The Equity Premium Puzzle, the consumption puzzle and the investment puzzle with Recursive Utility: Implications for optimal pensions

Knut K. Aase
Norwegian School of Economics
5045 Bergen, Norway

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1. Papers that this talk draws from

- (1) Knut K. Aase, "Recursive utility using the stochastic maximum principle." *Quantitative Economics; Journal of the Econometric Society* 7(3), 2016, pp 859-887.

- (2) Knut K. Aase, "Life Insurance and Pension Contracts I: The Time Additive Life Cycle Model." *Astin Bulletin*, Volume 45(1), 2015, pp 1-47.

- (3) Knut K.Aase, "Life Insurance and Pension Contracts II: The life cycle model with recursive utility." *Astin Bulletin*, Volume 46(1), 2016, pp 71-102.

2. Introduction

- Rational expectations, a cornerstone of modern economics and finance, has been under attack for quite some time.

- Questions like the following are asked: Are asset prices too volatile relative to the information arriving in the market? Is the mean risk premium on equities over the riskless rate too large? Is the real interest rate too low? Is the market's risk aversion too high?

- Some of these questions were raised after the paper by Mehra and Prescott (1985), which discovered the "Equity Premium Puzzle".

- Several other articles around this time arrived at similar conclusions with corresponding sets of questions (e.g., Hansen and Singleton).

- The puzzle is that they were unable to find a plausible parameter pair of the utility discount rate and the relative risk aversion to match the sample mean of the annual real rate of interest and of the equity premium over the 90-year period.

- These problems have received a lot of attention during the more than 30 years after the discovery of the problem in the early 1980ies.

- It is not my intention to discuss all the different approaches here. In the paper Aase (2016) there are, however, clear answers to all questions asked at the beginning; and they are:

  No, no, no, and no again!

- All this and more is now clear, and this what this talk is about.

- Simply told, the solution is to use recursive utility instead of additive and separable expected utility, but keep the rest of the standard model as it is.

- The advantage with this should be clear: If only changing the preferences solves the problem, then we know what was wrong, and we have *learnt* something.

2.1. Outline

- We look at; Equilibrium: risk premiums, the equity premium, the real interest rate;
- Optimal consumption;
- Application to pensions;
- Applications to portfolio choice.

3. The continuous-time representation

- Recursive utility in continuous time: $U : L_+ \to \mathbb{R}$ is defined by two primitive functions: $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $A : \mathbb{R} \to \mathbb{R}$.

- The function $f(c_t, V_t)$ represents a felicity index at time t, and A is associated with a measure of absolute risk aversion (of the Arrow-Pratt type) for the agent. $V_t := U_t(c)$. In discrete time: $V_t := U(c_t, \cdots, c_\tau)$.

- In addition to current consumption c_t, the function f also depends on future utility V_t time t, a stochastic process with volatility Z_t at time t.

- The representation is: $V_\tau = 0$, and

$$V_t = E_t\Big\{ \int_t^\tau \big(f(c_s, V_s) - \frac{1}{2}A(V_s)\, Z_s' Z_s\big) ds \Big\}, \quad t \in [0, \tau] \qquad (1)$$

- Since $V_\tau = 0$ and $\int Z_t dB_t$ is is assumed to be a martingale, (1) has the backward stochastic differential equation representation (BSDE)

$$\begin{cases} dV_t = \Big(- f(t, c_t, V_t) + \frac{1}{2} A(V_t)\, Z_t' Z_t \Big)\, dt + Z_t' dB_t \\ V_\tau = 0. \end{cases} \tag{2}$$

- If, for each consumption process c_t, there is a well-defined *pair* (V, Z) of a utility process V *and* volatility Z_t satisfying the associated BSDE, the stochastic differential utility U is defined by $U(c) = V_0$.

- The pair (f, A) generating V is called an aggregator.

- One may think of the term $m(V_t) = -\frac{1}{2} A(V_t) Z_t'\, Z_t$ as the Arrow-Pratt approximation to the certainty equivalent of V_t. For continuous processes in continuous time there is no element of approximation involved.

- The defining equation for utility is associated to the quadratic BSDE, and existence and uniqueness of solutions to such equations is in general far from granted.

- These topics have been dealt with in the original paper Duffie and Epstein (1992b).

