2nd AFIR Colloquium 1991, 1: 225-236

Yield Curve Fluctuations
Does French Market Fit the Ho and Lee's Model

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Summary

The development of futures contracts and of options on the French government bonds OAT indicates an active risk interest rates management. Valuation of these products can be realized with various models whose assumptions are not always verified.

In this paper we present empirical results on Hoo and Lee term structure model used for pricing interest rate contingent claims. Statistical tests have been performed in order to compare the assumptions of the model and the reality on the French Treasury market.

We outline the situations where the Ho and Lee's model can be used and where it can be misleading, we then try to devise the outlook for a candidate-model for the French yield curve flactuation.

Résumé

Fluctuations de la Courbe des Taux
Tests Empiriques du Modèle de Ho et Lee

Le développement de contrats "futures" et d'options sur les Obligations Assimilables du Trésor, montre l'intérêt porté pour une gestion active du risque de taux. L'évaluation de ces différents produits peut être réalisée à l'aide de nombreux modèles dont les hypothèses ne sont pas toujours verifiées.

Dans ce papier nous présentons des résultats empiriques sur le modéle de Ho et Lee, qui permet d'évaluer le prix des actifs contingents aux taux d'intérêt. Un ensemble de tests statistiques a été mis en oeuvre pour évaluer la vraissemblance des hypothèses du modèle sur le marché des titres du Trésor français.

Nous indiquons les configurations les plus appropriées pour l'utilisation du modèle de Ho et Lee et précisons ses limites. Nous traçons enfin le portrait-robot d'un bon candidat pour la modélisation de fluctuations de la courbe de taux française.

Yield curve fluctuations

Does French Market fit the Ho and Lee's Model ?

Introduction

Fixed income securities risk profile was often disregarded in France. The reason was that investors were used to follow buy-and-hold strategies. The implementation of modern auction techniques and the development of a futures contract on the French government bonds OAT have resulted in a very liquid market and generally more active fund management. Volatility could be seen as the price to pay, but historical data show that risk on interest rate instruments was there before MATIF ! The real evolution is more in the investor's mind : nowadays interest rate risk is no longer overlooked ; instead, duration and convexity are basic steps in fixed income investment. Among the many consequences of this major change, one has to realize that bond risk profile is not exactly the same as the one on ordinary stock and that dedicated risk management is necessary. Moreover options on interest rate, like caps and floors, are very useful for managing the risk exposure but they clearly cannot be priced directly from the bond volatility. So what ?

During the 80's a lot of theoretical work was carried out for modeling the interest rate risk. A complete literature review is beyond the scope of this paper (see Gibson [1] and Augros [2] (in French) for detailed discussions of the advantages and drawbacks of these models).

Let us just recall some of the most popular : Black [3] offered some sort of rough but simple approach ; Vasicek [4] gave the first way out of the Black and Scholes paved framework by changing the stochastic process describing the short term interest rate variations ; then came a two factors approach proposed by Brennan and Schwartz [5] in which the call money rate and the consol bond rate followed joint stochastic processes. Unfortunately this model is a too beautiful one : in addition to options pricing it also gives you the price of all bonds. In other words, the yield curve is an endogeneous factor of the model, a common feature with Vasicek's model [4]. Of course the market does not trade for the dream fulfilment of any modeler...and when trying the other way round to find out parameters of any model from the market prices, many difficulties arise and one is not even sure the model is right ! Eventually, Ho and Lee [6] proposed a very simple and astute model. Starting from the actual zero-coupons curve they make it move with the most simple model one can imagine - a binomial up and down fluctuation adapted for all the maturities. We shall stop here for a while - Augros[2] shows how efficient the approach could be for pricing the many different options one can find in our crazy financial markets. If it works so well we might really check out if the basic assumptions behind are true at least on the French OAT market.

The objective of this paper is then to test whether Ho and Lee's assumptions hold true. We did not try to make any estimation on the parameters of the model. The first section recalls briefly the model and its consequences for some related variables of interest. Section two presents the data sets and examines in detail the results of each empirical test we could implement. In the conclusion we try to devise the outlook for a candidate-model for the French yield curve fluctuation but also outline the situations where Ho and Lee's model can be used and when it can be misleading.

I The theorical framework of Ho and Lee's model.

