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A Model for Investment Return
Asymptotic Behaviour

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Summary

A stochastic model for investment return is presented. The model is described as a product of two factors: the effects of inflation and the inflation-independent factor. The asymptotics of these factors are studied.

Résumé

Modèle de Rentabilité d'Investissement
Comportement Asymptotique

Nous présentons un modèle stochastique de rentabilité d'investissement. Le modèle est décrit comme un produit de deux facteurs: les effets de l'inflation et le facteur indépendant de l'inflation. Le comportement asymptotes de ces facteurs est étudié.

1. Introduction

We present a model for return on investment. Using autoregressive processes as a tool, we describe the investment returns as a product of two factors: the effects of inflation and the inflation–independent factor, which is assumed to include the effects of all economic background factors other than inflation. In the model the latter factor has a drift to the expected value whereas the former is of a random walk type, having no drift. The main aim of this paper is to study the asymptotic behaviour of these two factors.

When introducing the model, our purpose is to describe investment returns, including investment income and change in value, in such a way that, above all, the average level and the fluctuation of the returns should correspond to the experienced ones. The model does not aim at forecasting future returns. Like Wilkie [6] we are striving for charting "possible futures" for actuarial use. Our model is closely related to those of Wilkie [6] and Pentikäinen et al. [5].

It is typical of inflation that after a change in the value of money there does not exist any drift to the previous value of the money. Accordingly, it is natural to model inflation in a way which leads to a random walk type of behaviour for the value of money, in proportion to its "expected" value (expressed by means of the mean inflation rate), cf. (2.7). As to the value of investments, the situation is somewhat different. If e.g. share prices have been driven far from their "true" values, market forces have a tendency to return the prices closer to those values. Accordingly, it is reasonable to model the development of the value of shares so that there exists a drift towards the "true" or mean value. The model to be presented in this paper includes that kind of drift. Note, however, that random walk type of models, having no drift to the center, are often applied when returns are considered in a shorter run. In the short run, these models need not essentially differ from those with drift, but differences come out in the long run considerations and especially in the limit behaviour.

In Chapter 2 we present the investment model. The model differs from that of Pentikäinen et al. [5] mainly in the long term behaviour.

In Chapters 3 and 4 we deal with the asymptotic behaviour of the model. It turns out that the (suitably normed) inflation–independent factor possesses a limit distribution with a finite variance, whereas the effect of the inflation factor, considering shares and real estate, has no limit distribution. Thus, for nominal returns, the effect of the inflation factor involves, in the long run, an essentially larger uncertainty than that of the inflation–independent factor.

Some simulation examples are presented in the Appendix.

2. The investment model

By investment return we refer both to the investment income and the change in value of assets which are treated separately by similar models. When applying the model, it is appropriate to divide the assets into homogeneous subcategories, such as bonds, shares and real estate. The different returns on different investment categories can be described by adjusting the parameters of the model. Inflation is used as the main explanatory factor for the returns. By calibrating the parameters of the model, the degree of the inflation–linkage and the length of the time lags involved can be adjusted. In each investment category the change in value and the income (separately) are presented as a product of two factors: the effects of inflation and the inflation–independent factor, including the effects of economic background factors other than inflation.

We turn to describing the model in detail and consider mainly the change in value of an investment category (e.g. shares, real estate, bonds).

Denote the *value* of the investment category under consideration, at year n, by A_n and, the *change in value* $A_n - A_{n-1}$ by C_n. Further, define the *rate of change in value*

$$r_n = \frac{C_n}{A_{n-1}}. \tag{2.1}$$

We describe the development of A_n as follows

$$A_n = A_o I_n G_n,$$

where A_o is the initial value of the portfolio, I_n the cumulative effect of inflation on the value and G_n the inflation–independent effect on the value. We consider first G_n.

