Quantile-based Risk Sharing and Equilibria

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Based on joint work with Haiyan Liu (Waterloo) and Ruodu Wang (Waterloo)

Outline

Risk Sharing

General setup

- n agents sharing a total risk (or asset) $X \in \mathcal{X}$ (set of rvs)
- $\rho_1, \ldots, \rho_n$: underlying risk measures (objectives to minimize)

The set of allocations of X:

$$\mathbb{A}_n(X) = \left\{ (X_1, \ldots, X_n) \in \mathcal{X}^n : \sum_{i=1}^n X_i = X \right\}.$$

Two basic risk sharing problems

- Pareto-optimal risk sharing: an allocation impossible to strictly improve
- Competitive risk sharing: an equilibrium arrived at via each agent optimizing their objectives individually

Value-at-Risk and Expected Shortfall

Value-at-Risk (VaR) at level $\alpha \geq 0$

$\mathrm{VaR}_\alpha : L^0 \to (-\infty, \infty]$,

$$\mathrm{VaR}_\alpha(X) = F_X^{-1}(1-\alpha) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq 1-\alpha\}.$$

Note: for $\alpha \geq 1$, $\mathrm{VaR}_\alpha(X) = -\infty$.

Expected Shortfall (ES/TVaR/CVaR/AVaR) at level $\beta \in (0,1)$

$\mathrm{ES}_\beta : L^1 \to (-\infty, \infty]$,

$$\mathrm{ES}_\beta(X) = \frac{1}{\beta}\int_0^\beta \mathrm{VaR}_\alpha(X)\mathrm{d}\alpha.$$

$$\mathrm{ES}_0(X) = \mathrm{VaR}_0(X); \quad \mathrm{ES}_1(X) = \mathbb{E}[X].$$

Remark: small α convention in this talk ...

Value-at-Risk and Expected Shortfall

The ongoing co-existence of VaR and ES:

- Basel III
- Solvency II
- Swiss Solvency Test

Quoting **Basel Committee on Banking Supervision**: Standards, January 2016, Minimum capital requirements for Market Risk. Page 1. *Executive Summary*:

> "... A shift from Value-at-Risk (VaR) to an Expected Shortfall (ES) measure of risk under stress. Use of ES will help to ensure a more prudent capture of "tail risk" and capital adequacy during periods of significant financial market stress."

Some Questions

Some questions on risk sharing:

- Explicit forms of the optimal risk allocation and the equilibrium?
- What property does an optimal allocation have (such as comonotonicity)?
- How does an optimal allocation react to model uncertainty (robustness)?

How do the answers to the above questions vary with respect to different underlying risk measures?

- More importantly, implications for regulation, VaR and/or ES?

Progress of the Talk

Range-Value-at-Risk (RVaR)

A two-parameter family of risk measures, for $\alpha, \beta \in \mathbb{R}_+ := [0, \infty)$,

$$\mathrm{RVaR}_{\alpha,\beta}(X) := \begin{cases} \frac{1}{\beta}\int_\alpha^{\alpha+\beta} \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma & \beta > 0, \\ \mathrm{VaR}_\alpha(X) & \beta = 0, \end{cases} \quad X \in L^1,$$

where $\mathrm{VaR}_\alpha(X) = -\infty$ for $\alpha \geq 1,\ X \in L^0$.

