2nd AFIR Colloquium 1991, 4: 1-25

# Generalized Arrow Pricing to Understand Financial Markets

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## Summary

Most of the concepts that are used in modern theory of financial markets are contained in a paper published by Arrow in 1953. Arrow's model generalizes to non finite set of states describing uncertainty so as to encompass general financial assets pricing.

We present several theorems of equivalence between General Equilibrium and Perfect Foresight Equilibrium (PFE), a concept adapted to financial assets markets. These results put forward several points:

- The welfare properties of PFE, or in Arrow's term, the "role of securities in the optimal allocation of risk".

- The role of the complete market hypothesis (CMS) and the reason why it takes an abstract mathematical form in modern finance.

- The probabilistic interpretation of assets prices under the CMS hypothesis. This interpretation extends to dynamic models (as the equivalent martingale property) and allows the pricing of assets by their expected payments.

- The necessary properties of equilibrium prices which are well defined by a linear, positive, continuous form. These properties are equivalent to three "no arbitrage" conditions that can be found in finance models without reference to equilibrium.

# Résumé

## Généraliser la Tarification Arrow pour Comprendre les Marchés Financiers

La plupart des concepts utilisés dans la théorie moderne des marchés financiers sont contenus dans un article publié en 1953 par Arrow. Le modèle Arrow généralise pour englober des groupes d'états non finis décrivant l'incertitude de façon à englober la tarification générale des actifs financiers.

Nous présentons plusieurs théorèmes d'équivalence entre l'Equilibre Général et l'Equilibre de Prévoyance Parfait (PFE), un concept adapté aux marchés des actifs financiers. Ces résultats mettent en avant plusieurs points:

- Les propriétés positives du PFE ou pour citer Arrow, le "rôle des valeurs dans la répartition optimale du risque".
- Le rôle de l'hypothèse complète de marché (CMS) et la raison pour laquelle elle prend une forme mathématique abstraite dans les finances modernes.
- L'interprétation probabiliste des prix des actifs d'après l'hypothèse CMS. Cette interprétation s'étend à des modèles dynamiques (comme la propriété de martingale équivalente) et permet la tarification des actifs par leurs paiements prévus.
- Les propriétés nécessaires des prix d'équilibre qui sont bien définies par une forme linéaire, positive et continue. Ces propriétés sont équivalentes à trois conditions de "non arbitrage" qui peuvent être trouvées dans des modèles financiers sans référence à l'équilibre.

## Generalizing Arrow pricing to understand financial markets

Arrow published in 1953 a seminal paper : "Le rôle des valeurs boursières dans la répartition la meilleure des risques".[1] There are very few papers in economics of uncertainty or in financial economics which can get away without quoting it. Indeed in no more than six pages, Arrow managed to express most of the concepts that have been used and developed in economics and finance since then :

- contingent goods and consumption plans,
- general equilibrium of contingent goods
- perfect foresight equilibrium of spot and financial markets,
- no arbitrage properties of equilibrium prices,
- pricing of redundant assets (portfolios) by marketed assets (Arow's assets)
- extension of general equilibrium welfare theorems to financial markets,
- complete markets and completing markets with financial assets,
- probabilistic interpretation of assets prices

... and we must forget many more.

However, many of these concepts have no name in Arrow's paper, some hypothesis are implicit, and a lot of properties are mixed up in the overflow of the genial prescient spring.

After thirty years of extensions of this model, we felt a need to make clearly appear several concepts used by Arrow which have become the basis of modern finance.

For instance finance models use no arbitrage conditions rather than equilibrium concepts. The complete markets assumption has taken the sense of a perfect hedging possibility for all relevant risk. Perfect foresight equilibrium defined by Radner [1972] is the natural extension of Arrow's notion of equilibrium and it underlies most financial models. The probabilistic interpretation of assets prices (when markets are complete) has been extended to the "equivalent martingale properties" of continuous time dynamic financial markets models.

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[1]In *Econométrie*, 40, pp 41-47, Cahiers du CNRS, English version 1964 "The role of securities in the optimal allocation of risk bearing", *Review of Economic Studies*, 31, pp 91-96.

Precise relationships between Arrow's concepts and concepts used in financial models will be established here. Perhaps the most important ones are those allowing welfare interpretations of equilibrium. For instance although Duffie (in Duffie and Sonnenschein [1989]) criticizes Arrow's interpretation of his model in term of the first welfare theorem, we shall show that Arrow is perfectly right as long as equivalence between general and perfect foresight equilibria is established instead of the usual one-way implication (actually neither the equivalence nor the implication is explicitly proved in Arrow, indeed perfect foresight equilibrium is not even defined).

One of the most fascinating property of Arrow's financial market is the revelation by markets equilibrium of a probability distribution over the states of nature, which is unrelated to any individual subjective probabilities agents can have. A similarly fascinating property appeared in financial economics where pricing of assets by arbitrage considerations suggests that assets prices are the expectations of their payments with respect to an equilibrium distribution : "pricing by the risk neutrality argument" (Jarrow and Rudd [1982]).

In order to compare, analyse and understand these results we reproduce Arrow's model and the generalization to non finite set of states of the world we need to understand modern finance. This is done in section 1. In section 2 we investigate the meanings of the "No arbitrage" conditions (we sort out three of them) imposed on the market structure, and we spell the properties it gives to asset prices. We come back to equilibria under a complete markets assumption in section 3. The complete markets definition we use seems to be the closest both to Arrow's (implicit) assumption and to various versions found in modern finance (for instance Harrison-Kreps [1979], Duffie [1988]). Contrary to other definitions found in economic theory (Wiesmeth [1988], Geanokoplos [1990]) it makes no appeal to equilibrium prices nor to efficiency properties. On the contrary our complete market assumption gives equilibria (when they exist) their efficiency property. Complete markets and Arrow's interpretation of asset's prices as probabilities are generalized in section 4.

