{0}\nu(t)$$

where:

$${}^{0}\nu(t) = \frac{{}^{0}V(t)}{V(t)}$$

Note that (8) and (9) allow to solve the direct problem of the calculation of $\gamma(t)$ knowing the value of α and when the value of G is defined. For the inverse problem the solution will be more simple or less simple depending on the expression of G.

5. We have already said that many expressions of the value of G have been given. They allow to achieve different approximations in the calculations.
One of the suggested approaches consists in writing:

$$(10) \qquad G = a_0\, f(0) + a_1\, f(t/2) + a_2\, f(t)$$

where the weights a_i are functions of the moments of the $\phi(s)$ and that is:*

$$a_0 = \frac{t^2 - 3m_1 t + 2m_2}{t^2}$$

(11)
$$a_1 = - \frac{- 4m_1 t + 4m_2}{t^2}$$

$$a_2 = \frac{- m_1 t + 2m_2}{t^2}$$

where: $a_0 + a_1 + a_2 = 1$

being:

(12)
$$m_r = \frac{\int_0^t s^r K(s)\, e^{\delta(t-s)}\, ds}{\int_0^t K(s)\, e^{\delta(t-s)}\, ds} \qquad (r = 0, 1, ..)$$

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* The procedure is due to R. E. Beard (see reference 4). One should note that a simpler formulae exist for G: (as example $G = a_0\, f(0) + a_1\, f(t)$). They imply a lower degree of accuracy not acceptable for the problem examined. We will come back to this in the next paragraph.

Now, remembering (7) the risult will be:

$$(13)\qquad \begin{aligned} f(0) &= e^{-(1/2)\alpha t^2} ; \\ f(t/2) &= e^{-(1/2)\alpha (3/4)t^2} ; \\ f(t) &= 1 \end{aligned}$$

So, from (8) and (10) for $V(0) = 0$

$$(14)\quad \gamma(t) \cong a_0\, e^{-(1/2)\alpha t^2} + a_1\, e^{-(1/2)\alpha (3/4)t^2} + a_2$$

While for $V(0) \neq 0$ we obtain:

$$(15)\quad \gamma(t) = e^{-(1/2)\alpha t^2} (a_0 + {}^0\nu(t)) + + a_1\, e^{-(1/2)\alpha (3/4)\, t^2} + a_2 - {}^0\nu(t)$$

6. We will now go back to the argument on the sufficient approximation of the procedure used for the integration, to observe that the error included in (14) may be expressed through(*):

$$(16)\qquad R = \frac{f^{(3)}(\xi)}{3!}\, M\,(t) \qquad\qquad 0 < \xi < t$$

(*) See R.E. BEARD.

where:

$$M(t) = m_3 - (3/2)\, m_2\, t + (1/2)\, m_1\, t^2$$

by relation (7), we obtain:

$$R = \frac{\alpha^3 \rho^3 + 3\alpha^2 \rho}{6}\, e^{-(1/2)\alpha (t^2 - \rho^2)}\, M(t)$$

R is an increasing function over the interval (0,t) when is ρ increasing, for $\alpha > 0$.
For $\rho = t$ a first rough exstimation of the upper bound of the error commited using the (14) or (15) may be obtained and precisesly:

$$(17) \qquad \mathrm{Max}\ R = \frac{\alpha^3 t^3 + 3\alpha^2 t}{6}\, M(t)$$

As expected Max R depends on the moments, on α and t.
It is worth noting that the upper bound of the error is exstimated in the continuous field and it may vary when discrete evaluations are made. The same analysis may be made for $\alpha < 0$.