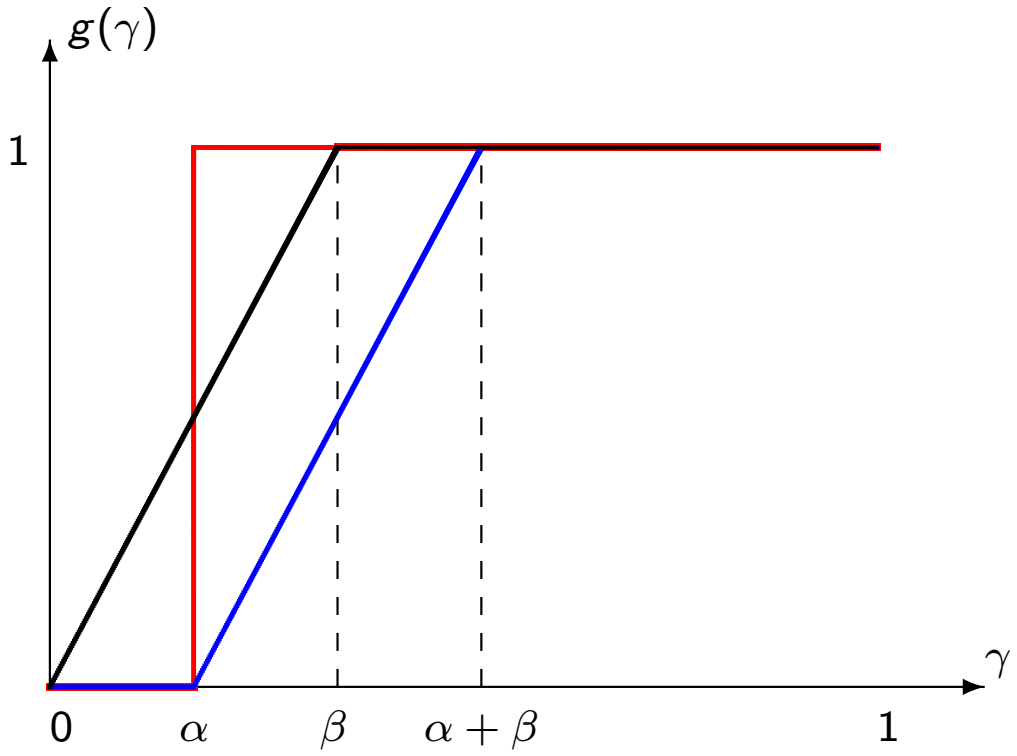
RVaR bridges the gap between VaR and ES: For $X \in L^1$,

- $\mathrm{VaR}_\alpha(X) = \mathrm{RVaR}_{\alpha,0}(X) = \lim_{\beta\to 0^+} \mathrm{RVaR}_{\alpha,\beta}(X),\ \alpha \in \mathbb{R}_+$.
- $\mathrm{ES}_\beta(X) = \mathrm{RVaR}_{0,\beta}(X) = \lim_{\alpha\to 0^+} \mathrm{RVaR}_{\alpha,\beta}(X),\ \beta \in [0, 1)$.

Practical values of (α, β) are $\alpha, \beta \geq 0$ and $\alpha + \beta \leq 1$.

- $\mathrm{RVaR}_{\alpha,\beta}(X) = -\infty$ for $\alpha + \beta > 1$, $X \in L^1$.

Range-Value-at-Risk (RVaR)



Distortion functions of VaR_α (red), ES_β (black) and $\mathrm{RVaR}_{\alpha,\beta}$ (blue)

in the form of $\int_0^1 \mathrm{VaR}_\gamma(X)\mathrm{d}g(\gamma)$

Range-Value-at-Risk (RVaR)

For $\alpha, \beta > 0$ and $\alpha + \beta < 1$,

- $\mathrm{RVaR}_{\alpha,\beta}$ is a distortion risk measure (Yaari's dual utility functional in decision theory): monetary, comonotonic additive, positive homogeneous, ...
- $\mathrm{RVaR}_{\alpha,\beta}$ is robust (continuous wrt weak convergence)
  - VaR_α and ES_β are not continuous wrt weak convergence (VaR_α is "almost continuous")

Some papers on risk measures robustness issues: Cont-Deguest-Scandolo 2010, Kou-Peng-Heyde 2013, Krätschmer-Schied-Zähle 2014, Embrechts-Wang-Wang 2015

Range-Value-at-Risk (RVaR)

- Take $\mathcal{X} = L^1$ (the common domain) in this talk
- The set of risk measures

$$\mathcal{G} = \left\{ \mathrm{RVaR}_{\alpha,\beta} : (\alpha, \beta) \in \mathbb{R}_+^2 \right\}$$

can be arranged into three categories:

