Introduction
Pricing Pension Buy-outs
Financial Market Model
Liability Model
Application
Case Study I
Case Study II
Numerical Results
Conclusion

On the Pension Buy-out Pricing in Presence of Stochastic Interest and Mortality Rates

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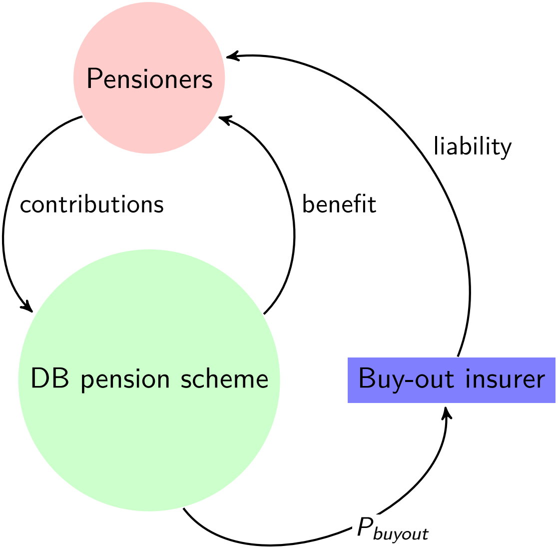
Content

1. Introduction
2. Pricing Pension Buy-outs
   - Financial Market Model
   - Liability Model
3. Application
   - Case Study I
   - Case Study II
4. Numerical Results
5. Conclusion

What is a Pension Buy-out?

Introduction
Pricing Pension Buy-outs
Financial Market Model
Liability Model
Application
Case Study I
Case Study II
Numerical Results
Conclusion



Motivation

- Discussion on the correlation between financial and actuarial risks by *Nicolini (2004); Miltersen and Persson (2006); Bauer et al. (2008); Jalen and Mamon (2009); Hoem et al. (2009) and Neyer et al. (2012), Dhaene et al. (2013).*

Aim of the Study

Pricing pension buy-outs under dependence assumption of financial and insurance markets using stochastic interest and mortality rate models

A Proposed Model for Pricing of the Pension Buy-outs under the Dependence Assumption

Introduction

Pricing Pension Buy-outs

Financial Market Model

Liability Model

Application

Case Study I

Case Study II

Numerical Results

Conclusion

- A combined modelling framework $(\Omega, \mathcal{G}, (\mathcal{G}_t), \mathbb{P})$ s.t. $\mathcal{G}_t = \mathcal{M}_t \vee \mathcal{F}_t$
  1. A probability space $(\Omega_1, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}_1)$ for financial market model
  2. A probability space $(\Omega_2, \mathcal{M}, \{\mathcal{M}_t\}, \mathbb{P}_2)$ which carries a pure insurance risk process
  3. $\Omega = \Omega_1 \times \Omega_2$
  4. $\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2$
- The deal guarantees to eliminate any potential asset-liability mismatching.
- The difference between asset and liability processes=one year put option spreads (*Lin et al. (2016)*)
- Redefine the liability process by extending the study of *Jalen and Mamon (2009)*

The Model Framework

Introduction
Pricing Pension Buy-outs
Financial Market Model
Liability Model
Application
Case Study I
Case Study II
Numerical Results
Conclusion

Proposition

(The Fair Price of the Buy-out Deal) *Under the dependence assumption of interest and mortality rate risks, the fair price of the buy-out deal at time t associated with both risks, conditioning on the filtration $\mathcal{G}_t$, is given by*

$$\begin{aligned} P_{buyout}(t) &= \frac{\sum_{t_i > t}^{M} E^{\mathbb{Q}}\left[e^{-\int_t^{t_i} r(s)ds} max(L(t_i) + N(t_i).C - PA(t_i), 0) | \mathcal{G}_t \right]}{L(t)} \\ &= \frac{\sum_{t_i > t}^{M} B(t, t_i, r(t)) E^{t_i}\left[max(L(t_i) + N(t_i).C - PA(t_i), 0) | \mathcal{G}_t \right]}{L(t)}, \end{aligned} \quad (1)$$