7. As for the inverse problem of the calculation of α if $V(0) = 0$,

given: $\qquad x = e^{-(1/2)\alpha t^2}$

(14) becomes:

$$(18) \qquad a_0\, x + a_1\, x^{(3/4)} + a_2 - \gamma(t) = 0$$

or, also, in a first approximation:

$$(19) \qquad a_0 x + a_1 (1 + (3/4)(x-1)) + a_2 - \gamma(t) = 0$$

and finally, solving the relation in respect of x

$$(20) \qquad x = \frac{\gamma(t) - a_2 - (1/4)\, a_1}{(3/4)\, a_1 + a_0}$$

and taking logarithms:

$$(21) \qquad \alpha = -(2/t^2) \ln \frac{\gamma(t) - a_2 - (1/4)\, a_1}{(3/4)\, a_1 + a_0}$$

Note that the passage from (18) to (19) can be done because, for the values which really concern, the approximation is irrelevant as one will better understand through the examples in the next paragraph.

We suppose now that $V(0) \neq 0$, (15) may be written:

$$(22) \quad x\,(a_0 + {}^0\nu(t)) + a_1 x^{3/4} + a_2 - {}^0\nu(t) - \gamma(t) = 0$$

and applying the semplification inserted in the (18):

$$(23) \quad x\,(a_0 + {}^0\nu(t)) + a_1 (1 + (3/4)(x-1)) + $$
$$+ a_2 - {}^0\nu(t) - \gamma(t) = 0$$

hence:

$$(24) \qquad x = \frac{\gamma(t) + {}^0\gamma(t) - a_2 - (1/4)\, a_1}{(3/4)\, a_1 + a_0 + {}^0\gamma(t)}$$

and taking logarithms:

$$(25) \qquad \alpha = - (2/t^2) \ln \frac{\gamma(t) + {}^0\gamma(t) - a_2 - (1/4)\, a_1}{(3/4)\, a_1 + a_0 + {}^0\gamma(t)}$$

8. We provide now a numerical example of what we have demonstrated.
Assuming first of all that the technical-financial management in the first ten years of a pension fund is described in the table 1, drawn by a concrete case. The choice of a period equal to 10 years, is justified by the fact that a greater interval should not allow any reasonable forecast on the investment yields, while a shorter period of time would imply less significant variations in the ratio between funds and actuarial reserves of the pension fund.
Table 1 sets forth the calculation of $\gamma(t)$. The first two columns show the projections of contributions income and of benefits expenses; the fourth column gives the interest income; the fifth column gives the funds accumulated year by year while the sixth column gives the ratio $\gamma(t)$. By projection assumptions the yield on assets is set at 9 percent constant in time while the contribution rate, variable from year to year, is calculated in such a way to ensure, for each year of the projection period, besides the benefit payment, funds equal to the actuarial reserves of the pension fund.
Suppose now a linear decrease in the yiels on asset is taking place to reduce it to half its value in ten years

(α = 0,005). The financial status of the plan varies as shown in table 2.
It is evident that the minor yield on assets causes, when the contributions are equal, a riduction of the accumulated assets from year to year and consequently a riduction in the ratio $\gamma(t)$ from 1 to 0, 8495 over ten years, as shown in the last column of table 2.
This example, obtained with the normal actuarial methods, allows now to verify what would have been the result, in terms of ratio assets and actuarial reserves at the tenth year if consideration is being given to the use of the procedure suggested before.
In this case, by formulae (11) and (13) it is easy to find:

$a_0 = 0,1847$ $\qquad$ $f(0) = 0,7788$

$a_1 = 0,6370$ $\qquad$ $f(t/2) = 0,8290$

$a_2 = 0,1783$ $\qquad$ $f(t) = 1,0000$

and, then, by applying the (14) we obtain:

$$\gamma(10) = 0,8502$$

It should be noted that the difference with the value 0,8495 previously calculated is very small, about + 0,0007.
We return now to the inverse problem, or the problem to determine α (see par. 7); the formulae (20) may be written:

$$X = \frac{0,8495 - 0,1783 - 0,25 \times 0,6370}{0,75 \times 0,6370 + 0,1847}$$

and, hence, by (21) we obtain:

$$\alpha = 0,0052$$

As the effective value of α is equal to 0,0050, the error is equal to 0,0002.