Let

$$G_n = (1+j)^n(1+X_n), \tag{2.2}$$

where j is a constant describing the mean inflation–independent growth of the value, characteristic of the investment category in question, and (X_n) is a second order autoregressive process

$$X_n = b_1 X_{n-1} + b_2 X_{n-2} + c\varepsilon_n, \tag{2.3}$$

where b_1, b_2 and $c > 0$ are constants and ε_n are i.i.d. random variables with zero mean and finite variance. We assume that the coefficients b_1 and b_2 are such that the roots of the characteristic equation

$$x^2 - b_1 x - b_2 = 0 \tag{2.4}$$

lie inside the unit circle of the complex plane. Further, we assume that the noise terms ε_n have a positive density on some interval $|x| < \eta$, $\eta > 0$.

Note that the coefficients b_1, b_2 can be chosen so that (X_n) has a certain type of cyclic character. This is the case when the roots of the characteristic equation (2.4) are complex. For this and other basic properties of autoregressive processes the reader is referred e.g. to Cox and Miller [2].

The process (X_n) has a drift to zero, which prevents it from a random walk type of behaviour, i.e. wandering long times far from its mean value. As a consequence, (X_n) has a limit distribution to which it converges rapidly, see Chapter 4.

We turn to the effects of inflation. Like in Wilkie [6] we present inflation by means of a first order autoregressive process. Denote the inflation rate in year n by i_n. Let i be a constant. We assume that

$$1 + i_n = (1+i)K_n, \tag{2.5}$$

where $\log K_n = Y_n$ is a first order autoregressive process

$$Y_n = aY_{n-1} + c'\eta_n, \tag{2.6}$$

where a and c' are constants, $0 < a < 1$, $c' > 0$ and the noise terms η_n are i.i.d. with zero mean, finite variance $\sigma_\eta^2 > 0$ and finite third absolute moment $E|\eta_n|^3$. We also assume that the variables η_n are independent of the variables ε_n. The constant i represents the mean inflation rate, in the sense of the assumptions above.

We call the product $\prod_{k=1}^n (1+i_k)$ the *price index induced by inflation* and

$$H_n = (1+i)^{-n} \prod_{k=1}^{n} (1+i_k) = \prod_{k=1}^{n} K_k, \tag{2.7}$$

the *normed price index.*

We now move on to the cumulative effect of inflation on the value of an investment category. The consideration can be divided into the following cases:

1° Inflation does not affect the value,

2° Inflation has, possibly with a lag, a full effect on the value,

3° Inflation has a partial effect on the value.

In case 1° we set, of course, $I_n \equiv 1$. In case 2° we define

$$\begin{aligned} I_n &= (1 + a_0 i_n)(1 + (a_0 + a_1)i_{n-1}) \cdots \\ &\quad (1 + (a_0 + \ldots + a_s)i_{n-s}) \cdot \prod_{k=1}^{n-s-1} (1+i_k), \end{aligned} \tag{2.8}$$

where $0 \leq a_k \leq 1$, $k = 0, 1, \ldots, s$ and $\sum a_k = 1$.

The coefficients a_k describe the lag with which inflation affects the value of the investment category in question. In case 3° it is assumed that $\sum a_k < 1$. It might be realistic to include real estate and shares in case 2°.

The investment income can be described in a similar way as the development of the value above. In case of shares or real estate, it must be taken into consideration when choosing the parameters of the model, cf. (2.3), that the fluctuation of the investment income is generally smaller than that of the value of the corresponding assets. Also the interaction between the income and the value of assets ought to be taken into consideration, see e.g. Wilkie [6]. However, we ignore this aspect here.

In every investment category, both as regards change in value and investment income, the parameters of the model must, of course, be chosen so that they correspond to reality as closely as possible. The estimation of the parameters is not dealt with in this paper.

Note. The possibility that part of the investment portfolio loses its value finally, for example as a consequence of a bankruptcy, might be taken into account e.g. by means of a suitable compound Poisson process.