where $r(\cdot)$ is the stochastic interest rate for $t \geq 0$. $L(\cdot)$ is the liability of the pension scheme as defined in Definition 2 according to the fair price of a life annuity $a(.)$ and the number of survivors $N(.)$ at the relevant time. $a(.)$ is given by Proposition 3 in terms of the fair price of pure endowment deals $B_S(.)$. C is the annual survival benefit and $B(.)$ shows the fair price of a zero coupon bond as defined by Proposition 3. $PA(\cdot)$ is the value of pension portfolio which follows (2).

Introduction
Pricing Pension Buy-outs
Financial Market Model
Liability Model
Application
Case Study I
Case Study II
Numerical Results
Conclusion

The Model Framework (Continued)

- The value of pension asset portfolio at time t_i

$$\begin{aligned}P_{buyout}(t) &= \frac{\sum_{t_i>t}^{M} E^{\mathbb{Q}}\left[e^{-\int_t^{t_i} r(s)ds} max(L(t_i) + N(t_i).C - PA(t_i), 0)|\mathcal{G}_t\right]}{L(t)} \\ &= \frac{\sum_{t_i>t}^{M} B(t, t_i, r(t))E^{t_i}[max(L(t_i) + N(t_i).C - PA(t_i), 0)|\mathcal{G}_t]}{L(t)}\end{aligned}$$

Financial Market Model

Introduction
Pricing Pension Buy-outs
Financial Market Model
Liability Model
Application
Case Study I
Case Study II
Numerical Results
Conclusion

- The value of synthetic pension portfolio under $\mathbb{Q}$

$$PA(t) = PA(0)\exp\left(\left(r(t) - \frac{1}{2}\sigma_W^2(t)\right)t + \sum_{i=1}^{3}\pi_i(t)\sigma_i W_i^{\mathbb{Q}}(t)\right). \quad (2)$$

1. $r(t)$ is the risk free rate (stochastic).
2. $\sigma_W^2(t) = \sum_{i,j=1}^{3}\pi_i(t)\pi_j(t)\rho_{ij}\sigma_i\sigma_j$ is the correlation coefficient between asset i and j
3. $\pi(t) = (\pi_1(t), \pi_2(t), \pi_3(t))'$ are the weights of the assets at time t (*Fernholz (2002)*).
4. Moreover,

$$dW_i^{\mathbb{P}}(t) = dW_i^{\mathbb{Q}}(t) - \frac{\sum_1^3 \pi_i(t)\alpha_i - r(t)}{\sum_1^3 \pi_i(t)\sigma_i}.$$

Financial Market Model (Continued)

Introduction
Pricing Pension Buy-outs
Financial Market Model
Liability Model
Application
Case Study I
Case Study II
Numerical Results
Conclusion

- The dynamics of pension assets are represented by $A_1(t)$, $A_2(t)$ and $A_3(t)$ respectively for t values starting from 0 to a fixed value T.

$$dA_i(t) = A_i(t)[\alpha_i\, dt + \sigma_i\, dW_i(t)] \tag{3}$$

1. α_i :drift term and σ_i :instantaneous volatility for the asset i where $i = 1, 2, 3$.
2. $dW_i(t)$ is the Wiener process with zero mean and t variance under $\mathbb{P}_1$.
3. $Cov(dW_i(t), dW_j(t)) = \rho_{ij}\, t,\;\; i = 1, 2, 3;\;\; j = 1, 2, 3;\;\; i \neq j$
4. ρ_{ij} :the correlation coefficient between assets i and j
5. $Cov(A_i(t), A_j(t)) = \rho_{ij}\sigma_i\sigma_j t,\;\; i = 1, 2, 3;\;\; j = 1, 2, 3;\;\; i \neq j$

The Model Framework (Continued)