To illustrate the impact on the results of the choice of the value of α or, in the inverse problem, of $\gamma(t)$, we show the results obtained under different assumptions operating with the traditional methods or with those here suggested.

α	$\gamma(10)$		
	actual	estimated	error
- 0,010	1,3814	1,4097	+ 0,0283
- 0,005	1,1760	1,1838	+ 0,0078
- 0,001	1,0330	1,0338	+ 0,0008
+ 0,001	0,9680	0,9675	- 0,0005
+ 0,005	0,8495	0,8502	+ 0,0007
+ 0,010	0,7213	0,7281	+ 0,0068

As for the inverse problem the data are set forth in the following table:

$\gamma(10)$	α		
	actual	estimated	error
1,3000	- 0,0081	- 0,0075	+ 0,0006
1,2000	- 0,0056	- 0,0053	+ 0,0003
1,1000	- 0,0029	- 0,0028	+ 0,0001
0,9000	+ 0,0032	+ 0,0033	+ 0,0001
0,8000	+ 0,0068	+ 0,0072	+ 0,0004
0,7000	+ 0,0109	+ 0,0121	+ 0,0012

The assumptions made on α and on $\gamma(10)$ are comprehensive of a wide range of possible cases, and for all the hypothesis made as emphasited in error column, the estimation of the values through the suggested pro cedure is widely acceptable.

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Table 1. Forecast of contributions income (1), of benefits outgo (2), interest income (4) and assets (5). (1) and (2) fall at the middle of the year, (5) falls at the end of the year (amounts in millions of Lire).

Year of Forecast	Contributions (1)	Benefits (2)	(1)-(2) (3)	Interest Income (4)	Assets (5)	$\gamma(t)=\frac{\text{Assets}}{\text{Reserves}}$ (6)
1	6280,0(*)	259,0	6021,0	270,9	6291,9	1,00
2	1585,1	280,3	1304,8	625,0	8221,7	1,00
3	3615,7	331,7	3284,0	887,7	12393,4	1,00
4	3843,4	389,2	3454,2	1270,8	17118,4	1,00
5	4132,5	458,6	3673,9	1706,0	22498,3	1,00
6	4611,4	572,9	4038,5	2206,6	28743,4	1,00
7	4903,4	704,1	4199,3	2775,9	35718,6	1,00
8	5139,9	837,4	4302,5	3408,3	43429,4	1,00
9	5448,5	991,4	4457,1	4109,2	51995,7	1,00
10	6106,1	1176,3	4929,8	4901,5	61827,0	1,00

(*) By hypothesis the value of the initial reserves is included in this amount. This shall allow to use simpler formulae under the assumption $V(0) = 0$

Table 2. Forecast of contributions income (1), of benefits outgo (2), interest incme (4) and assets (5). (1) and (2) fall at the middle of the year, (5) falls at the end ofhe year (amounts in millions of Lire).

Year of Forecast	Contributions (1)	Benefits (2)	(1)-(2) (3)	Interest Income (4)	sset (5)	$\gamma(t)=\frac{\text{Assets}}{\text{Reserves}}$ (6)
1	6280,0	259,0	6021,0	263,4	614,4	0,9988
2	1585,1	280,3	1304,8	572,3	811,5	0,9927
3	3615,7	331,7	3284,0	759,8	1215,2	0,9848
4	3843,4	389,2	3454,2	1010,1	1619,5	0,9738
5	4132,5	458,6	3673,9	1249,2	2112,6	0,9597
6	4611,4	572,9	4038,5	1475,7	2716,8	0,9431
7	4903,4	704,1	4199,3	1679,4	3215,5	0,9235
8	5139,9	837,4	4302,5	1844,7	3912,7	0,9011
9	5448,5	991,4	4457,1	1964,7	4514,4	0,8761
10	6106,1	1176,3	4929,8	2040,8	5215,1	0,8495

R E F E R E N C E S

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