A The basic assumptions

In this first part we recall the main hypothesis of Ho and Lee's model developed in [6]. We consider the financial market as a standard capital market with a discrete time approach.

i The market is frictionless. there are no taxes and no transaction costs. Anyone can borrow or lend at the market rates.

ii We consider the market at discrete points in time, sampled at regular intervals. The time unit period is denoted by δt.

iii The bond market is complete : there exists a discount bond for each maturity $n.\delta t$ $(n=1,2,...)$

iiii At each time $n\delta t$ there is a finite number of states of the world. For state i, we denote the equilibrium price of the discount bond of maturity T by $G_i^n(T)$. $G_i^n(.)$ is the discount function. It is decreasing and verifies [1]:

$$G_i^n(0)=1 \qquad \text{for all } i,n$$

$$\lim_{T\to\infty} G_i^n(T) = 0 \qquad \text{for all } i,n$$

the price of a discount bond and the continuous interest rate for the maturity T, are linked by the following equation

$$r(T) = -\frac{\log(G(T))}{T}$$

B The binomial lattice

Ho and Lee's model makes the initial zero coupon bond price as given. At time 0 we observe the discount function in the market. This data is exogenuous and enables us to take into account the implicit information of the initial curve through the forward rates. Secondly Ho and Lee let the entire zero-coupon bond prices fluctuate randomly across time, in an arbitrage-free manner, according to a single discrete time binomial process. This process is specified at time $n\delta t$ by the following up or down motion :

$$G_i^n(T) \begin{cases} G_{i+1}^{n+1}(T) = \dfrac{G_i^n(T+\delta t)}{G_i^n(\delta t)} \times h(T) \text{ with probability } p \\ G_i^{n+1}(T) = \dfrac{G_i^n(T+\delta t)}{G_i^n(\delta t)} \times h^*(T) \text{ with probability } 1-p \end{cases}$$

In Ho and Lee's model, $h(T)$ and $h^*(T)$ are perturbation functions depending only on the time to maturity T of the bond. They specify the deviation of the discount functions from the implied forward function. The binomial assumption requires the discount function attained by an upstate followed by a downstate to be equal to the discount function reached by a downstate followed by an upstate.

C The arbitrage-free model

Using the standard arbitrage-free condition, Ho and Lee find a restriction on the perturbation functions at each vertex (n,i). Specifically they found there must exists some constant π independent of time T and initial discount function G(.) such that

$$\pi_i^n h(T) + (1-\pi_i^n) h^*(T) = 1 \qquad \text{for all } T$$

The parameter π defines a probability measure change, where all the bonds have the same return for a single period. Using then the path independent condition, it can be shown that π is independent of the vertex (n,i) and we have the complete formulation of h(.) and $h^*(.)$

$$h(T) = \frac{1}{\pi+(1-\pi)\delta^T} \qquad h^*(T) = \frac{\delta^T}{\pi+(1-\pi)\delta^T} \qquad \text{for all } T$$

where δ is a strictly positive constant smaller than 1.

Therefore Ho and Lee's model is a three parameters model with

p : binomial probability of a rise of the discount function

π : implied binomial probability in a risk neutral world

δ : a risk parameter such that $0 \le \delta \le 1$, $\delta=1$ being the certainty case.

We observe that Ho and Lee's model is a single factor model where all default-free bonds of different maturities have perfectly correlated returns. However, unlike one factor models it does not attempt to endogenize the equilibrium term structure.

II Some facts implied by the model

Derivation of a new variable

Starting from the formula which gives the price of a discount bond of maturity T at the vertex (n,i), we derive a new variable Y, whose properties make it easy to test.

Using the previous notation we have

$$G_i^n(T) = \frac{G(T+n\delta t)}{G(n\delta t)} \cdot \frac{h(T+(n-1)\delta t)h(T+(n-2)\delta t)\ldots h(T)}{h((n-1)\delta t)h((n-2)\delta t)\ldots h(\delta t)} \cdot \delta^{T(n-i)}$$

then we set

$$Y = \log \frac{G_i^n(T)}{G(T+n\delta t)} \cdot G(n\delta t) = \log \frac{h(T+(n-1)\delta t)h(T+(n-2)\delta t)\ldots h(T)}{h((n-1)\delta t)h((n-2)\delta t)\ldots h(\delta t)} + (n-i)\log\delta$$