Introduction
Pricing Pension Buy-outs
Financial Market Model
Liability Model
Application
Case Study I
Case Study II
Numerical Results
Conclusion

- Liability value of the pension scheme at time t_i
- Number of survivors at time t_i

$$
\begin{aligned}
P_{buyout} &= \frac{\sum_{t_i>t}^{M} E^{\mathbb{Q}}\left[e^{-\int_t^{t_i} r(s)ds} max(L(t_i) + N(t_i).C - PA(t_i), 0)|\mathcal{G}_t\right]}{L(t)} \\
&= \frac{\sum_{t_i>t}^{M} B(t, t_i, r(t))E^{t_i}[max(L(t_i) + N(t_i).C - PA(t_i), 0)|\mathcal{G}_t]}{L(t)}
\end{aligned}
$$

Liability Model

Introduction
Pricing Pension Buy-outs
Financial Market Model
Liability Model
Application
Case Study I
Case Study II
Numerical Results
Conclusion

Proposition

(Liability Process) *The pension liability process $L(t)$ at time t is derived using the number of survivors process $N(t)$ and immediate annuity factors at the relevant time as follows:*

$$L(t) = N(t) \times a(t, T, C)$$

where $a(t, T, C)$ is given by (4). Here $N(t)$ process is determined according to the force of mortality rate dynamics.

Liability Model (Continued)

Introduction
Pricing Pension Buy-outs
Financial Market Model
Liability Model
Application
Case Study I
Case Study II
Numerical Results
Conclusion

Proposition

The fair price of an annuity contract which guarantees to make survival benefits C under the dependence assumption is asserted as

$$a(t, T, C) = \sum_{t_i=t+1}^{T} B_S(t, t_i, C) \tag{4}$$

where

$$B_S(t, t_i, C) = 1_{\tau > t} \times C \times B(t, t_i, r_t) \times \tilde{p}(t, t_i, x)$$

Here

- $B_S(t, t_i, C)$ *:the fair price of a pure endowment contract for a constant survival benefit C to an individual aged x at time t until period t_i where $t_i \in \{t+1, t+2, \dots, T = M\}$ and $M = min\{t : N(t) = 0\}$.*
- $\tau(x)$ *:the future lifetime of an individual aged x.*
- $B(t, t_i, r_t) = E^{\mathbb{Q}}\left[\exp\left(-\int_t^{t_i} r(s)ds\right) |\mathcal{G}_t\right].$
- $\tilde{p}(t, t_i, x) = E^{t_i}\left[\exp\left(-\int_t^{t_i} \mu(s, x+s)ds\right) |\mathcal{G}_t\right].$

Model Framework (Continued)

Introduction
Pricing Pension Buy-outs
Financial Market Model
Liability Model
Application
Case Study I
Case Study II
Numerical Results
Conclusion

Remark

(The Value of Pension Portfolio after Annuity Payments) *Let $PA(t^+)$ be the value of the pension portfolio right after possible adjustments such as any potential funding contributions in the case of underfunded event or the annuity payments at time t. $PA(t^+)$ is calculated at the end of time t as below:*

$$PA(t^+) = max\{PA(t) - N(t) \times C, L(t)\},$$

for $t = 1, 2, \ldots, T$. The number of survivors at time t, $N(t)$, is obtained according to the force of mortality rates.

Application

- We obtain the numerical results according to the below assumptions:
  1. Short rate is assumed to follow Vasicek model.
  2. OU process is applied to mortality dynamics.

Aim

1. Attain zero-coupon bond prices under measure $\mathbb{Q}$ and survival rates under measure $\mathbb{P}^T$ in order to drive the liability process.
2. Derive the payoff under measure $\mathbb{P}^T$.