3. Asymptotics of the inflation effect factor

In this Chapter we deal with the asymptotic behaviour of the inflation effect factor (I_n). Since the length of the time lag involved is finite, it is sufficient to study the normed price index (H_n) (see 2.7 and 2.8) when considering the asymptotic behaviour of (I_n). Consider first the logarithm of H_n and denote

$$S_n = \log H_n = \log \prod_{k=1}^{n} K_k = \sum_{k=1}^{n} \log K_k = \sum_{k=1}^{n} Y_k.$$

Recall that we assumed that (Y_n) is a first order autoregressive process, cf. (2.5) and (2.6). Denote $s_n = (\text{Var } S_n)^{1/2}$. Further, denote by F_n the distribution function of S_n/s_n. *We shall show that for (S_n) the following central limit theorem convergence is valid:*

(3.1) $$F_n \to \Phi \quad \text{uniformly,}$$

where Φ is the standard normal distribution function and

$$s_n = ((1 + a + \cdots + a^{n-1})^2 + \cdots + (1 + a)^2 + 1)^{1/2} \sigma_\eta.$$

Let $Y_1 = \eta_1$. Then

$$Y_n = a^{n-1}\eta_1 + a^{n-2}\eta_2 + \cdots + a\eta_{n-1} + \eta_n.$$

Hence

$$S_n = \sum_{k=1}^{n} Y_k = (1 + a + \cdots + a^{n-1})\eta_1 + \cdots + (1 + a)\eta_{n-1} + \eta_n \tag{3.2}$$

and

$$s_n = ((1 + a + \cdots + a^{n-1})^2 + \cdots + (1 + a)^2 + 1)^{1/2}\sigma_\eta. \tag{3.3}$$

Define a triangular matrix (Y_{nj}), $j \le n$ as follows

$$\begin{pmatrix} Y_{11} & & \\ Y_{21} & Y_{22} & \\ \cdots & & \\ Y_{n1} & Y_{n2} & Y_{nn} \end{pmatrix} = \begin{pmatrix} \eta_1/s_1 & & \\ (1+a)\eta_1/s_2 & \eta_2/s_2 & \\ \vdots & & \\ (1 + \cdots + a^{n-1})\eta_1/s_n & (1 + \cdots + a^{n-2})\eta_2/s_n & \cdots \quad \eta_n/s_n \end{pmatrix}$$

Clearly, the random variables in each row Y_{nj}, $j = 1, \ldots, n$ are independent and $EY_{nj} = 0$. By (3.2),

$$\sum_{j=1}^{n} Y_{nj} = S_n/s_n.$$

Thus Var $(\sum_{j=1}^{n} Y_{nj}) = 1$. Denote

$$\gamma_{nj} = E|Y_{nj}|^3 = \frac{(1 + \cdots + a^{n-j})^3}{s_n^3}\gamma,$$

where $\gamma = E|\eta_k|^3 < \infty$ (cf. the assumptions in Chapter 2). Furthermore, denote $\Gamma_n = \sum_{j=1}^{n} \gamma_{nj}$. It is easy to see that

$$\Gamma_n = \frac{\gamma}{\sigma_\eta^3} \frac{(1 + \cdots + a^{n-1})^3 + \cdots + (1 + a)^3 + 1}{(1 + \cdots + a^{n-1})^2 + \cdots + (1 + a)^2 + 1)^{3/2}} \to 0$$

when $n \to \infty$. Accordingly, by Theorem 7.4.1 of Chung [1], the assertion (3.1) follows. By (3.3), $s_n \ge \sqrt{n}\sigma_\eta$. Hence, $s_n \to \infty$, when $n \to \infty$. Thus, it follows from (3.1) that (S_n), and hence (H_n), does not converge to any probability distribution.