- These questions are also part of contemporary research in applied mathematics, see e.g., Øksendal and Sulem (2014), or Peng (1990).

- For the particular BSDE that we end up with, existence and uniqueness follows from Duffie and Lions (1992). See also Schroder and Skiadas (1999) for the life cycle model.

- We work with the following specification, which corresponds to the aggregator (f, A) with the constant elasticity of substitution (CES) form

$$f(c, v) = \frac{\delta}{1-\rho} \frac{c^{(1-\rho)} - v^{(1-\rho)}}{v^{-\rho}} \quad \text{and} \quad A(v) = \frac{\gamma}{v}. \tag{3}$$

- The parameters have the same interpretations as for the discrete-time model.

  $\delta =$ the impatience rate; $\rho =$ resistance to consumption substitution in a deterministic world; $\gamma =$ relative risk aversion.

  $EIS := \psi := \frac{1}{\rho} =$ elasticity of intertemporal substitution in consumption.

- This preference fall in the Kreps-Porteus class when the certainty equivalent is derived from expected utility.

- This is the equivalence of the Epstein-Zin (1989-91) aggregator in discrete time. I consider it as the most minimalistic, non-trivial extension of the standard expected utility model.

- The discrete time utility:

$$V_t = f(u(c_t), m_{t+1}) = \Big((1-\beta)c_t^{1-\rho} + \beta\big(E_t(V_{t+1}^{1-\gamma})\big)^{\frac{1-\rho}{1-\gamma}}\Big)^{\frac{1}{1-\rho}} \quad (4)$$

with $0 < \beta < 1$, $1 \neq \gamma > 0$, $\rho > 0, \rho \neq 1$.

- The parameter γ is the relative risk aversion, ρ is time preference, the inverse of the EIS-parameter ψ, and β is the impatience factor, with impatience rate $\delta = -\ln(\beta)$. When the parameter β is large, the agent puts more weight on the future and less weight on the present, in accordance with the impatience interpretation of this parameter.

4. Equilibrium

- The consumer is characterized by a utility function U and an endowment process e. The representative agent's problem is to solve

$$\sup_{c \in L} U(c)$$

subject to

$$E\Big\{\int_0^\tau c_t \pi_t dt\Big\} \leq E\Big\{\int_0^\tau e_t \pi_t dt\Big\}.$$

- Here π_t is the state price deflator (as of time zero). It represents the Arrow-Debreu state prices in units of probability, and plays a major role in this theory.

4.1. The first order conditions

- We use Recursive utility and the stochastic maximum principle.

- Maximizing the Hamiltonian with respect to c gives the first order equation

$$\alpha\, \pi_t = Y(t) \frac{\partial \tilde{f}}{\partial c}(t, c_t^*, V(t), Z_t) \quad \text{a.s. for all } t \in [0, \tau]. \tag{5}$$

where

$$\tilde{f}(t, c_t, V_t, Z_t) = f(c_t, V_t) - \frac{1}{2} A(V_t)\, Z_t' Z_t.$$

- Notice that the state price deflator π_t depends, through the adjoint variable Y_t, on the entire optimal paths (c_s, V_s, Z_s) for $0 \leq s \leq t$.

- This follows from the representation

$$Y_t = \exp\Big(\int_0^t \big(\frac{\partial f}{\partial v}(s, c_s) + \frac{1}{2}\frac{\gamma(1-\gamma)}{V_s^2} Z'(s) Z_s\big) ds - \int_0^t \frac{\gamma}{V_s} Z_s dB_s \Big). \tag{6}$$

- Compare to the additive and separable expected utility preference:

$$U(c) = E\Big(\int_0^\tau u(c_t) e^{-\delta t} \Big)$$

- The FOC are then

$$\alpha \pi_t = e^{-\delta t} \frac{du(c_t^*)}{dc}.$$

It depends only on the present time t.

4.2. Connecting to the market

- The market-price-of-risk η_t is given by

$$\sigma_t \eta_t = \nu_t$$

where the n-th component of ν_t equals $(\mu_n(t) - r_t)$ the excess rate of return on security $n, n = 1, 2, \cdots, N$.

- Recall: $\pi_t = \xi_t e^{-\int_0^t r_u du)}$, $\xi_t = \xi_0 e^{-\frac{1}{2}\int_0^t \eta_s^2 ds - \int_0^t \eta_s dB_s}$.