Assumptions

1. Monte Carlo simulation using 5000 sample paths
2. No annual contributions
3. No pension gap at inception

Introduction
Pricing Pension Buy-outs
Financial Market Model
Liability Model
Application
Case Study I
Case Study II
Numerical Results
Conclusion

Case Study I

1. Let us suppose that the short rate dynamics follow the Vasicek model as

$$dr(t) = a^r(b^r - r(t))dt + c^r dW^r(t). \tag{5}$$

2. Let us also suppose the force of mortality rate dynamics satisfy the below construction:

$$d\mu(t) = a^\mu \mu(t)dt + c^\mu dW^\mu(t), \tag{6}$$

where $dW^\mu(t) = \rho dW^r(t) + \sqrt{1-\rho^2}dW(t)$ under measure $\mathbb{Q}$. Here $W^r(t)$ and $W(t)$ are independent Wiener processes and ρ is a correlation coefficient.

Case Study I (Continued)

Introduction
Pricing Pension Buy-outs
Financial Market Model
Liability Model
Application
Case Study I
Case Study II
Numerical Results
Conclusion

Proposition

(Vasicek Term Structure) *Let us assume that the short rate dynamics satisfy (5). The zero coupon bond prices under Vasicek model are obtained for* $t \in [0, t_i]$ *where* $t_i \in \{t+1, t+2, \ldots, M\}$ *by*

$$B(t, t_i, r(t)) = \exp(-A^r(t, t_i) r(t) + B^r(t, t_i)),$$

where

$$\begin{aligned} A^r(t, t_i) &= \frac{1 - e^{-a^r(t_i - t)}}{a^r}, \\ B^r(t, t_i) &= \left(b^r - \frac{c^{2r}}{2a^{2r}}\right)[A^r(t, t_i) - (t_i - t)] - \frac{c^{2r} A^r(t, t_i)^2}{4a^r}. \end{aligned}$$

Case Study I (Continued)

Introduction
Pricing Pension Buy-outs
Financial Market Model
Liability Model
Application
Case Study I
Case Study II
Numerical Results
Conclusion

Proposition

(Vasicek Dynamics under forward measure) *Let us assume that short rates follow a Vasicek model as described by (5). The dynamics of the short rate under the forward measure* $\mathbb{P}^{t_i}$ *is*

$$dr(t) = (a^r b^r - a^r r(t) - (c^r)^2 A^r(t, t_i))dt + c^r d\tilde{W}^r(t), \quad (7)$$

where $d\tilde{W}^r(t) = dW^r(t) + c^r A^r(t, t_i)dt$ *for all* $t_i \in \{t+1, \dots, M\}$.

Case Study I (Continued)

Introduction

Pricing Pension Buy-outs

Financial Market Model

Liability Model

Application

Case Study I

Case Study II

Numerical Results

Conclusion

Proposition

(The OU Dynamics using Vasicek Model under forward measure) *Let us assume that the force of mortality rate dynamics satisfy expression (6). When we choose Vasicek model for the short rate dynamics, the behaviour of formula (6) under the forward measure $\mathbb{P}^{t_i}$ is attained as*

$$d\mu(t) = (a^{\mu}\mu(t) - c^{\mu}\rho A^{r}(t, t_i)c^{r})dt + c^{\mu} d\tilde{W}^{\mu}(t), \tag{8}$$

where

$$d\tilde{W}^{\mu}(t) = \rho d\tilde{W}^{r}(t) + \sqrt{1-\rho^2} d\tilde{W}(t). \tag{9}$$

Here, $d\tilde{W}^{r}(t) = dW^{r}(t) + c^{r}A^{r}(t, t_i)dt$ *and* $d\tilde{W}(t) = dW(t)$ *for all* $t_i \in \{t+1, \ldots, M\}$.