When we define Y with the bond prices, there are only observable terms, whereas if we introduce the factors h and h^*, the formula depends only on the parameters of the model. Setting

$$P(T,n) = \log\left(\frac{h(T+(n-1)\delta t)h(T+(n-2)\delta t)\ldots h(T)}{h((n-1)\delta t)h((n-2)\delta t)\ldots h(\delta t)} \right)$$

it can be proved that P(T,n) has a limit value P(T), when n goes to infinity. Moreover we have the following inequalities :

$$\int_0^T \log\left(1+\frac{(1-\pi)}{\pi}\delta^x\right)dx \le P(T) \le \int_0^T \log\left(1+\frac{(1-\pi)}{\pi}\delta^x\right)dx + \log\frac{1}{\pi+(1-\pi)\delta^T}$$

so we have

$$P(T) = \int_0^T \log(1+\frac{(1-\pi)}{\pi}\delta^x)dx + \varepsilon(\pi,T)$$

where the value of the last term $\varepsilon(\pi,T)$ is small compared to the integral (the error of approximation we found, was always smaller than 0.1% for realistic values of π, and δ close to 1).

We shall now give a financial interpretation of the variable Y

* Y as a function of the interest rates

We set : $R_T^{n,i}$ is the random future interest rate to borrow at time $n\delta t$ for a maturity T, in the state of the world i.

F_T^n the implied forward rate a time $n\delta t$ for a maturity T

we have :

$$G_i^n(T) = \exp(-T.R_T^{n,i}) \quad \text{and} \quad \frac{G(n\delta t)}{G(T+n\delta t)} = \exp(T.F_T^n)$$

we deduce that $Y = T.(F_T^n - R_T^{n,i})$

Therefore Y is a measure of the distance between the forward rate and the future rate.

* Y as an excess return

If we invest at time 0 for a period $n\delta t$, buying a discount bond of maturity $T+n\delta t$, the random return from our investment would be

$$\tilde{r} = \frac{G_i^n(T) - G(T+n\delta t)}{G(T+n\delta t)}$$

over the same period the riskless return is $r = \dfrac{1 - G(n\delta t)}{G(n\delta t)}$

but we have $Y = \log \frac{1+\tilde{r}}{1+r} \simeq \tilde{r} - r$ = *excess return*

Therefore, at the first order, we can identify the variable Y with the risk premium.

III Empirical Tests

a. Data sets

We have tested the Ho and Lee's model on the French treasury market. With these assets we eliminate any default risk. We have disposed of a weekly sample of French Treasury bonds, beginning in February 1987 and ending in January 1990. Then the discount function was computed according to the Vasicek-Fong model [7]. It gives a good approximation of zero coupon prices with maturities over one year.

b. Tests of the Normal distribution for the random variable Y

As the number of states of the world follows a binomial distribution, so does the variable Y. Then, when the value of n is large enough, it can be approximated by a Normal distribution, with mean $\mu(T)$ and variance $\sigma^2(T)$ functions of time to maturity T.
A brief computation gives

$$\mu(T) = \int_0^T \ell og\left(1+\frac{(1-\pi)}{\pi}\delta^x\right)dx + \varepsilon(p,T) + T.n.p.\ell og(\delta)$$

$$\sigma^2(T) = (T.\ell og(\delta))^2 n.p(1-p)$$

Using the Chi-squared statistic, we have tested the Normal hypothesis with a 1% confidence level for all the maturities. Results, reported in fig n° 1 and 2 show that it is not possible to reject the Normal hypothesis for maturities over 5 years. But for short term maturities, tests values are beyond the 1% critical value. On the other hand, using the Kolmogorov-Smirnov test, we can't reject the Normal hypothesis, whatever the maturities.

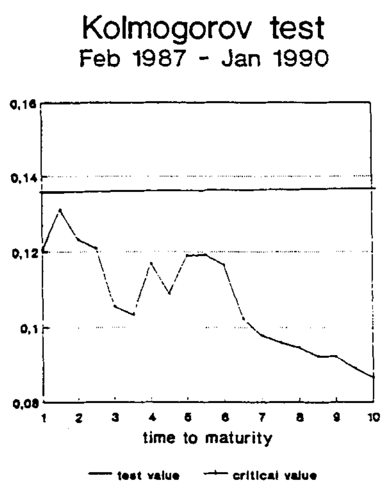

Fig.1

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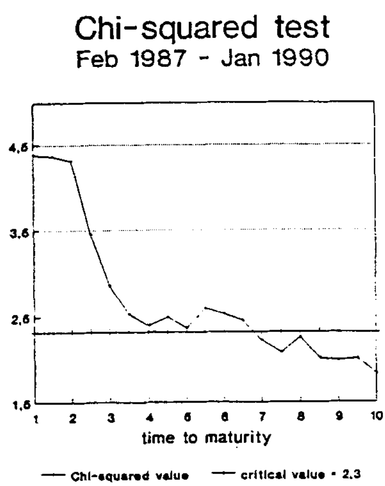