- ES: $\alpha = 0$
- true VaR: $\beta = 0$, $\alpha > 0$
- true RVaR: $\beta > 0$, $\alpha > 0$

Quantile Inequalities

Theorem 1

For any $X_1, \ldots, X_n \in \mathcal{X}$ and $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{R}_+$, we have

$$\mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \bigvee_{i=1}^n \beta_i}\left(\sum_{i=1}^n X_i\right) \leq \sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i).$$

- $\bigvee_{i=1}^n \beta_i = \max\{\beta_1, \ldots, \beta_n\}$.
- RVaR satisfies a special form of subadditivity ($+$ and $\vee$ can both be viewed as additive operations on $\mathbb{R}$).

Quantile Inequalities

Corollary: Taking $\beta_1 = \cdots = \beta_n = 0$,

$$\mathrm{VaR}_{\sum_{i=1}^n \alpha_i}\left(\sum_{i=1}^n X_i\right) \leq \sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(X_i).$$

(Also valid for $\mathcal{X} = L^0$)

"Corollary": Taking $\alpha_1 = \cdots = \alpha_n = 0$ and $\beta_1 = \cdots = \beta_n = \beta$,

$$\mathrm{ES}_\beta\left(\sum_{i=1}^n X_i\right) \leq \sum_{i=1}^n \mathrm{ES}_\beta(X_i).$$

(Classic subadditivity of ES)

Seven proofs of subadditivity of ES: Embrechts-Wang 2015

Progress of the Talk

Risk Sharing

Pareto-optimal allocation

An allocation $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$ is Pareto-optimal if for any $(Y_1, \ldots, Y_n) \in \mathbb{A}_n(X)$, $\rho_i(X_i) \geq \rho_i(Y_i)$, $i = 1, \ldots, n$ implies equality.

For monetary (monotone and cash-invariant) risk measures:

A Pareto-optimal allocation $\Leftrightarrow$ an optimal allocation

which is defined as one that minimizes

$$\sum_{i=1}^{n} \rho_i(X_i) \quad \text{subject to } X_1 + \cdots + X_n = X.$$

Allocations and Inf-convolution

The inf-convolution of n risk measures is a risk measure $\square_{i=1}^{n} \rho_i$ mapping $\mathcal{X}$ to $[-\infty, \infty]$:

$$\mathop{\square}_{i=1}^{n} \rho_i(X) = \inf \left\{ \sum_{i=1}^{n} \rho_i(X_i) : (X_1, \ldots, X_n) \in \mathbb{A}_n(X) \right\}.$$

An optimal allocation $(X_1^*, \ldots, X_n^*)$ of X satisfies

$$\sum_{i=1}^{n} \rho_i(X_i^*) = \mathop{\square}_{i=1}^{n} \rho_i(X).$$

From now on, $\rho_i = \mathrm{RVaR}_{\alpha_i, \beta_i}$, $i = 1, \ldots, n$.

Some classic references (mainly on convex objectives): Barrieu-El Karoui 2005, Jouini-Schachermayer-Touzi 2008, Delbaen 2012, Rüschendorf 2013

Solution to the Risk Sharing Problem

Theorem 2

For $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{R}_+$ *and* $X \in \mathcal{X}$*, we have*

$$\bigsqcup_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X) = \mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \bigvee_{i=1}^n \beta_i}(X).$$

Proof of Theorem:

- "$\leq$": by construction
- "$\geq$": by the previous RVaR inequality

Remark:

- $(\mathcal{G}, \sqcup)$ is a commutative monoid (semi-group), isomorphic to the monoid $(\mathbb{R}_+^2, (+, \vee))$.