Case Study I (Continued)

Introduction
Pricing Pension Buy-outs
Financial Market Model
Liability Model
Application
Case Study I
Case Study II
Numerical Results
Conclusion

Proposition

(Survival Probabilities using Vasicek Model under $\mathbb{P}^{t_i}$) *Let us suppose that the dynamics of the force of mortality rate is given by (8). Then, the survival rates under the forward measure $\mathbb{P}^{t_i}$ satisy the below formula:*

$$\tilde{p}(t, t_i, x) = \exp(-G^\mu(t, t_i)\mu(t) + H^\mu(t, t_i)),$$

where

$$G^\mu(t, t_i) = \frac{e^{a^\mu(t_i - t)} - 1}{a^\mu},$$

and

$$H^\mu(t, t_i) = \left(\frac{\rho c^r c^\mu}{a^r a^\mu} - \frac{(c^\mu)^2}{2(a^\mu)^2}\right)[G^\mu(t, t_i) - (t_i - t)] + \frac{\rho c^r c^\mu}{a^r a^\mu}[A^r(t, t_i) - \Phi(t, t_i)] + \frac{(c^\mu)^2}{4a^\mu}(G^\mu(t, t_i))^2,$$

$$\Phi(t, t_i) = \frac{(1 - e^{-(a^r - a^\mu)(t_i - t)})}{(a^r - a^\mu)} \quad \text{for all} \quad t \leq t_i.$$

Case Study I (Continued)

Introduction
Pricing Pension Buy-outs
Financial Market Model
Liability Model
Application
Case Study I
Case Study II
Numerical Results
Conclusion

- According to Elliott and Kopp (2004), $(r(t), \mu(t))$ is a bivariate normal random variable under measure $\mathbb{P}^{t_i}$ with the following parameters:

$$
\begin{aligned}
E^{t_i}[r(t)] &= e^{-a^r t} r(0) + b^r (1 - e^{-a^r t}) - \frac{(c^r)^2}{(a^r)^2} \left[(1 - e^{-a^r t}) + \frac{1}{2} e^{-a^r t_i} (e^{a^r t} - e^{-a^r t}) \right], \\
Var^{t_i}(r(t)) &= \frac{(c^r)^2}{2a^r} (1 - e^{-2a^r t}),
\end{aligned}
$$

and

$$
\begin{aligned}
E^{t_i}[\mu(t)] &= e^{a^\mu t} \mu(0) + \frac{c^\mu \rho c^r}{a^r a^\mu} (1 - e^{a^\mu t}) + \frac{c^\mu c^r \rho e^{-a^r t_i}}{a^r (a^r - a^\mu)} (e^{a^r t} - e^{a^\mu t}), \\
Var^{t_i}(\mu(t)) &= \frac{(c^\mu)^2}{2a^\mu} (e^{2a^\mu t} - 1),
\end{aligned}
$$

where

$$
Cov[r(t), \mu(t)] = \frac{c^r c^\mu \rho}{a^r - a^\mu} (1 - e^{-(a^r - a^\mu) t}).
$$

Introduction
Pricing Pension Buy-outs
Financial Market Model
Liability Model
Application
Case Study I
Case Study II
Numerical Results
Conclusion

Case Study II

Proposition

(The CIR Term Structure) *Let us assume that the short rate follows a CIR model as*

$$dr(t) = a^r(b^r - r(t))dt + \sigma^r \sqrt{r(t)}dW(t).$$

The price of a zero coupon bond at time t with maturity $t_i \in \{t+1, \dots, M\}$ under CIR model is

$$B(t, t_i, r(t)) = \exp(-A^r(t_i - t)r(t) + B^r(t_i - t)),$$

where

$$\begin{aligned} A^r(t_i - t) &= \frac{2(e^{\gamma(t_i - t)} - 1)}{(\gamma + a^r)(e^{\gamma(t_i - t)} - 1) + 2\gamma}, \\ B^r(t_i - t) &= \frac{2a^r b^r}{\sigma^{2r}} \log\left[\frac{2\gamma e^{(a^r + \gamma)\frac{(t_i - t)}{2}}}{(\gamma + a^r)(e^{\gamma(t_i - t)} - 1) + 2\gamma}\right], \end{aligned}$$

and $\gamma = \sqrt{a^{2r} + 2\sigma^{2r}}$.