## 1. Arrow's model and its generalization

Let $\varepsilon = (X_i, \succcurlyeq_i, w_i \in X_i)_{i \in I}$ be a pure exchange economy, where I is a finite set of agents, $X_i$, $\succcurlyeq_i$, and $w_i$, are the choice set, preference relation over $X_i$ and endowments of agent i. Here $X_i$ is a set of consumption plans, contingent on some states s in a set S. A state s can be understood as a couple of time t and elementary event $\omega$ say $s = (\omega,t)$ where $\omega$ belongs to a set $\Omega$ (states of nature) and t (time) belongs to a subset of $\mathbb{R}$.

There are H physical goods, so each $X_i$ is a set of functions $x_i : S \rightarrow \mathbb{R}^H$ (consumption plan : $x_i(s)$ is the bundle agent i is planning to consume if state s occurs).

The classical Arrow-Debreu model (Arrow [1953], Debreu [1959], chap. 7) is a fantasy of markets for all consumptions plans, called contingent goods, taking place in an abstract time and location. Then, under some mathematical assumptions on the structure and properties of $X_i$ ($X_i \subset L$ a topological vector space) and $\succcurlyeq_i$, an equilibrium (general equilibrium, or Arrow-Debreu equilibrium) is proved to exist. A price P can be defined in the dual of L such that agent i would have a wealth : $p.w_i$ to exchange for consumption goods such that $p.x_i = p.w_i$. Instead of considering markets for all contingent goods, Arrow proposed to consider markets for financial assets in the same abstraction. Financial assets are contracts under which a certain amount of wealth (i.e. units of purchase power) is delivered in some states $s \in S$ if they occur.

An asset is a function $y : S \rightarrow \mathbb{R}$. In the abstract location and time markets for a set Y of assets can be open, each asset y in Y delivers a quantity y(s) of wealth if state s occurs.

Y will be called the set of marketed assets.

If not all contingent goods can be traded, markets for goods will be reopened when state s occurs. In state s, agents will trade their endowments $w_i(s)$ in order to achieve a consumption basket $x_i(s)$. In equilibrium this will define a spot price $\Pi(s)$ such that $\Pi(s).x_i(s) = \Pi(s)w_i(s)$. If there is a market for assets, agents can transfer some of their wealth from one state to another. Indeed if an agent i buys an asset y, he will be able to add y(s) to his ensowment in state s. Then agent i will be able to consume $x_i(s)$ such that $\Pi(s)\ x_i(s) = \Pi(s)\ w_i(s) + y(s)$.

Assets are contracts (mere sheet of paper) to give or receive wealth in state s, that are exchanged between agents according to their preferences for consumption in the different states. Relative prices of these assets reflect demands for goods in the different states.

In order to achieve a consumption plan x when not all contingent good markets are opened, agents will have the opportunity to trade assets of the set Y of marketed assets. They will therefore form portfolios, i.e. a finite list of quantities of marketed assets, $\theta = (\theta(y))_{y \in Y_\theta}$ where $Y_\theta$ is a finite subset of Y.

Actually Arrow's story is slightly different. It refers to two periods : present time at which markets for physical goods and financial markets are opened, and a future time where physical goods markets are reopened and financial assets pay. In present time agents buy their portfolios of assets using their present wealth, so their budget contraint is $\Pi(s_0)\ x_i(s_0) + q(\theta) = \Pi(s_0)\ W_i(s_0)$, where $s_0$ is present state and $q(\theta)$ is the cost of portfolio $\theta$. In state s, their budget constraint is $\Pi(s)\ x_i(s) = \Pi(s)\ w_i(s) + \theta(s)$ where $\theta(s)$ is the payment of portfolio $\theta$ in state s. This story is appealing because it has some realistic feature : transfer of money, or wealth at present time toward future time. But it gives rise to problems that are unecessary : why would wealth in state $s_0$ be the same as wealth in state s ? What is the meaning of intertemporal arbitrage considerations ?

To avoid this kind of difficulty which cannot be solved without reference to a general equilibrium, we shall define assets in the modern financial way : an asset is a function $y : S \to \mathbb{R}$. In each state $s \in S$ y(s) is an amount of wealth in state s (defined by a spot price $\Pi(s)$). In the abstract location and time where assets are exchanged no endowment, and hence no wealth is available yet. To buy an asset y at price q(y) one must buy or sell some others, say y' and y" at prices q(y') and q(y") such that $q(y) = q(y') + q(y")$.

In Arrow's model Y is finite, it is the set of assets $y_s$, $s \in S$, such that $\forall s' \in S$ $y_s(s') = 0$ if $s' \neq s$ $y_s(s) = 1$, and S is assumed to be finite. Here we do not assume that S nor Y are necessarily finite.

With a portfolio an agent will get an endowment $\sum_{y \in Y_\theta} \theta(y)\, y(s)$ of state s wealth (i.e. they will be able to afford consumption $x_i(s)$ such that

$\Pi(s)x_i(s) = \Pi(s)w_i(s) + \sum_{y \in Y_\theta} \theta(y)\, y(s)$, if $\Pi(s)$ is the spot price when state s occurs).

Any portfolio defines a flow of endowments $\Theta = \sum_{y \in Y_\theta} \theta(y)\, y$, where $\Theta : S \to \mathbb{R}$. Such a flow will be called a marketable asset, meaning that, although not traded (marketed), it is tradable (marketable) using the traded (marketed) assets in Y.