$\frac{dQ}{dP} = \xi_T$, $\xi_t = E_t(\xi_T)$. ξ_t is the "density" process of the change of measure.

- The optimal consumption is given by (follows from the first order conditions)

$$c_t^* = \left(\frac{\alpha \pi_t}{\delta Y_t}\right)^{-\frac{1}{\rho}} V_t.$$

We use the notation $Z(t) = V_t \sigma_V(t)$ from now on.

- By the multidimensional version of Ito's lemma we can now calculate the dynamics of the optimal consumption as follows:

$$dc_t^* = \frac{\partial c}{\partial \pi} d\pi_t + \frac{\partial c}{\partial v} dV_t + \frac{\partial c}{\partial y} dY_t + \frac{1}{2}\frac{\partial^2 c}{\partial \pi^2} d\pi_t^2 + \frac{1}{2}\frac{\partial^2 c}{\partial v^2} dV_t^2 + \frac{1}{2}\frac{\partial^2 c}{\partial y^2} dY_t^2 + \frac{\partial^2 c}{\partial \pi \partial v} d\pi_t dV_t + \frac{\partial^2 c}{\partial \pi \partial y} d\pi_t dY_t + \frac{\partial^2 c}{\partial v \partial y} dV_t dY_t. \quad (7)$$

- The stochastic representation for the consumption growth rate is given

by

$$\frac{dc_t^*}{c_t^*} = \mu_c(t)\,dt + \sigma_c(t)\,dB_t. \tag{8}$$

- We now use the representations for the processes π_t, V_t and Y_t.

- After a fair amount calculations, the result is

$$\mu_c(t) = \frac{1}{\rho}\,(r_t - \delta) + \frac{1}{2}\frac{1}{\rho}\,(1 + \frac{1}{\rho})\,\eta_t'\eta_t - \frac{(\gamma - \rho)}{\rho^2}\,\eta_t'\,\sigma_V(t) + \frac{1}{2}\frac{(\gamma - \rho)\gamma(1 - \rho)}{\rho^2}\sigma_V'(t)\sigma_V(t) \tag{9}$$

and

- $$\sigma_c(t) = \frac{1}{\rho}\Big(\eta_t + (\rho - \gamma)\sigma_V(t)\Big). \tag{10}$$

- Here $\sigma_V(t)$ and V_t exist as a solution to the the backward stochastic differential equation for V.

- When $\rho = \gamma$ (or $\gamma = 1/\psi$), the optimal consumption dynamics for the conventional model results.

- When consumption is considered as aggregate consumption in society, and the consumer is the representative agent, we obtain that

$$\varphi_t' \sigma_t \eta_t = \mu_W(t) - r_t.$$

where $\varphi_t' \sigma_t = \sigma_W'(t)$ is the volatility of the wealth portfolio, by market clearing.

- Using (10), this gives the following expression for the risk premium of the wealth portfolio

$$\mu_W(t) - r_t = \rho \sigma_W'(t) \sigma_c(t) + (\gamma - \rho) \sigma_W'(t) \sigma_V(t). \tag{11}$$

- The idea is here that the representative agent optimally consumes the aggregate endowment, and prices and the interest rate adjust such that this is the optimal solution.

- Using that the utility function V is homogeneous of degree one in consumption, we can determine the volatility $\sigma_W(t)$ of the wealth portfolio in terms of the utility volatility $\sigma_V(t)$, the parameter ρ and the volatility of the aggregate consumption process.

- Turning this relationship around, we have at the same time the volatility of utility in terms of the volatility of the wealth portfolio and the volatility of the aggregate consumption process.

- Thus $\sigma_V(t)$ is connected to quantities that may be estimated from market and consumption data.

- By market clearing again, the property that recursive utility is homogeneous of degree 1, and by diffusion invariance we can show that

$$\sigma_W(t) = (1-\rho)\sigma_V(t) + \rho\sigma_c(t)$$

where $\sigma_W(t)$ is the volatility of the return of the wealth portfolio (see Aase (2016)). This internalizes the probability distribution of wealth.

- From this relationship we get

$$\sigma_V(t) = (\sigma_W(t) - \rho\sigma_c(t))/(1-\rho)$$

connecting $\sigma_V(t)$ to 'observables' and the given preference parameter ρ.