Fig.2

c. The variable Y as a function of the maturity T

We now make the assumption that the parameter δ is very close to 1 (when we estimate the value of δ, we always find $\delta > 0.95$). In the formula giving the mean $\mu(T)$ of the variable Y, we can then approximate the expression

$$\int_0^T \ell og(1+\frac{1-\pi}{\pi}.\ \delta^x)dx \qquad with \qquad \ell og(1+\frac{1-\pi}{\pi}).T$$

Therefore, neglecting the coefficient $\varepsilon(\pi,T)$, at the first order the mean $\mu(T)$ should be a linear function of the maturity T.
We should have the same approximation with the standard deviation $\sigma(T) = \log(\delta).(np(1-p))^{\frac{1}{2}}.T$.

Using maximum likelihood estimates of the mean and the variance, we have tested the hypothesis of linearity on various subperiods in our sample, and for different values of the shift n. As can be seen from figures 3 and 4, results of the experiments are not decisive. When we look at the mean μ, there is a large spread for high maturities between the model and the observed values. The estimations of the variance point out a fitting error more important on the short term.

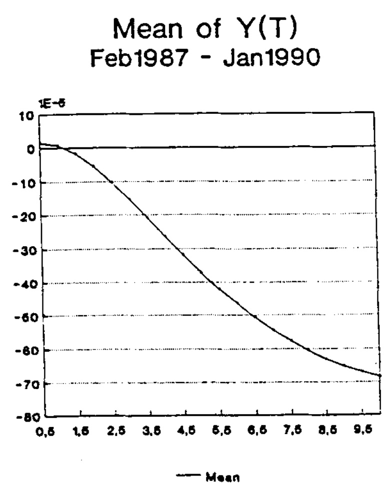

Fig.3

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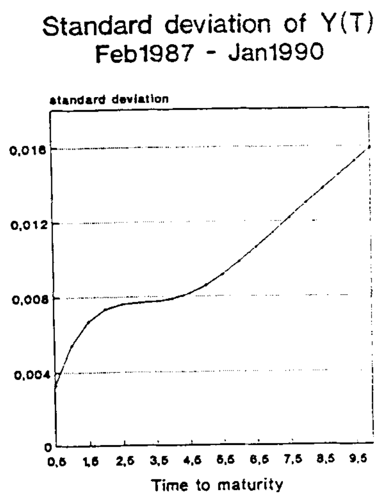


Fig.4

In order to identify the common factors which affect the values of Y for all the maturities T, we performed a principal component analysis

on the covariance matrix $(Y(T_i)Y(T_j))$, where $(T_i,T_j) \varepsilon \{1..10\}$. Results shown that with two uncorrelated factors only, we can explain more than 99% of the total variance. The first factor accounts for 85% of total explained variation, and agrees with Ho and Lee's factor. It affects the variables $Y(T_i)$ quasi linearly with the maturity T. The second factor, whose interpretation is not obvious, has a maximum effect on the short term (1 to 3 years) and is not taken into account by the model.