Notice that amoung marketed assets some might be redundant. A redundant (marketed) asset is an asset $\bar{y}$ such that there exists a portfolio $\bar{\theta}$ such that

$\bar{y} = \sum_{y \in Y_{\bar{\theta}}} \bar{\theta}(y)\, y$. This leads to a slackness in the definition of marketable asset because several portfolios may give rise to the same flow of endowments.

Let us call Span Y the set of all marketable assets.

Formally, Span Y = $\{\Theta: S \to \mathbb{R} / \exists \theta \;\; Y_\theta \text{ finite} \subset Y \;\; \Theta = \sum_{y \in Y_\theta} \theta(y)\, y\}$ is the set of all finite linear combinations of marketed assets. Let $Y_b$ be a subset of linearly independent assets[2] in Y such that Span $Y_b$ = Span Y ($Y_b$ is a basis for Span Y). Then any marketable asset $\Theta \in$ Span Y is uniquely defined by a portfolio $\theta_b$ such that $\Theta = \sum_{y \in Y_{\theta_b}} \theta_b(y)\, y$.

In Arrow's model the set S is finite and the all (Arrow's) assets are linearly independant . Therefore a consumption plan (or contingent good) is a vector of $\mathbb{R}^{S\times H}$. A price p in contingent good market is defined by a (dual) vector $\Pi$ in $\mathbb{R}_+^{S\times H}$ the value of a consumption good x, is $p.x = \sum_{s \in S} \Pi(s)\, x(s)$.

In more general models, S is not assumed to be finite. This is necessary if one wants to take into account the fact that all future possible exchange rates of Ecu against US dollar is part of the uncertainty an agent faces when there is no contingent claims of Ecu in terms of dollars for year 1999 (for instance !).

In order to have tractable mathematical properties that are not too far out of the intuition of Arrow's model, let us assumed that S is endowed with a probability space structure $(S, \mathcal{S}, \mu)$ and $X_i$, the choice space of agent i is included in $L^2(S, \mathbb{R}^H, \mu)$ the vector space of square integrable random variables on S. ($L^2$ is its own dual as is $\mathbb{R}^{H\times S}$ when S is finite).

A price P in contingent goods markets is defined by a dual vector $\Pi$ in $L^2(S, \mathbb{R}^H, \mu)$, the value of a contingent good is $p.x = \int_S \Pi(s)\, x(s)\, \mu(ds)$.

A marketed asset $y \in Y$ has a price $q(y) \in \mathbb{R}_+$. We define the cost of a portfolio $\theta = (\theta(y))_{y \in Y_\theta}$ as $K(\theta) = \sum_{y \in Y_\theta} \theta(y)\, q(y)$.

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[2]By this we mean that for any portfolio $\theta_b = (\theta(y))_{y \in Y_{\theta_b}}$ where $Y_{\theta_b}$ is a finite subset of $Y_b$, $\sum_{y \in Y_{\theta_b}} \theta_b(y) = 0 \Rightarrow \forall y \in Y_{\theta_b}\ \theta_b(y) = 0$. Otherwise stated, there are no redundant assets in $Y_b$, but still Span $Y_b$ = Span Y.

Notice that this is a static model although s can include time, meaning that when state s occurs there are no future transactions, and therefore no more room for an asset market (a dynamic extension of this model is Radner's 1972 when set S is finite. Nearly all financial markets models consider dynamic trading since Black and Scholes [1973] model with S a continuous time stochastic process).

## 2. Financial assets market structure

When setting a financial market a first difficulty arises if redundant assets are marketed. Assume $\bar{y} = \sum_{y \in Y_\theta} \theta(y)\, y$ is marketed and has a price $q(\bar{y})$. Assume $\Sigma\theta(y)q(y) < q(\bar{y})$ then any agent (assuming no transaction costs) could sell $\bar{y}$ at price $q(\bar{y})$ and buy portfolio $\theta$ at cost $K(\theta) = \Sigma\theta(y)\, q(y)$ achieving a net positive profit. This is an arbitrage opportunity that will cause market to misfunction.

More generally if a marketable asset $\Theta$ can be obtained by two different portfolios with different costs, arbitrage opportunities will prevent one of these portfolio to be traded. We shall therefore assume No Arbitrage 1 *(one good, one price)* :

$NA_1 : \forall\Theta, \Theta' \in \text{Span } Y \quad \Theta = \sum_{y \in Y_\theta} \theta(y)\, y = \sum_{y \in Y_{\theta'}} \theta'(y) y = \Theta \Rightarrow K(\theta) = K(\theta').$

**Consequence 1** : Under $NA_1$, given a marketable asset $\Theta \in \text{Span } Y$, for any basis $Y_b$, if $\Theta = \sum_{y \in Y_{\theta_b}} \theta_b(y)\, y$ with $Y_{\theta_b} \subset Y_b$, $K(\theta_b)$ does not depend on b.

Indeed, assume $Y_{b'}$ to be an other basis and so that in that basis :

$\Theta = \sum_{y' \in Y_{\theta_{b'}}} \theta_{b'}(y)y'$ with $Y_{\theta_{b'}} \subset Y_{b'}$. But any $y' \in Y_{\theta_{b'}}$ can be written as $y' = \sum_{y \in Y_{\theta_b}} \theta'_b(y)y$ and

$$\Theta = \sum_{y \in Y_{\theta_b}} \theta_b(y)\, y = \sum_{y' \in Y_{\theta_{b'}}} \theta_{b'}(y')y' = \sum_{y \in Y_{\theta_b}} \theta''_b(y)y$$

then because of $NA_1$, $K(\theta_b) = K(\theta''_b) = K(\theta_{b'})$

**Consequence 2** : Under $NA_1$, any marketable asset $\Theta \in \text{Span } Y$ has a unique cost $K(\theta)$ where $\theta$ is a portfolio defined on any basis $Y_b$ of Span Y.