Solution to the Risk Sharing Problem

Assumptions

(a) $\sum_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i < 1$: existence of an optimal allocation

(b) $\beta_n = \bigvee_{i=1}^n \beta_i$: agent n is the remaining-risk bearer (most tolerant to risk beyond α-level)

(c) $X \geq 0$ and $\mathbb{P}(X > \mathrm{VaR}_{\sum_{i=1}^n \alpha_i}(X)) = \sum_{i=1}^n \alpha_i$ (not essential)

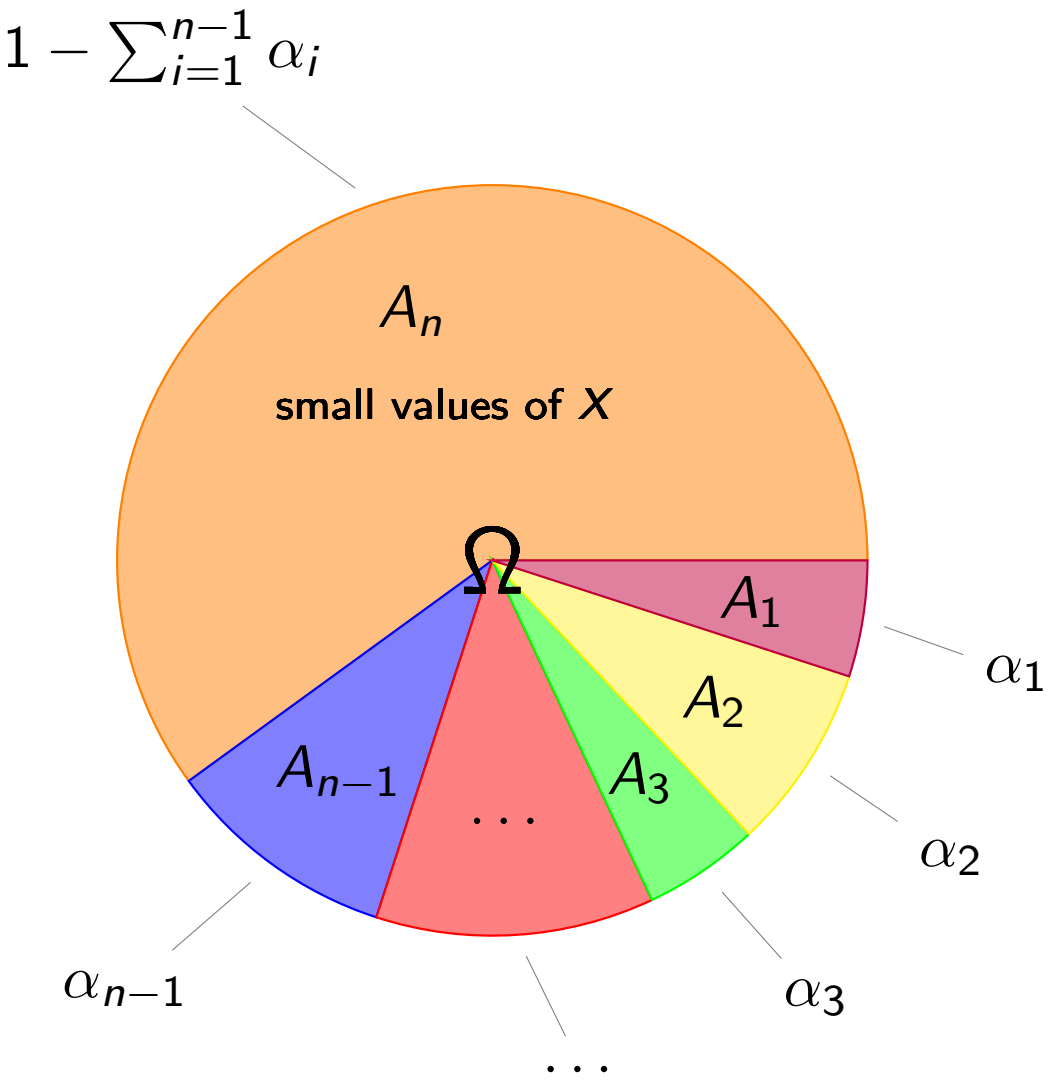
Optimal allocation

Under (a)-(c), an optimal allocation $(X_1^*, \ldots, X_n^*)$ of X is

$$X_i^* = X \mathrm{I}_{A_i}, \; i = 1, \ldots, n. \qquad (\star)$$

where $A_1, \ldots, A_n$ form a partition of Ω with $\mathbb{P}(A_i) = \alpha_i$ for $i = 1, \ldots, n-1$ and $\bigcup_{i=1}^{n-1} A_i \subset \{X > \mathrm{VaR}_{\sum_{i=1}^n \alpha_i}(X)\}$.