- This finally gives the expression for the risk premium of any risky asset denoted by R:

4.3. The Equity Premium

- The expression for the risk premium of any risky security with return rate $\mu_R(t)$ is given by the following formula

- The result for the risk premium of any risky security is

$$\mu_R(t) - r_t = \frac{\rho(1-\gamma)}{1-\rho}\sigma_{c,R}(t) + \frac{\gamma-\rho}{1-\rho}\sigma_{W,R}(t) \tag{12}$$

- The first term: the consumption based CAPM of Breeden (1979); the second term: the market based CAPM of Mossin (1966).

- The market based CAPM is only valid in a "timeless" setting, i.e., a one period model with consumption only on the terminal time, in its original derivation.

- Consider the last term, and let $\gamma > \rho$. This term may grow large by letting ρ approach 1 from below.

- If $\gamma < \rho$, the same conclusion follows by leting ρ approach 1 from above.

- That is, the expression (12) can explain a wide range of possible equity premiums for plausible values of these two parameters.

- The trouble with the standard model is that it could not explain the 'large' equity premium of 6% in the 90 years of data.

- Here the *equity premium* $(\mu_M(t) - r_t)$ was estimated to 6% and $\sigma_{c,M}(t)$ was estimated to 0.0023.

- Setting $\gamma = \rho$ we obtain the expected utility based premium:

$$\mu_M(t) - r_t = \gamma\, \sigma_{c,M}(t) \tag{13}$$

- The implication of this is that γ is larger than 26, for this 'to add up'.

- The general model contains certain parameters and relationships between various quantities. Some of these we know from the 'outside' of the model, and such facts may be used to tell us something about other parameters in the model, which is what we do when we *calibrate* the model. The number $\gamma = 26$ is a result of this line of thought.

- The above relates to the data in the Mehra and Prescott-study, who used key consumption and market (S&P-500) US-data for the period 1889-1978.

- Has the world finally adjusted to the model?

- Newer data, also including the wealth portfolio does not indicate that this has taken place.

- We have used key US-data for the period 1960-2015. The standard model then gives $\gamma = \rho = 197$ and $\delta = 0.22$

4.4. The equilibrium risk-free interest rate

- It is given by

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2}\frac{\rho(1-\rho\gamma)}{1-\rho}\sigma_c'(t)\sigma_c(t) + \frac{1}{2}\frac{\rho-\gamma}{1-\rho}\sigma_W'(t)\sigma_W(t). \quad (14)$$

- The two first terms on the right-hand side are the familiar terms in the Ramsey (1928)-model (a deterministic model).

- The third term corresponds to the "precautionary savings term" in the conventional model:

$$\frac{1}{2}\rho(1+\rho)\,\sigma_c'(t)\sigma_c(t).$$

If $\rho = \gamma$ in (14), this expression results.

- The last term in (14) is new, and comes from the recursive specification of utility.

- When $\gamma > \rho$ and $\rho < 1$ this term is negative, and can be low by letting ρ approach 1 from below.

- When $\gamma < \rho$ and $\rho > 1$ this term is also negative, and can be low by letting ρ approach 1 from above.

- This helps explaining the "low" real interest rate observed during the 90-year period of the data.

- When the wealth uncertainty increases, the "prudent" recursive utility maximizer saves, and the interest rate falls.

- Let $\rho = \gamma$, and we get the traditional expression:

$$r_t = \delta + \gamma\mu_c(t) - \frac{1}{2}\gamma(1+\gamma)\,\sigma_c'(t)\sigma_c(t). \tag{15}$$

- Even when γ is as large as 26, δ will still have to be *negative* for this formula to provide an interest rate below 1%.

- Table 1 presents the summary statistics of the data used in the Mehra and Prescott (1985)-paper:

	Expectation	Standard dev.	covariances
Consumption growth	1.81%	3.55%	$\hat{\sigma}_{cM} = .002268$
Return S&P-500	6.78%	15.84%	$\hat{\sigma}_{Mb} = .001477$
Government bills	0.80%	5.74%	$\hat{\sigma}_{cb} = -.000149$
Equity premium	5.98%	15.95%	

Table 1: Key US-data for the time period 1889 -1978. Continuous-time compounding. $\hat{\kappa}_{M,c} = .4033$.