**Proposition 1** : Under $NA_1$, $\bar{q} : \text{Span } Y \rightarrow \mathbb{R}_+$ $\bar{q}(\Theta) = K(\theta)$ is a well-defined function, it is linear and its restriction to Y is q.

**Proof** :

• that $\bar{q}$ is a well-defined function is an obvious consequence of consequences 1 and 2.

• $\bar{q}$ is linear : if a and b are any two numbers $a\Theta + b\Theta'$ is a marketable asset defined by a portfolio $a\theta + b\theta'$ where $\Theta = \sum\theta(y)y$ $\Theta' = \sum\theta'(y)y$. Then
$\bar{q}(a\Theta + b\Theta') = K(a\theta + b\theta') = \sum(a\theta(y) + b\theta(y))\, q(y) = a\bar{q}(\Theta) + b\bar{q}(\Theta')$

• $\bar{q}$ is an extension of q as $Y \subset$ Span Y. Notice however that $\bar{q}$ is not in general the unique linear extension of q to Span Y.■

From now we shall always assume $NA_1$ (one good, one price) the most common (implicit) assumption in all microeconomics models.

However, even under $NA_1$, a financial market may offer other arbitrage opportunities.

Assume you could form a portfolio, costing zero, which gives you a positive return in each state : $\Theta(s) \geq 0$ and at least one strictly positive : $\Theta \neq 0$. Do you think you will resist telling your best friends to buy it ? Then knowing they will tell all their friends and family to do the same the market will not function. Morality : you can't have *"le beurre et l'argent du beurre"*, this is No Arbitrage 2 :

**$NA_2$** : $\forall \Theta \in \text{Span } Y, \quad \Theta \geq 0 \;\; \Theta \neq 0 \Rightarrow \bar{q}(\Theta) > 0$

**Proposition 2 and obvious consequence** : $\bar{q}$ is a positive linear form on Span Y.

Knowing how people are, you can imagine that even thought they cannot make a sure profit for no cost, they'll try to get close to it. Assume that, although $NA_2$ is satisfied, you could find a sequence of marketable assets (that is portfolios, under $NA_1$) $\Theta_n$, with payments getting very small (for some N, if n>N $|\Theta_n| < \varepsilon$) but that you could sell for a positive price ($\forall n \; \bar{q}(\Theta_n) \geq K > 0$). Then you could assure yourself positive returns in all states by buying other portfolios with the positive amount K you receive for delivering nearly nothing ($\varepsilon$) when you sell a portfolio $\Theta_n$ with n > N.

A market organizer might want to exclude this kind of arbitrage opportunity. Given its similarity to Kreps' definition of a free lunch (Harrisson-Kreps [1979]) and a cat way of sneaking food, we could call it *"no catimini free lunch"* $NA_3$ :

$NA_3 : (\Theta_n)_{n \in N} \qquad \Theta_n \underset{n}{\to} 0 \Rightarrow \bar{q}(\Theta_n) \underset{n}{\to} 0$

**Proposition 3 and obvious consequence** : Under $NA_3$, q is a continuous (assuming Span Y is separable).

Let us summarize and conclude about our three No Arbitrage conditions.

- From $NA_1$, $\bar{q}$ is defined as a linear form on all marketable assets. When S is finite (Arrow's model) this means there exists a vector $\gamma$ of $\mathbb{R}^S$ (dual) such that for any

$$\Theta \in \text{Span } Y \qquad \bar{q}(\Theta) = \sum_{s \in S} \gamma(s)\, \Theta(s).$$

In particular the price of any marketed asset can be written as $q(y) = \sum_{s \in S} \gamma(s)\ y(s)$.

Assume Y is formed of Arrow's assets $y_s$ yielding one unit in stat s and zero elsewhere, then $q(y_s) = \gamma(s)$.

- From $NA_2$ we derive that $\gamma(s)$ must be positive. Assume furthermore that as in Arrow's model we have exactly card(S) linearly independent Arrow's assets $y_s$ (complete financial market). Let $B = \sum_{s \in S} y_s$ then $\forall s\ B(s) = 1$, B is a riskless asset paying one unit of state s wealth in every state s. It is usually assumed in finance models that $\bar{q}(B)$ is one. In Arrow's model this is proved using a conservation of wealth argument. This argument amounts to a no arbitrage between wealth in two states which we would rather avoid. In section 3 we shall prove indeed that normalize $\bar{q}(B)$ to be 1 at equilibrium is always possible. In any cases, when $\bar{q}(B) = 1$, as $\bar{q}(B) = \sum_{s \in S} \gamma(s)$, $\gamma(s)$ can be interpreted as a probability of state s. This probability has nothing to do with any subjective probabilities agents may asses to S, it is merely revealed by equilibrium assets prices.

This result is generalized to non finite set of states.

Assume S is a probability space $(S, \mathcal{S}, \mu)$ and $Y \subset L^2(S, \mathbb{R}, \mu)$.