Solution to the Risk Sharing Problem



- $X_i^* = X \mathrm{I}_{A_i}$
- For $i = 1, \dots, n-1$, $\mathbb{P}(X_i^* > 0) = \alpha_i \Rightarrow \mathrm{RVaR}_{\alpha_i, \beta_i}(X_i^*) = 0$; "Agent i walks away thinking the risk is free"
- All the remaining risk is taken by agent n (most tolerant)
- "Neglecting the tail risk" ($\alpha_i > 0$) vs "capturing the tail risk" ($\alpha_i = 0$)

Case of VaR

Corollary: For $\alpha_1, \ldots, \alpha_n \geq 0$ and $X \in \mathcal{X}$,

$$\mathop{\Box}_{i=1}^{n} \mathrm{VaR}_{\alpha_i}(X) = \mathrm{VaR}_{\sum_{i=1}^n \alpha_i}(X).$$

Moreover, if $\sum_{i=1}^n \alpha_i < 1$ and (c) holds, an optimal allocation of X is given by $(\star)$.

Progress of the Talk

Competitive Equilibria

Question: Can the optimal allocation ($\star$) be achieved in a competitive market? Our setup:

- Agent i has an initial risk $\xi_i \in \mathcal{X}$. Assume $X = \sum_{i=1}^n \xi_i \geq 0$.
- An rv $\psi \geq 0$: the pricing rule. Each agent i may trade ξ_i for $X_i \in \mathcal{X}$ and receive the monetary amount $\mathbb{E}[\psi(X_i - \xi_i)]$.
- For a given ψ, agent i minimizes

$$\mathrm{RVaR}_{\alpha_i,\beta_i}\Big(\underbrace{X_i}_{\text{risk held}} - \underbrace{\mathbb{E}[\psi(X_i - \xi_i)]}_{\text{payment received}} \Big).$$

  (ξ_i is irrelevant in this optimization.)
- Assume one is not allowed to take a risk more than the total risk X or less than zero (no short selling).

Competitive Equilibria

Under our setting, the objective of agent i is

$$\text{to minimize} \quad \mathcal{V}_i(X_i) = \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i - \mathbb{E}[\psi X_i]) \qquad \text{(E)}$$

over $X_i \in \mathcal{X}$ satisfying $0 \leq X_i \leq X$. To reach an equilibrium, the market clearing equation

$$\sum_{i=1}^{n} X_i^* = X = \sum_{i=1}^{n} \xi_i$$

needs to be satisfied, where X_i^* solves (E), $i = 1, \ldots, n$.

e.g. Föllmer-Schied 2016

Arrow-Debreu Equilibria

Arrow-Debreu equilibrium

A pair $(\psi, (X_1^*, \ldots, X_n^*))$ is an Arrow-Debreu (AD) equilibrium if

$$X_i^* \in \arg\min \{\mathcal{V}_i(X_i) : X_i \in \mathcal{X},\ 0 \leq X_i \leq X\},\ i = 1, \ldots, n$$

and $X_1^* + \cdots + X_n^* = X$ (i.e. $(X_1^*, \ldots, X_n^*) \in \mathbb{A}_n(X)$).

Proposition 3

Assume (a) and $(\psi, (X_1^*, \ldots, X_n^*))$ is an AD equilibrium. Then $(X_1^*, \ldots, X_n^*)$ is necessarily an optimal allocation.