4. Asymptotics of the inflation–independent factor

When considering the asymptotics of the inflation–independent factor (G_n) it is sufficient to study the process (X_n), cf. (2.2). In the following, we shall treat (X_n), under assumptions stated in Chapter 2. It turns out that (X_n) converges in total variation, with a geometric rate, to a limit distribution with a finite variance. In fact, denote μ_n the probability distribution of X_n, conditioned on some fixed initial values $X_1 = x_1$, $X_o = x_o$. *Then the distributions* μ_n *tend to a limit distribution* μ *(in total variation norm of measures, and with a geometric rate). The mean of* μ *equals to zero and the variance of* μ *is finite. Moreover, the expectation and variance of* X_n *converge (with a geometric rate) to the mean and variance of* μ*, respectively.*

The proof of these assertions uses the theory of general Markov chains. For a general reference concerning Markov chains, see e.g. Nummelin [4].

Since the process (X_n) depends on its two previous states, it is not a Markov chain as such. However, $Z_n = (X_n, X_{n-1})$, the pair of two successive values of the process (X_n), is a Markov chain with state space R^2. The process (Z_n) can be expressed in the vector form as follows

$$Z_n = \begin{pmatrix} b_1 & b_2 \\ 1 & 0 \end{pmatrix} Z_{n-1} + \begin{pmatrix} c\varepsilon_n \\ 0 \end{pmatrix},$$

where $Z_n = \begin{pmatrix} X_n \\ X_{n-1} \end{pmatrix}$, cf. (2.3).

Denote by

$$P^n(z, A) = Pr(Z_n \in A | Z_o = z), \quad z \in R^2, \ A \subset R^2$$

the n–step transition probabilities of (Z_n). It follows (as a special case) of Theorem 3 of Feigin and Tweedie [3] that there exist a probability measure π in R^2 and a constant ρ, $0 < \rho < 1$ such that for every initial value $z_o = (x_1, x_o)$

$$\lim \rho^{-n} \| P^n(z_o, \cdot) - \pi(\cdot) \| = 0, \tag{4.1}$$

where $\| \cdot \|$ denotes the total variation. The probability measure determined by (4.1) is *invariant*, i.e.

$$\pi(\cdot) = \int P^n(z, \cdot) \pi(dz).$$

Denote the marginal distributions of π by π_1 and π_o and the corresponding marginal distributions of $P^n(z_o, \cdot)$ by $\mu_{1,n}, \mu_{o,n}$, where $z_o = (x_1, x_o)$ is some initial value. Clearly,

$$\begin{aligned} \mu_{1,n}(\cdot) &= Pr(X_n \in \cdot \mid Z_o = z_o), \\ \mu_{o,n}(\cdot) &= Pr(X_{n-1} \in \cdot \mid Z_o = z_o). \end{aligned} \tag{4.2}$$

It follows from (4.1) that

$$\begin{aligned} \lim_n \rho^{-n} \| \mu_{1,n} - \pi_1 \| &= 0, \\ \lim_n \rho^{-n} \| \mu_{o,n} - \pi_o \| &= 0. \end{aligned} \tag{4.3}$$

Thus, by (4.2) and (4.3), first part of the assertion holds true with $\mu = \pi_1$. (Since $\mu_{1,n} = \mu_{o,n+1}$, it follows from the uniqueness of the total variation limit that $\pi_o = \pi_1$). Further, it is well known for autoregressive processes that, for all initial states $z = z_o$, EX_n tends to zero with a geometric rate. On the other hand, it follows from the invariance of π and the assumption concerning the roots of the characteristic equation that $E_{\pi_1} = 0$. The result for the variances follows from Theorem 4 of Feigin and Tweedie [3].

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Appendix

Simulation examples

In the following we present some simulation examples. First an inflation rate realization was simulated, according to (2.5) and (2.6) with parameters specified in the table below, see Figure 1. Using this fixed inflation, we then simulated the rate of change in value (cf. 2.1) of two investment categories of different type, see Figures 2, 3. The parameters specifying the investment categories are given in the table below. Note that the roots of the characteristic equation (2.4) are complex in case of category 1, and real in case of category 2. (The noise terms was taken such that $\varepsilon_n + 2$ was Gamma (4,2) and $\eta_n + 4$ Gamma (16,4)-distributed, cf. Pentikäinen et al. [5], p. 121.) Using the parameters specified for inflation in the table we simulated a bundle of realizations of the process (H_n), see Figure 4. Correspondingly, a bundle of the process (X_n) was simulated using the parameters of category 1, see Figure 5.