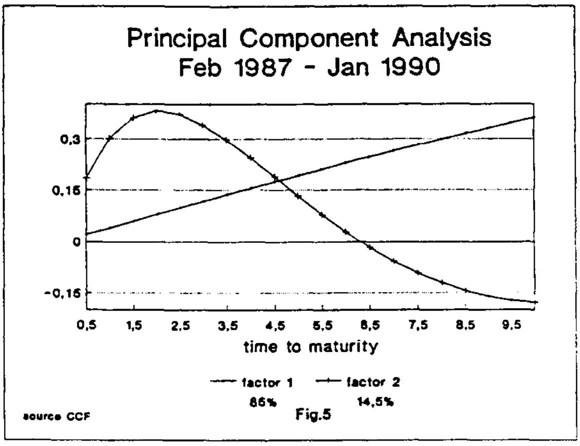


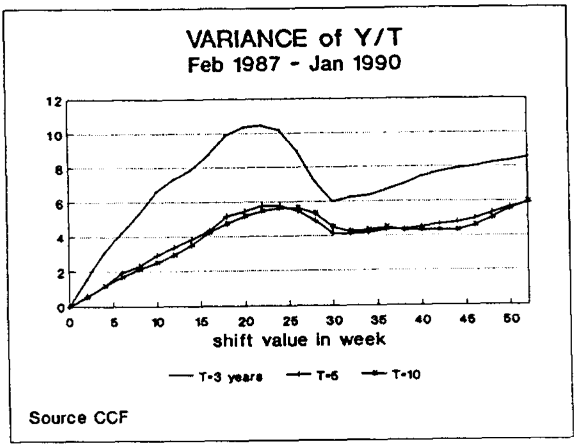
Fig.5

d. The variable Y as a function of the horizon time

In this part we have tested the linear correspondance implied by the model, between the variance of Y(T) and the time shift n. Figure 6 shows the variation of σ for increasing values of n and for the maturities
T= 3, 5, 10 years. Results don't confirm the model : the variance is not linear with maturity. In the long run the variance seems to go to an asymptotic value. This drawback of the Ho and Lee's model can also be observed when we compute the variance of the insterest rates

$$\sigma^2(R_T^{i,n}) = \log^2(\delta)p(1-p).n$$

this variance, which should be independent of the maturity, should also diverges as n goes to infinity. But actually we observe on the market that first short term rates have larger variances than long term rates and secondly these variances never diverge which seems an assumption economically reasonable. In the long run interest rates variances remain in a flat collar.



IV Conclusions

In this paper we present some experiments to check the empirical validity of Ho and Lee's Arbitrage-free Rates movement model. Are the main assumptions credible ?

Ho and Lee's model is a one factor model applied to the yield curve, where all contingent claims, including bonds, are priced by the short-rate movement, in a risk neutral world. Empirical tests have shown the limits of this hypothesis. At least two factors are necessary to take into account uncertainties in French treasury markets. However the first factor identified and which explains more than 85% of the total variance, induces yield changes basically constant across maturities, and agrees with Ho and Lee's factor. The second factor, usually called steepness, is neglected by the model and may induce problems in a position hedged only with duration. The same troubles arise when pricing options with a long time to expiration, compared to the time to maturity of the underlying asset, like warrants on short bonds, caps and floors.. , the mere passing of time changing overpriced instruments into underpriced instruments, other things being equal.

In Ho and Lee's model, the variance of interest rates is a constant function of time to maturity. The Market data contradict this assumption, as shown by the existence of a second factor leading to underprice options on short instruments. The hypothesis of a divergent variance for interest rates, as a function of the horizon time, is not verified. Beyond a six months horizon, the reliability of the model seems small.

This first model, proposed by Ho and Lee, has, beyond its simplicity, some very attractive features. Discount bond prices, which are in principle observable, are fully utilized by the model. The binomial lattice is helpful to price American contingent claims. However improvements are needed to fit market empirical tests. Features of a good model for the French Treasury market should take into account the two identified factors, an interest rates volatility function of time to maturity, a bounded interest rates variance as in Vasicek model [4].

Ho and Lee propose to make the perturbation functions h, h^* and the implied probability π, time and state dependent. Heath, Jarrow and Morton[8] propose a new approach, with a time continuous multi-factors model. Other metodologies are evolved by Black-Derman-Toy[9] or Courtadon-Weintraub[10]. Then which is the good candidate ? Further empirical studies are to be undertaken to answer this question !

References

[1] R. Gibson, "Obligations et Clauses Optionnelles", Presses Universitaires de France, Paris 1990.

[2] J.C. Augros, "Les Options sur taux d'intérêt", Economica, Paris 1989.

[3] F. Black, " The Pricing of Commodity Contracts", Journal of Financial Economics, Vol 3 , 1976.

[4] Vasicek O., An Equilibrium Characterization of the Term Structure, Journal of Financial Economics, 5, 1977.

[5] M. J. Brennan and E. S. Schwartz, " An Equilibrium Model of Bond Pricing and a Test of Market Efficiency", Journal of Financial and Quantitative Analysis, 17, September 1982.

[6] Ho T. S. and Lee S. B., Term Structure Movement and Pricng Interest Rate Contingent Claims, The Journal of Finance, Vol 41, n° 5, December 1986.

[7] Vasicek O. and Fong H. G., Term Structure Modeling Using Exponential Splines, The Journal of Finance, Vol 33, n° 2 May 1982.

[8] D. Heath, R. Jarrow and A. Morton, "Bond Pricing and the Term Structure of Interest Rates : A New Methodology", working paper of Cornell University, Octobre 1988.

[9] F. Black, E. Derman and W. Toy, "A One Factor Model of Interest Rates and Its Application to Treasury Bond Options", Goldman Sachs discussion paper, June 1988.

[10] G. Courtadon and K. Weintraub, " An Arbitrage Free Debt Option Pricing Model Based on Lognormally Distributed Forward Rates", Citicorp, October 1989.