- By fixing the impatience rate δ to some reasonable value, $\delta = 0.03$ say, one solution to the two equations (12) and (14) with $R = M$ (M is the market portfolio) and $W = M$, this yields

$$\delta = 0.03, \ \gamma = 1.74 \quad \text{and} \quad \rho = 0.48.$$

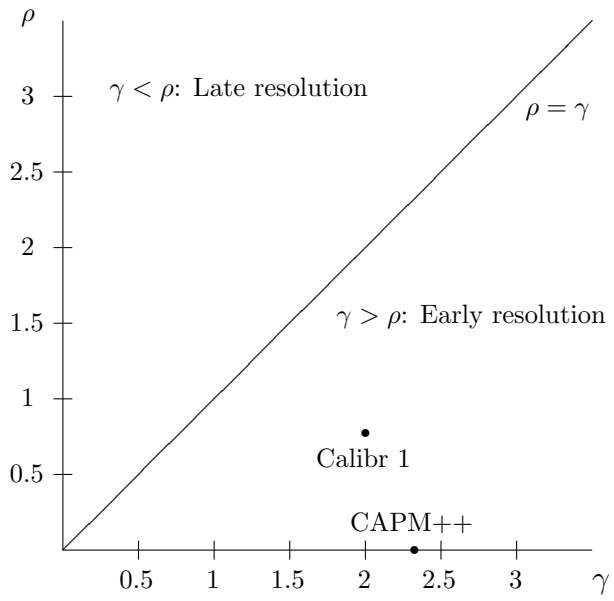

Figure 1: Calibration points in the (γ, ρ)-space

- In contrast, a similar calibration of the expected utility model leads to the (unique) values $\gamma = 26$ and $\delta = -.015$.

  This is the Equity Premium Puzzle.

- It is tempting to cite Albert Einstein: "When studying a problem one should apply the simplest model possible, not simpler."

- For the newer data set we find, for example:
  $\gamma = 2.00$, $\rho = 0.99$, $EIS = 1.01$, and $\delta = 0.01$.

- For more details, see Aase, K. K (2016): "Recursive utility using the stochastic maximum principle". *Quantitative Economics* 7(3), (2016).

5. Optimal Consumption: The Problem

- Consider an individual (U, e), with utility function $U(c)$ for a life-time consumption stream $c = \{c_t,\ 0 \le t \le \tau\}$, and with an endowment process $e = \{e_t,\ 0 \le t \le \tau\}$.

- Here $U : L_+ \to \mathbb{R}$, where

$$L = \Big\{c : c_t \text{ is } \mathcal{F}_t\text{-adapted, and } E\Big(\int_0^\tau c_s^2 ds\Big) < \infty\Big\}.$$

- For a price π_t of the consumption good, again the problem is to solve

$$\sup_{c \in L} U(c), \tag{16}$$

subject to

$$E\Big\{\int_0^\tau \pi_t c_t \, dt\Big\} \le E\Big\{\int_0^\tau \pi_t e_t \, dt\Big\} := w. \tag{17}$$

- State price π_t reflects what the representative consumer is willing to pay for an extra unit of consumption; in particular is π_t high in "times of crises" and low in "good times".

6. Pensions

- Let T_x be the remaining life time of a person who entered into a pension contract at age x. Let $[0, \tau]$ be the support of T_x.

- The single premium of an annuity paying one unit per unit of time is given by the formula

$$\bar{a}_x^{(r)} = \int_0^{\tau} e^{-rt} \frac{l_{x+t}}{l_x} dt, \tag{18}$$

where r is the short term interest rate.

- The single premium of a "temporary annuity" which terminates after time n is

$$\bar{a}_{x:\bar{n}|}^{(r)} = \int_0^{n} e^{-rt} \frac{l_{x+t}}{l_x} dt. \tag{19}$$

- Consider the following income process e_t:

$$e_t = \begin{cases} y, & \text{if } t \leq n; \\ 0, & \text{if } t > n \end{cases} \tag{20}$$

- Here y is a constant, interpreted as the consumer's salary when working, and n is the time of retirement for an x-year old.

- Equality in the budget constraint can then be written

$$E\Big(\int_0^{\tau} (e_t - c_t^*) \pi_t P(T_x > t) dt \Big) = 0.$$

(The Principle of Equivalence).

- The pension insurance element secures the consumer a consumption stream as long as needed, but only if it is needed. (This makes it possible to compound risk-free payments at a higher rate of interest than r_t.)