- a version of the First Welfare Economics Theorem (an equilibrium allocation is Pareto-optimal under some conditions)

Arrow-Debreu Equilibria

Theorem 4

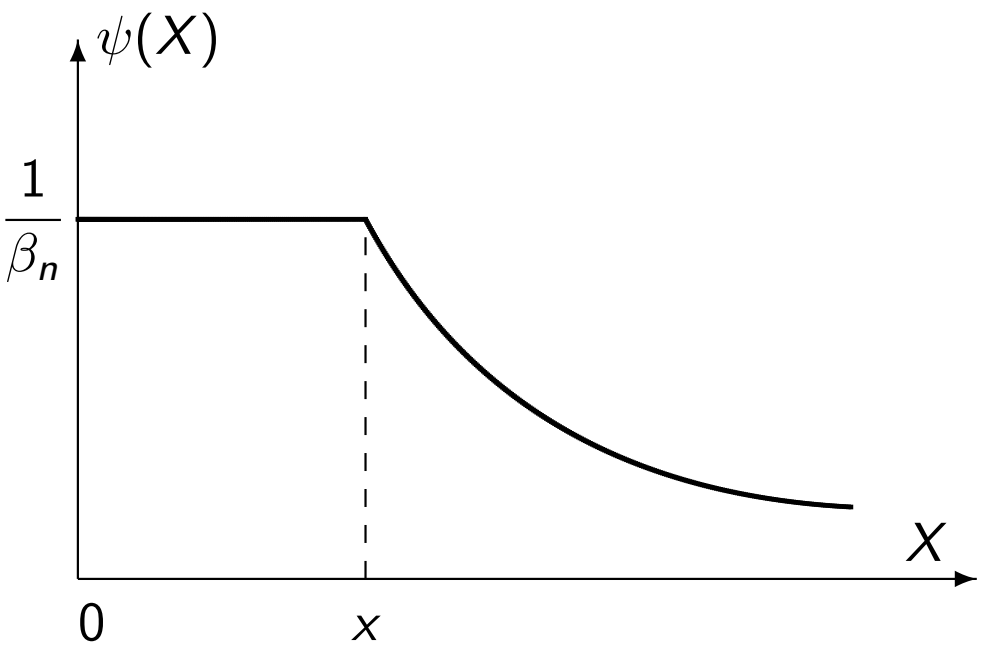
Assume $X \geq 0$ *satisfies* $\mathbb{P}(X > 0) \leq \max\{\bigwedge_{i=1}^n \alpha_i + \beta_n, \sum_{i=1}^n \alpha_i\}$ *and (a)-(b). Let* $(X_1^*, \ldots, X_n^*)$ *be given by* $(\star)$*, and*

$$\psi = \min\left\{\frac{x}{X\beta_n}, \frac{1}{\beta_n}\right\} \mathrm{I}_{\{X\beta_n > 0\}} \qquad \textit{where } x = \mathrm{VaR}_{\sum_{i=1}^n \alpha_i}(X).$$

Then $(\psi, (X_1^*, \ldots, X_n^*))$ *is an AD equilibrium.*

- We assumed $\mathbb{P}(X > 0)$ is not too large - e.g. credit portfolio
- Results for some more general settings are available
- The pricing rule ψ is a reciprocal function of X pasted to a constant

Equilibrium Pricing Rule



The equilibrium pricing rule ψ as a function of X

$$\psi = \min\left\{\frac{x}{X\beta_n}, \frac{1}{\beta_n}\right\} \mathrm{I}_{\{X\beta_n>0\}}, \text{ where } x = \mathrm{VaR}_{\sum_{i=1}^n \alpha_i}(X)$$

Progress of the Talk

Robustness

How does one know about a risk?

- Typically through a model/distribution: simulation, parametric models, expert opinion, ...
- Information asymmetry, model misspecification, data sparsity, random errors ...
- Model uncertainty!

Robustness

$\mathbb{F}_n(X)$: the set of sharing principles $(f_1, \dots, f_n)$ where each $f_i : \mathbb{R} \to \mathbb{R}$ has at most finitely many discontinuous points and $(f_1(X), \dots, f_n(X)) \in \mathbb{A}_n(X)$.