- From $NA_1$, $NA_2$ and $NA_3$, $\bar{q}$ is a continuous linear form on Span Y. Therefore it is defined by a random variable in the dual of Span Y, say $\gamma \in L^2(S, \mathbb{R}, \mu)$ such that $\bar{q}(\Theta) = \int_S \gamma(s)\ \Theta(s)\ \mu(ds)$ (This is one of the Riesz decomposition theorem, $\gamma$ is unique). In particular the price of any marketed asset can be written as $q(y) = \int_S \gamma(s)\, y(s)\ \mu(ds)$.

Assume futhermore that there is a riskless marketable asset paying one in ($\mu$-almost) every states. This will be true if we assume for instance that Span Y = $L^2(S, \mathbb{R}, \mu)$ (an assumption which we will explain later in section 3 to be the natural extension of the finite case complete market hypothesis). Then, calling B the riskless asset $\bar{q}(B) = \int_S \gamma(s)\mu(ds)$. If $\bar{q}(B) = 1$ then $\gamma\mu$ is a probability distribution (equivalent to $\mu$), the analogue of Arrow's probabilities of the states defined by (Arrow's) assets prices. Notice again that probability distribution $\gamma\mu$ is revealed by market prices and has nothing to do with agents subjective beliefs. (We needed $\mu$ to give Span Y a topological structure, but we did not assume $\mu$ to be known by the agents. However, observing assets prices they might learn about $\gamma\mu$, a probability distribution equivalent to $\mu$).

This is as far as we want to go concerning the structure of financial assets markets. Indeed without even defining an equilibrium notion we have been able to capture through 'no arbitrage' and 'complete markets' conditions, one of the most striking result of Arrow's model : the emergence of a probability distribution over the set of states of nature which is imposed by the structure of prices. What is the implication of the existence of this distribution ?

Look at the prices on the financial market (with no arbitrage and complete) you have a probability distribution. With respect to this distribution the price of a portfolio (marketable asset) is its expected return. Assume agents have preferences on returns (implied by their preferences on consumption plans in a way we cannot determine now) and that these preferences satisfy the expected utility axioms. Taking as given the distribution defined by assets prices these agents behave as if all were risk neutral, their expected utility of any return being the expected return.

### 3. Equilibrium

In a contingent goods market we call general equilibrium (G.E.) of the exchange economy $(X_i, \succcurlyeq_i, w_i)_{i\in I}$ a pair $(p^*, (x^*_i)_{i\in I})$ of prices and consumption plans such that :

**GE :**

- Budget constraints (BC) : $\forall i\in I \; p^*.(x_i - w_i) = 0$

- $\forall i \in I$ $x_i$ maximizes $\succsim_i$ under (BC)

- Markets are clear $\sum_{i \in I} (x_i - w_i) = 0$

In the case where S is finite (Arrow-Debreu general equilibrium) p* is defined by a vector $\Pi^* \in \mathbb{R}^{HxS}$ so that (BC) is :

$$\sum_{s \in S} \sum_{h \in H} \Pi^{h*}(s)(x^h{}_i(s) - w^h{}_i(s)) = 0, \text{ or } \sum_{s \in S} \Pi^*(s).\,(x_i(s) - w^h{}_i(s)) = 0$$

In the case where $(S, \mathcal{S}, \mu)$ is a probability space and $X_i \subset L^2(S, \mathbb{R}^H, \mu)$ p* is defined by a random variable $\Pi^*$ in $L^2(S, \mathbb{R}^H, \mu)$ so that (BC) is : $\int_S \left[ \sum_{h \in H} \Pi^{h*}(s)(x^h{}_i(s) - w^h{}_i(s)) \right] \mu(ds)$, or $\int_S \Pi^*(s).(x_i(s) - w_i(s))\, \mu(ds) = 0$.

Existence conditions for a general equilibrium when $(S, \mathcal{S}, \mu)$ is a probability space are studied in Mas Collel [1986], see also Duffie [1988] proposition 11G.

If there are spot markets for physical goods and a market for a set Y of financial assets[3] we call Perfect Foresight Equilibrium (PFE) of the exchange economy $((X_i, \succsim_i, w_i)_{i \in I}, Y)$ a pair $((\Pi, q), (\theta_i, x_i)_{i \in I})$ of goods prices, assets prices, portfolios and consumption plans such that :

**PFE** :

- Budget constraints $(B_1) : \forall i \in I, \Pi.(x_i - w_i) = \sum_{y \in Y_\theta} \theta_i(y)y$

$$(B_2) : \forall i \in I, \bar{q}(\theta_i) = \sum_{y \in Y_\theta} \theta_i(y)q(y) = 0$$

[3]Satisfying $NA_1$ as always, and we exclude redundant assets from Y.

- $\forall i \in I$, $x_i$ maximizes $\succsim_i$ under $B_1$ and $B_2$

- Markets are clear : $\sum_{i \in I} (x_i - w_i) = 0$ and $\sum_{i \in I} \theta_i = 0$

In the Arrow-Debreu model markets are said to be complete because every contingent good can be traded (S is finite). The consequence of this is that any consumption plan is conceivable and all feasible consumption plans can be realized. Therefore the Pareto optimality of equilibrium is significantly interpreted as an optimal risk sharing allocation. In Arrow's model no contingent goods are traded at all. However any feasible consumption plan can be realized if spot prices are perfectly anticipated. This is because the spendings in each state (in wealth defined by spot prices) can be met by the payments of assets. Indeed, there is exactly one asset paying one unit of wealth in each state in Arrow's model ; so it is possible to build a portfolio paying any planned spending. Arrow's model is called as well a model of complete markets, or of "complete market structure", because, through the set of all Arrow's assets it is possible to obtain the same allocations of consumption plans than in the Arrow-Debreu equilibrium model ... at least in a perfect foresight equilibrium. We shall give a precise result about this equivalence between allocations obtained by a General Equilibrium and Perfect Foresight Equilibrium, under a "complete market structure hypothesis".