Case Study II (Continued)

Introduction

Pricing Pension Buy-outs

Financial Market Model

Liability Model

Application

Case Study I

Case Study II

Numerical Results

Conclusion

Proposition

(The OU Dynamics using CIR Model under $\mathbb{P}^{t_i}$) *Let us assume the force of mortality rate dynamics satisfy the OU process as described by expression (6) under $\mathbb{Q}$. When we choose CIR model to represent the short rate dynamics, the SDE of formula (6) under the forward measure $\mathbb{P}^{t_i}$ is provided by*

$$d\mu(t) = [a^{\mu}\mu(t) - c^{\mu}\rho\sigma^{r}\sqrt{r(t)}A^{r}(t_i - t)]dt + c^{\mu}d\tilde{W}^{\mu}(t), \tag{10}$$

where the Wiener process under $\mathbb{P}^{t_i}$ is

$$d\tilde{W}^{\mu}(t) = \rho d\tilde{W}^{r}(t) + \sqrt{1-\rho^2}d\tilde{W}(t). \tag{11}$$

Here, $d\tilde{W}^{r}(t) = dW^{r}(t) + \sigma^{r}\sqrt{r(t)}A^{r}(t_i - t)dt$ and $d\tilde{W}(t) = dW(t)$ for all $t_i \in \{t+1, \dots, M\}$.

Numerical Results

Introduction
Pricing Pension Buy-outs
Financial Market Model
Liability Model
Application
Case Study I
Case Study II
Numerical Results
Conclusion

- The plan funds are assumed to be invested in the *S&P 500 index* $A_1(t)$, the *Merrill Lynch corporate bond index* $A_2(t)$ and the *3-month T-bill* $A_3(t)$.

Table 1: Parameter Estimates of Three Pension Assets

Parameter	Estimate	Parameter	Estimate
α_1	0.1097	σ_1	0.1458
α_2	0.0959	σ_2	0.0770
α_3	0.0631	σ_3	0.0286

- Estimated correlation coefficients

$$\rho = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.2905 & 0.0615 \\ 0.2905 & 1 & 0.0129 \\ 0.0615 & 0.0129 & 1 \end{bmatrix}$$

Numerical Results (Continued)

Introduction
Pricing Pension Buy-outs
Financial Market Model
Liability Model
Application
Case Study I
Case Study II
Numerical Results
Conclusion

Table 2: Parameter values used in the numerical analysis

Parameter set for the application
Contract details: $C = 60000$, $N(0) = 10000$, $PA(0) = L(0)$, $x = 65$
Interest rate model: $a^r = 0.045398$, $b^r = 0.090070$, $c^r = 0.003789$
Mortality model: $a^\mu = 0.078282$, $c^\mu = 0.002271$
N=5000, $dt = 1/252$, $w = 111$

Numerical Results (Continued)

Introduction

Pricing Pension Buy-outs

Financial Market Model

Liability Model

Application

Case Study I

Case Study II

Numerical Results

Conclusion

Table 3: Actuarial fair prices of a buy-out deal under dependence assumption

ρ	Buy-out Price	
	Case Study I	Case Study II
-0.7	0.3602789	0.2524002
-0.2	0.3568392	0.2516408
0	0.3554243	0.2513238
0.2	0.3539935	0.2510038
0.7	0.3503689	0.2502191

Introduction
Pricing Pension Buy-outs
Financial Market Model
Liability Model
Application
Case Study I
Case Study II
Numerical Results
Conclusion

Concluding Remarks and Future Research

- Pricing buy-outs under the dependence assumption $\equiv$ utilize the change of measure technique to simplify the general pricing formula
- The impact of correlation coefficient ρ on the buy-out price $P_{buyout}(.)$
- Confidence intervals for the Monte Carlo simulations
- Solvency II conditions from the perspective of the buy-out insurer

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Introduction

Pricing Pension Buy-outs

Financial Market Model

Liability Model

Application

Case Study I

Case Study II

Numerical Results

Conclusion

Thanks for your attention.