Definition 5

For n risk measures $\rho_1, \dots, \rho_n$, a (pseudo-)metric π on $\mathcal{X}$ and $X \in \mathcal{X}$, an allocation $(f_1(X), \dots, f_n(X)) \in \mathbb{A}_n(X)$ with $(f_1, \dots, f_n) \in \mathbb{F}_n(X)$ is π-robust if $\sum_{i=1}^n (\rho_i \circ f_i)$ at X is continuous with respect to π.

- Robustness $\Rightarrow$ small model misspecification does not ruin the optimality of a sharing principle
- metrics: e.g. L^1, L^∞, Wasserstein, $\pi_W =$ Lévy, ...

Robustness

Theorem 6

Assume (a) and $X \in \mathcal{X}$ has continuous cdf and inverse cdf.

(i) *There exists an L^1-robust optimal allocation of X if and only if $\beta_1, \dots, \beta_n > 0$ (all ES or true RVaR).*

(ii) *If X is bounded, then there exists an L^∞-robust optimal allocation of X if and only if $\beta_1, \dots, \beta_n > 0$ (all ES or true RVaR).*

(iii) *There exists a π_W-robust optimal allocation of X if and only if $\beta_1, \dots, \beta_n > 0$ and $\alpha_i > 0$ for some $i = 1, \dots, n$ (all ES or true RVaR, and at least one true RVaR).*

- No true VaR is allowed for robust optimal allocations
- True RVaR is the most robust, and ES is in between

Comonotonicity

Theorem 7

Assume (a) and $X \in \mathcal{X}$ has continuous cdf. There exists a comonotonic optimal allocation of X if and only if there exists $i = 1, \ldots, n$, such that for all $j = 1, \ldots, n$, $j \neq i$, $\alpha_j = 0$ and $\beta_i \geq \beta_j$.

- To have a comonotonic optimal allocation, at most one true RVaR or true VaR is allowed.

Progress of the Talk

Implications

Is risk sharing of type ($\star$) realistic?

"Starting in 2006, the CDO group at UBS noticed that their risk-management systems treated AAA securities as essentially riskless even though they yielded a premium (the proverbial free lunch). So they decided to hold onto them rather than sell them. "

- From Feb 06 to Sep 07, UBS increased investment in AAA-rated CDOs by more than 10 times; many large banks did the same.
  - Take a risk of big loss with small probability, $X_i = X\mathrm{I}_{A_i}$
  - Treat it as free money - profit
  - Financial crisis?

quoted from Acharya-Cooley-Richardson-Walter 2010

Implications

Some issues with VaR as the regulatory risk measure, assuming arbitrary risk sharing is allowed in the market.

Recall that $\square_{i=1}^{n} \mathrm{VaR}_\alpha(X) = \mathrm{VaR}_{n\alpha}(X)$.

- A firm has incentives to split its risk: regulatory arbitrage
- Sharing among firms is not comonotonic: moral hazard
- Sharing is not robust: insolvency under model uncertainty
- Total regulatory capital after sharing is much smaller than $\mathrm{VaR}_\alpha(X)$: insufficient capital for the whole economy
- A firm takes big losses with small probability as free money: problematic risk management

Implications

Some partial solutions

- Regulate against (particular forms of) risk sharing
  - risk sharing does not reduce the total risk in the economy, but reduces the total regulatory capital
- Introduce costs for splitting the business of a firm
- Use ES (consistent with the Basel III proposals for Market Risk)

VaR should be taken with extra caution as a regulatory risk measure

Discussion paper: Embrechts et al. 2014

Conclusions

Mathematical contributions

- RVaR inequalities
- Inf-convolution of VaRs/RVaRs
- Pareto-optimal allocations
- Arrow-Debreu equilibria
- Robustness in risk allocation

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Thank You

Thank you

The manuscript can be downloaded at

`http://ssrn.com/abstract=2744142`