But before that we have to give a precise definition of "complete market structure" (CMS). Because the result we shall detail and generalize further down is only implicit in Arrow's paper, the role of the CMS hypothesis is not very clear and several definitions have been used in the litterature (generally in a implicit way).

The first idea is that CMS characterizes the fact that there are exactly as many Arrow's assets or states (finite) of the world.

Actually this genralizes very well to any set of assets (not necessarily Arrow's paying one unit of wealth in one state) that are linearly independent (Ross [1976]). Indeed the fact that the payments matrix is of rank card(S) plays a central role in the demonstration of our first theorem. So let our first definition be :

**$CMS_1$** : When S is finite the payment matrix of the marketed assets is of full rank : $\mathrm{rank}(y(s))_{s\in S, y\in Y} = \mathrm{card}(S)$. Oterwise stated: Y is a basis of $\mathbb{R}^S$.

The following theorem sums up some of Arrow's results, it is similar to Geanokoplos [1990] theorem 1:

**Theorem 1** (Equivalence between GE and PFE for a complete market structure when the set of states is finite)

Under $NA_1$, $NA_2$ and $CMS_1$ for an exchange economy $(X_i, \succ_i, w_i)_{i\in I}$ and a financial market Y :

1) If there exists a GE $(p^*, (x_i^*)_{i\in I})$ with $p^* = (\Pi^*(s))_{s\in S} \in \mathbb{R}^{H\times S}$, then there exists a PFE $(\Pi, q, (\theta_i, x_i)_{i\in I})$ such that

① $x_i = x_i^* \quad \forall i \in I$

② $\forall \Theta \in \mathrm{Span}\, Y \quad \bar{q}(\Theta) = \sum_{s\in S} \Theta(s)\, \gamma(s)$ where $\forall s \in S$ and $\forall h \in H$ $\gamma(s)\Pi^h(s) = \Pi^{h*}(s)$.

2) If there exists a PFE $(\Pi, q, (\theta_i, x_i)_{i\in I})$ then there exists a GE $(q^*, (\theta_i, x_i^*)_{i\in I})$ such that ① and ② hold.

Relation ① expresses the equality between allocations obtained in GE and PFE an important statement for welfare results.
Relation ② expresses $\bar{q}$ as a linear functional, and defines the dual vector $\gamma$, which will be proved not to depend on good h.

The proof goes as follows :
In a first step it is recalled that if a GE exists a PFE exists for an Arrow's assets market satisfying ① and ②' $\Pi^*(s) = \Pi(s)\, q(y_s^A)$ where $y_s^A$ is Arrow's asset paying 1 in state s and zero elsewhere. It is straightforward to check that budgets constraints are the same

and markets clear. ②' defines $\gamma(s)$ as Arrow's assets prices : $\gamma(s) = q(y_s^A) = \dfrac{\Pi^{*h}(s)}{\Pi^h(s)}$ for any good $h \in H$.

Because Arrow's assets form the cannonical basis of $\mathbb{R}^S$, any complete financial assets market Y is defined by a SxS matrix, call it Y again with colum vectors $y \in Y$. From ②',

for all $s \in S$ and all y, $q(y) = \sum_{y_s \in Y^A} y(s)\, q(y_s^A) = \sum_{s \in S} y(s)\, \gamma(s)$. Then because $\bar{q}$ is linear for any $\theta \in \Theta$, ② holds.■

The main interest of this theorem is its social welfare corollary as indicated by Arrow's paper title. The main point put forward by Arrow (and found since then is all microeconomics textbook) is the economy of markets needed to sustain a Pareto efficient allocation (H+S in a PFE instead of HxS in a GE). As emphasized by Duffie and Sonnenschein [1989]), Arrow's argument can be misleading in that it implicitely refers to the first welfare theorem when it uses the second one. In our opinion this is mainly due to the briefness and compactness of the paper.

Given a Pareto allocation of consumption plans, we know from the second welfare theorem a GE could be constructed to sustain it. From part ⇔ of theorem 1 a PFE would then exist and would sustain the same Pareto optimal allocation.

But from part ② of theorem 1, given a PFE of a complete financial market, its allocation of consumption plans is the same as the contingent good allocation of the GE with prices defined by ②. Then according to the first welfare theorem it is Pareto efficient.

Because $CMS_1$ is essential for theorem 1 to hold, and because the proofs of the welfare theorem go through relation ②, it has been thought by some authors (see Wiesmeth [1988] and his references) that the complete market hypothesis was a characterization of prices sustaining an efficient allocation. As the definition of Pareto efficiency for consumption plans generalizes to sets of states that are not finite a definition of complete market structure could go as the following :

$CMS_2$ : A financial assets market is complete if there exists a PFE and there is one PFE for which the equilibrium allocation is Pareto efficient. (see Wiesmeth for a detailed discussion of different definitions along this line).

In financial economics such a definition underlies many non theoretical litterature (without reference to a notion of equilibrium), often with a confusion between Pareto efficiency and informational efficiency (a PFE is a special case of rational expectations equilibrium, which in certain cases is informationally efficient : Grossmann [1978]).

However the definition used in theoretical work in finance as in Harrison-Kreps [1979] for instance, is not in the spirit of our $CMS_2$. Such a work does not deal with differentiated goods markets but only with one good, called numéraire, so that financial assets are set up to meet the demands for this good only. In other words the role of financial assets is to hedge investors against risks bore by the numéraire. In this sense the Black and Scholes model is thought of as a complete market model although it has only two financial assets (a bond and an option). This is because it allows agents to form completely hedged portfolios (portfolio strategies, actually).