7. Optimal consumption with RU

- We use the same technique as explained for the expected utility model, except that we employ the stochastic maximum principle instead of directional derivatives.

- The optimal consumption turns out to be (as we have seen before)

$$c_t^* = c_0\, e^{\int_0^t (\mu_c(s) - \frac{1}{2}\sigma_c(s)'\sigma_c(s))ds + \int_0^t \sigma_c(s) dB_s} \tag{21}$$

- where $\mu_c(t)$ and $\sigma_c(t)$ are as determined as

$$\sigma_c(t) = \frac{1}{\rho}\Big(\eta_t + (\rho - \gamma)\sigma_V(t)\Big) \tag{22}$$

- and

$$\mu_c(t) = \frac{1}{\rho}(r_t - \delta) + \frac{1}{2}\frac{1}{\rho}(1 + \frac{1}{\rho})\,\eta_t'\eta_t - \frac{(\gamma - \rho)}{\rho^2}\,\eta_t'\,\sigma_V(t) + \frac{1}{2}\frac{(\gamma - \rho)\gamma(1 - \rho)}{\rho^2}\sigma_V'(t)\sigma_V(t) \quad (23)$$

- From (21) and (22) we see the following: First, a shock to the economy via B has the conventional effect via the market-price-of-risk-term η_t;

- Second, if $\sigma_V(t) > 0$, the shock has the opposite effect via the recursive utility term

$$(\rho - \gamma)\sigma_V(t)$$

  provided $\gamma > \rho$.

- The volatility of wealth $\sigma_W(t)$ is a linear combination of $\sigma_c(t)$ and $\sigma_V(t)$.

- The interpretation is that the agent uses wealth to dampen the effects of market movements on consumption.

- The expected utility maximizer is simply unable to do this.

- The connection between volatilities can also be written

$$\sigma_c(t) = \frac{1}{\rho}\Big(\sigma_W(t) - (1-\rho)\sigma_V(t)\Big)$$

  and the interpretation is that the recursive utility maximizer stabilizes variations in c by using wealth W.

7.1. Pensions with RU

- The optimal life time consumption ($t \in [0, n]$) and pension ($t \in [n, \tau]$) is

$$c_t^* = y \frac{\bar{a}^{(r)}_{x:\bar{n}|}}{\bar{a}^{(\hat{r})}_x} \exp\Big\{ \big(\frac{1}{\rho}(r-\delta) + \frac{1}{2\rho}\eta^2 + \frac{1}{2\rho}(\gamma-\rho)(1-\gamma)\sigma_V^2\big)t + \frac{1}{\rho}(\eta + (\rho-\gamma)\sigma_V)B_t \Big\}, \quad (24)$$

provided the agent is alive at time t (otherwise $c_t^* = 0$).

- Here

$$\hat{r} = r - \frac{1}{\rho}(r-\delta) + \frac{1}{2}\frac{1}{\rho}\left(1 - \frac{1}{\rho}\right)\eta'\eta + \frac{1}{\rho}\left(\frac{1}{\rho} - 1\right)(\rho-\gamma)\,\eta\sigma_V - \frac{1}{\rho}(\gamma-\rho)\Big(\frac{1}{\rho}(\gamma-\rho) + \frac{1}{2}(1-\gamma)\Big)\sigma_V^2. \quad (25)$$

- The premium intensity is given by the F_t-adapted process $p_t := y - c_t^*$. $(c_t^* = y - p_t)$.

- As can be seen, the optimal pension with recursive utility is being "smoothened" in the same manner as the optimal per capita consumption.

- A positive shock to the economy via the term B_t increases the optimal pension benefits via the term ηB_t, which may be mitigated, or strenghtened, by the term $(\rho - \gamma)\sigma_V B_t$, depending on its sign.

- When $(\gamma > \rho)$, then $\sigma_V(t) > 0$ and shocks to the economy are smoothened in the optimal pension with RU.

- This indicates that the pensioner in this model is considerably more sophisticated than the one modeled in the conventional way when $\rho = \gamma$. We summarize as follows:

- **Theorem 1** *The individual with recursive utility will prefer a pension plan that smoothens market shocks provided the consumer prefers early resolution of uncertainty to late ($\gamma > \rho$).*

- This points in the direction of DB-pension plan rather than a DC-plan ($\gamma > \rho$).