Table

Parameters used in simulation

Inflation

i	a	c'
0.06	0.7	0.025

Investment returns

Category	j	a_1	a_2	a_3	a_4	a_5	b_1	b_2	c
1	0.02	0.2	0.2	0.2	0.2	0.2	0.75	−0.5	0.08
2	0.03	0.5	0.3	0.2	0	0	0.65	0.3	0.04

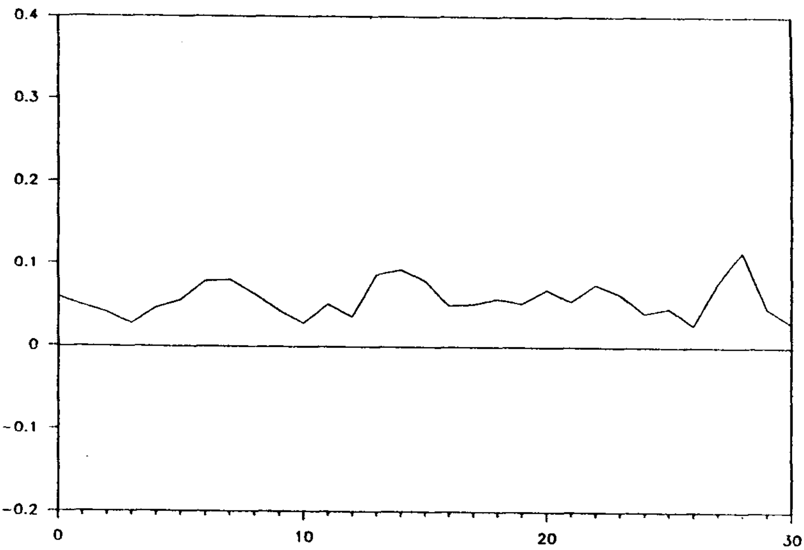

FIG. 1. A simulation realization of inflation rate, using the parameters in the table.

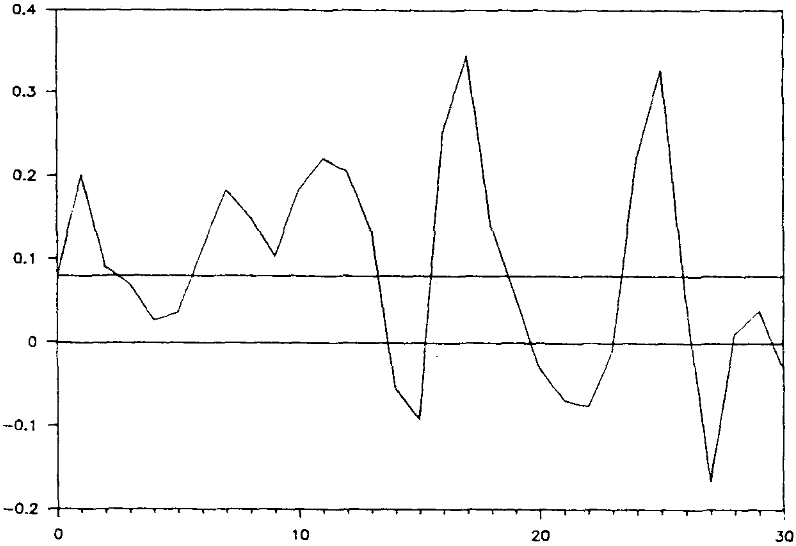

FIG. 2. A simulation realization of rate of change in value, category 1.