This led to the definition that can be found (when it is not implicit) in financial litterature of the 80's wich states bluntly that a market is complete if and only if Span Y (the set of payments of portfolios made with marketed assets in Y) is equal to L, the set of choices of all agents ($\forall i \;\; X_i \subset L$).

Here we want to generalize our theorem 1 to the case where S is a probability space, so it is indeed such a definition we shall use.

However agents make their choices on consumption plans dealing with H physical goods. Given the definition of a PFE, and looking at budget constraint 1 :

$\Pi(x_i - w_i) = \sum_{y \in Y_\theta} \theta(y)\, y$, we see that what we need in order to go from GE to PFE is that any spending like $\Pi(x_i - w_i)$ must be met by some portfolio of Span Y.

Let then $D : S \rightarrow \mathbb{R}$ be a spending, meaning that for $(\mu.a.e) s \in S$ , $D(s)$ is in the same unit as $\Pi(s)\, w_i(s)$. We shall furthermore assume that all D's are in $L^2(S, \mathbb{R}, \mu)$. We shall then define :

**$\mathrm{CMS}_3$** : A financial market structure Y is complete if Span $Y = L^2(S, \mathbb{R}, \mu)$, or (assuming Y is restricted to no redundant assets), Y is a (Hamel) basis of $L^2$.

This means that for any spending $D \in L^2$, there exists a portfolio $\theta$ and a finite set of non redundant assets $Y_\theta$ such that $D = \sum_{y \in Y_\theta} \theta(y) y$.

**Theorem 2** : (Equivalence between GE and PFE when the set of states is a probability space $(S, \mathcal{S}, \mu)$).

Under $NA_1$, $NA_2$, $NA_3$ and $CMS_3$ for an exchange economy $(X_i, \succ_i, w_i)_{i \in I}$ and a financial market $Y \subset L^2(S, \mathbb{R}, \mu)$:

1) If there exists a GE $(p^*, (x_i^*)_{i \in I})$, with $p^*.x = \int_S \Pi^*(s)\, x(s)\, \mu(ds)$, then there exists a PFE $(\Pi, q, (\theta_i, x_i)_{i \in I})$ such that

① $x_i = x_i^* \quad \forall i \in I$

② $\forall \Theta \in L^2(S, \mathbb{R}, \mu) \quad \overline{q}(\Theta) = \int_S \Theta(s)\, \gamma(s)\, \mu(ds)$ where for $\mu$.a.e $s \in S$, and

$\forall h \in H \qquad \gamma(s)\Pi^h(s) = \Pi^{*h}(s)$

2) If there exists a PFE $(\Pi, q, (\theta_i, x_i)_{i \in I})$ then there exists a GE $(p^*, (x_i^*)_{i \in I})$ such that ① and ② hold.

**Proof** :

1) Assume ① and ② hold for a given GE.

Then, because of $CMS_3$ $\forall i \ \exists \Theta_i \ \Pi(s)\, x_i(s) - w_i(s)) = \Theta_i(s)$ which is $BC_1$ of PFE. Multiplying by $\gamma(s)$ and integrating gives :

$$\int_S \gamma(s)\, \Pi(s)\, (x_i(s) - w_i(s))\, \mu(ds) = \int_S \Theta(s)\, \gamma(s)\, \mu(ds)$$
$$\int_S \Pi^*(s)\, (x_i(s) - w_i(s))\, \mu(ds) = \overline{q}(\Theta)$$

The first hand side is 0 because of GE budget constraint, so $BC_2$ holds.

Because of ① $x_i$ is optimal as budget constraints are the same, and markets for goods clear.

Markets for assets clear as well : $\forall y \in Y \ \sum_{i \in Y} \theta_i(y) = 0$

From $BC_1$ in PFE suming up over I, we have : $\sum_{i \in I} \Theta_i(s) = \sum_{i \in I} \Pi(s)\, (x_i(s) - w_i(s))$

$$\text{But} \sum_{i\in I} \Theta_i(s) = \sum_{i\in I} \sum_{y\in Y_{\theta_i}} \theta_i(y)\, y(s) = \sum_{y\in \bigcup_{i\in I} Y_{\theta_i}} y(s) \sum_{i\in I} \theta_i(y)$$

As Y is formed of non redundant assets i.e. linearly independent vectors, $\sum_{i\in I} \Theta_i(s) = 0$ implies $\forall y \in Y \ \sum_{i\in I} \theta_i(y) = 0$.

2) Assume ① and ② hold for a given PFE. As $\Pi(s)(x_i(s) - w_i(s)) = \theta_i(s)$ integrating after multiplying by $\gamma(s)$ gives GE budget constraint as $\int \theta_i(s)\, \gamma(s)\, \mu d(s) = 0$.
Because of ① $x_i^*$ is optimal as budget constraints are the same, and markets are clear. ■

## 4. Complete markets and probabilities

In Arrow's model there is exactly one Arrow's asset say $y_s$ for each state $s\in S$. As $y_s(s') = \begin{cases} 1 \text{ if } s' = s \\ 0 \text{ otherwise} \end{cases}$, the vector $(q(y_s))_{y_s \in Y}$ defines the linear form $\overline{q}$ (section 1) (Y is the cannonical basis of $\mathbb{R}^S$).