- When $\rho > \gamma$ the agent has preference for late resolution of uncertainty, and the same conclusion may still follow, now depending on the sign of $\sigma_V(t)$.

- The expected utility model, on the other hand, may be taken as support for unit linked pension insurance, or, defined contribution (DC)-plans. Here all the financial risk resides with the customers.

- One may wonder: Are ordinary pension insurance customers best equipped to carry aggregate financial risk? See Aase, K. K., (2015). Life Insurance and Pension Contracts I: The Time Additive Life Cycle Model. *ASTIN Bulletin*, Volume 45(1), pp 1-47.

- Recall the theory of syndicates: In a Pareto optimum is the risk tolerance (rt_λ) of the syndicate (i.e., an insurance company) the sum of the risk tolerances (rt_i) of the individual members of the syndicate (i.e., its insurance customers).

$$rt_\lambda(W) = \sum_{i=1}^{n} rt_i(c_i(W)).$$

- The interpretation of this result is precisely that the risk carrying capacity of a syndicate is larger than that of any of its members.

- Expected utility has problems with the axioms in a *temporal* context including uncertainty (e.g., Mossin (1969)): Derived utility does not satisfy the substitution axiom, which gives an internal inconsistency problem.

8. Optimal investment with RU

- Consider an agent with recursive utility who takes the market as given. In this setting we now discuss optimal portfolio choice.

- At the beginning of each period, the agent allocates a certain proportion of wealth to immediate consumption, and then invests the remaining amount in the available securities for future consumption. Accordingly, the the optimal portfolio choice will quite naturally depend on consumption.

- The growth rate of consumption, and its conditional variance, is known once consumption is determined. This is achieved by the agent using his/her preferences faced with the various market opportunities.

- **Theorem 2** *The optimal portfolio fractions in the risky assets are given by*

$$\varphi(t) = \frac{1-\rho}{\gamma-\rho}(\sigma_t\sigma_t')^{-1}\nu_t - \frac{\rho(1-\gamma)}{\gamma-\rho}(\sigma_t\sigma_t')^{-1}(\sigma_t\sigma_{c^*}(t)). \tag{26}$$

assuming $\gamma \neq \rho$.

- The volatility of the optimal consumption growth rate depends on preferences and market quantities only:

$$\sigma_{c^*}(t) = \frac{1}{\rho}\Big(\eta_t + (\rho-\gamma)\sigma_V(t)\Big). \tag{27}$$

- The agent first determines the optimal consumption growth rate and then the optimal portfolio choice in such a manner that the relationship (26) holds.

- The above formula answers the question of the insurance industry how to invest in order to satisfy the recursive utility pensioner.

- The optimal fractions with recursive utility depend on both risk aversion and time preference as well as the volatility σ_{c^*} of the optimal consumption growth rate dc_t^*/c_t^* of the agent.

- To illustrate, consider the standard situation with one risky and one risk-free asset, letting the S&P-500 index represent the risky security.

- Consider the market data of Table 1 with the S&P-500 index playing the role of the risky asset, with the associated estimate of $\sigma_{c^*} = 0.0355$.

- It is known that the average American household holds between 6% and 29% in risky securities.

- The standard model the predicts 120% in risky securities when $\gamma = 2$. To get a reasonable answer with the Eu-model, γ has to be large.

- The recursive model explains an average of 13% in risky securities for the following parameter values $\gamma = 2.6$ and $\rho = .90$.

- Given participation in the stock market, when $\varphi = .40$, this is consistent with $\gamma = 2.2$ and $\rho = .76$.

- If $\varphi = .60$, this can correspond to $\gamma = 2.0$ and $\rho = .66$, etc., a resolution of this puzzle.

- In addition to the insurance industry, other interesting applications would be to management of funds that invest public wealth to the benefits of the citizens of a country, or the members of a society.

- If this country/society is large enough for an estimate of the volatility of the consumption growth rate of the group to be available, the application becomes particularly simple, as the above exercise shows.

- One such example is the Norwegian Government Pension Fund Global (formerly the Norwegian Petroleum Fund).

- See Aase, K. K. (2016). "Life Insurance and Pension Contracts II: The life cycle model with recursive utility." *ASTIN Bulletin*, Volume 46(1), pp 71-102.