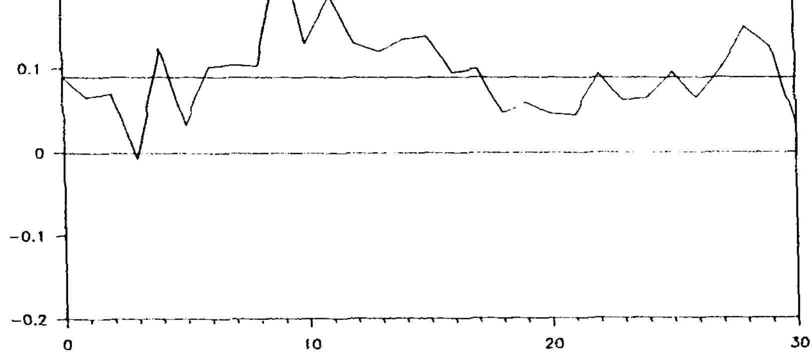

FIG. 3. A simulation realization of rate of change in value, category 2.

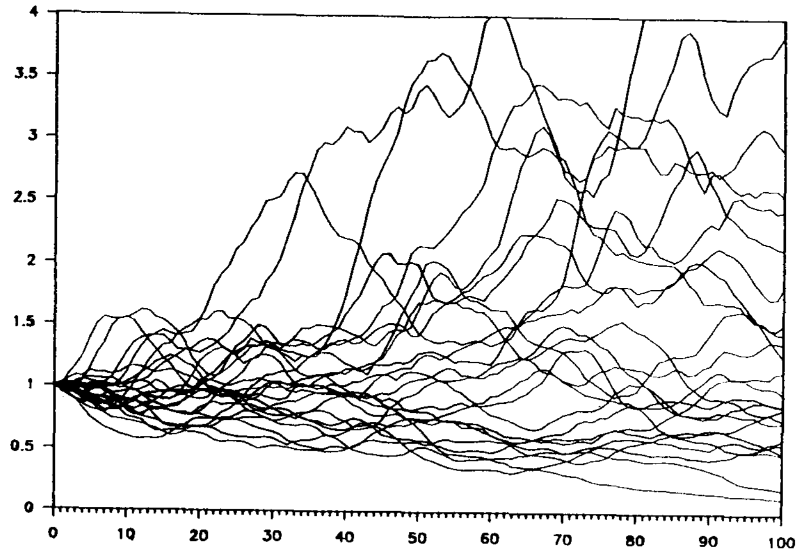

FIG. 4. Simulation of the normed price index (H_n), using the parameters for inflation in the table (24 realizations).

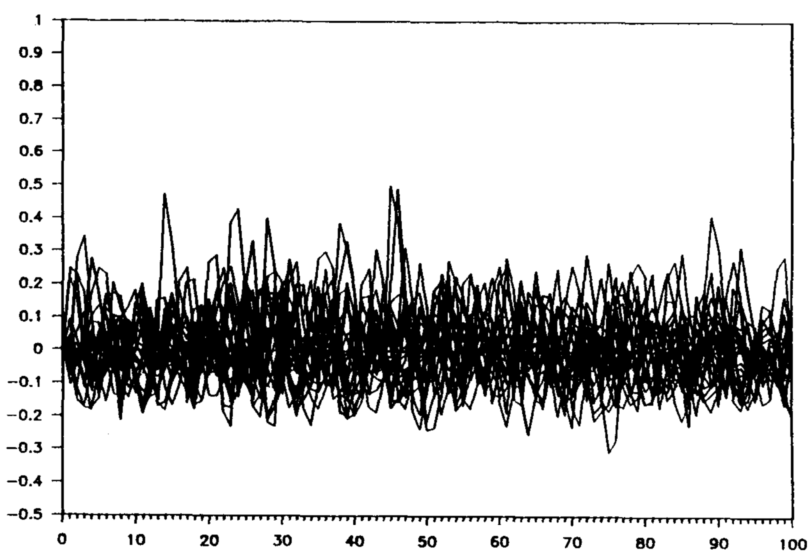

FIG. 5. Simulation of the process (X_n), using parameters of category 1 (24 realizations).