But from relation (2') in theorem 1 proof $\dfrac{q(y_s)}{y_s(s)} = \dfrac{\Pi^{h^*}(s)}{\Pi^h(s)} = \dfrac{\Pi^{h^*}(s)\, w_i^h(s)}{\Pi^h(s)\, w_i(s)}$, as $y_s(s) = 1$ we get $q(y_s).\Pi^h(s)\, w_i^h(s) = \Pi^{*h}(s)\, w_i^h(s)$ . Suming up for all h we have :

$q(y_s).\Pi(s)\, w_i(s) = \Pi^*(s)\, w_i(s)$. Suming up for all i we have :

$q(y_s).m(s) = m^*(s)$ where m(s) is the total amount of money in the PFE model and $m^*(s)$ in the GE model in state s.

Now it is argued in Arrow's paper that $\forall s \in S \qquad m(s) = \sum_{s\in S} m^*(s) = m^*$.

This is because in Arrow's model assets are written in money defined by a GE, so that this money (invested in the portfolios) is paid back by the portfolios when state s obtains. From this obviously

$$\sum_{s\in S} q(y_s)m^* = \sum_{s\in S} m^*(s) \qquad \text{and} \sum_{s\in S} q(y_s) = 1.$$

Assets prices are positive (the $y_s$ are positive) then q is a probability distribution over S : $q(y_s)$ is the probability "the market assesses" to state s, write it $q(y_s) = \gamma(s)$.

This distribution is revealed by equilibrium prices and has nothing to do what so ever with any subjective or anticipated personal distributions agents may hold over the states $s\in S$. The consequence of this result is that the value of any portfolio $\Theta$, $\overline{q}(\Theta)$ is its expected payment with respect to this revealed distirbution, as :

$$\overline{q}(\Theta) = \sum_{y\in Y} \theta(y)\, q(y) = \sum_{s\in S} \theta(y_s)\, \gamma(s), \text{ and given that } y_s \text{ pays } 1, \Theta(s) = \theta(s).$$

The functioning of assets market is not described in Arrow's model which only deals with equilibrium, and PFE is described from GE.
However induced preferences on money (wealth) can be derived from preferences on consumption plans. From these induced preferences demands for assets will be derived from risk aversions of the agents. The interpretation of assets prices as probabilities means that for the same economy (at equilibrium) with the same equilibrium prices, if all agents take assets prices as the probabilities of the states, their demands for assets would be the same as those of risk neutral agents.

The consequence of this is that, taking marketed assets prices as given and using them as probabilities, the value f any redundant asset is its expected payment. This is one way to derive Black and Scholes formula interpreting the distributions of the bond and the stock prices as equilibrium distributions and deriving the option price by figuring out its expected return by the so called "risk neutrality argument". In summary we can write :

**Proposition 1** : Assuming there is a GE and $CMS_1$, taking Arrow's assets prices as given and using them as probabilities, all assets (marketable) can be priced by their expected payments.

In our model marketed assets, even when s is finite, are not necessarily Arrow ones, but with $CMS_1$ it only amounts to change the basis of Span Y to get Arrow's assets

and hence the prices (or probabilities) of the states - More tricky seems the fact that we did not assumed any relation between GE wealth and PFE wealth, assets are "pure" contracts. The fact that $\sum_{s \in S} q(y_s) = 1$ will happen as soon as we assume there is one riskless bond paying one in each state the price of whom is one. But it is obvious that this is always possible after normalization of assets price. Indeed we see that $BC_1$ of PFE is not changed by multiplying assets prices by any factor. Notice that this factor will affect GE prices through relation ②. However spot prices cannot be normalized as $BC_2$ of PFE would be changed unless payments of asset are multiplied by the same factor as well.
When S is a probability space we have a similar characterization of complete market structure.

**Proposition 2** : Assuming there is a GE and $CMS_3$, there exists a unique probability distribution $\nu$ on $(S,\mathscr{S})$ with respect to which the price of any marketable asset is its expected payment.

**Proof** : From theorem 2, $\nu$ is $\gamma.\mu$. $\gamma.\mu$ is a probability distribution for exactly the same reason than the one used in proposition 1. If the price of a riskless bond paying 1 in ($\mu$-almost) every state s is normalized to 1, then $\int \gamma d\mu = 1$. Recall that the existence of $\gamma$ relies on $\bar{q}$ being a positive linear continuous form meaning that at PFE each of the no Arbitrage conditions hold (see section 2).

## Conclusion

Arrow's paper contains most of the concepts that are used in modern theory of financial markets. His model does generalize to non finite set of states describing uncertainty so as to encompass general financial assets pricing.

Our theorems of equivalence between General Equilibrium and Perfect Foresight Equilibrium should precise several points :

- The welfare properties of PFE, or in Arrow's term, the role of securities in the optimal allocation of risk.

- The role of the complete market hypothesis and the reason why it takes the abstract mathematical form (Span Y = $L^2$) in modern finance.

- The probabilistic interpretation of assets prices under the CMS hypothesis. This interpretation extends to dynamic models (as the equivalent martingale property) and allows the pricing of assets by their expected payments.

- The necessary properties of equilibrium prices which are well defined by a linear, positive, continuous form. These properties are equivalent to three "no arbitrage" conditions that can be found in finance models not refering to equilibrium.

Other generalizations will introduce dynamic trades. This was first done in Radner [1972] who defined PFE, and after Black and Scholes [1973] model of option pricing was generalized by Cox-Ingersoll and Ross [1985] in a special type of GE model. The weakness of all these models remains the fundamental role of the complete market hypothesis and the unrealism of no arbitrage conditions (which exclude transaction costs and asymetric information). Furthermore dynamics introduces difficult problems about the consistency of intertemporal preferences which we shall present in further work.

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