Quantile-based Risk Sharing and Equilibria

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Based on joint work with Haiyan Liu (Waterloo) and Ruodu Wang (Waterloo)
Outline

1. Risk measures and risk sharing
2. RVaR and quantile inequalities
3. Pareto-optimal risk sharing
4. Competitive equilibria
5. Properties of optimal allocations
6. Implications for regulation
7. References
Risk Sharing

General setup

- $n$ agents sharing a total risk (or asset) $X \in \mathcal{X}$ (set of rvs)
- $\rho_1, \ldots, \rho_n$: underlying risk measures (objectives to minimize)

The set of allocations of $X$:

$$\mathbb{A}_n(X) = \left\{ (X_1, \ldots, X_n) \in \mathcal{X}^n : \sum_{i=1}^{n} X_i = X \right\}.$$

Two basic risk sharing problems

- **Pareto-optimal risk sharing**: an allocation impossible to strictly improve
- **Competitive risk sharing**: an equilibrium arrived at via each agent optimizing their objectives individually
Value-at-Risk (VaR) at level $\alpha \geq 0$

$\text{VaR}_\alpha : L^0 \to (-\infty, \infty],$

$\text{VaR}_\alpha(X) = F_X^{-1}(1 - \alpha) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq 1 - \alpha\}.$

Note: for $\alpha \geq 1$, $\text{VaR}_\alpha(X) = -\infty$.

Expected Shortfall (ES/TVaR/CVaR/AVaR) at level $\beta \in (0, 1)$

$\text{ES}_\beta : L^1 \to (-\infty, \infty],$

$\text{ES}_\beta(X) = \frac{1}{\beta} \int_0^\beta \text{VaR}_\alpha(X) d\alpha.$

$\text{ES}_0(X) = \text{VaR}_0(X); \quad \text{ES}_1(X) = \mathbb{E}[X].$

Remark: small $\alpha$ convention in this talk ...
Value-at-Risk and Expected Shortfall

The ongoing co-existence of VaR and ES:

- Basel III
- Solvency II
- Swiss Solvency Test


*Executive Summary*:

“... A shift from Value-at-Risk (VaR) to an Expected Shortfall (ES) measure of risk under stress. Use of ES will help to ensure a more prudent capture of “tail risk” and capital adequacy during periods of significant financial market stress.”
Some Questions

Some questions on risk sharing:

- Explicit forms of the optimal risk allocation and the equilibrium?
- What property does an optimal allocation have (such as comonotonicity)?
- How does an optimal allocation react to model uncertainty (robustness)?

How do the answers to the above questions vary with respect to different underlying risk measures?

- More importantly, implications for regulation, VaR and/or ES?
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Range-Value-at-Risk (RVaR)

A two-parameter family of risk measures, for $\alpha, \beta \in \mathbb{R}_+ := [0, \infty)$,

$$\text{RVaR}_{\alpha,\beta}(X) := \begin{cases} \frac{1}{\beta} \int_{\alpha}^{\alpha+\beta} \text{VaR}_{\gamma}(X) d\gamma & \beta > 0, \\ \text{VaR}_{\alpha}(X) & \beta = 0, \end{cases} \quad X \in L^1,$$

where $\text{VaR}_{\alpha}(X) = -\infty$ for $\alpha \geq 1$, $X \in L^0$.

RVaR bridges the gap between VaR and ES: For $X \in L^1$,

- $\text{VaR}_{\alpha}(X) = \text{RVaR}_{\alpha,0}(X) = \lim_{\beta \to 0^+} \text{RVaR}_{\alpha,\beta}(X)$, $\alpha \in \mathbb{R}_+$.
- $\text{ES}_{\beta}(X) = \text{RVaR}_{0,\beta}(X) = \lim_{\alpha \to 0^+} \text{RVaR}_{\alpha,\beta}(X)$, $\beta \in [0, 1)$.

**Practical values** of $(\alpha, \beta)$ are $\alpha, \beta \geq 0$ and $\alpha + \beta \leq 1$.

- $\text{RVaR}_{\alpha,\beta}(X) = -\infty$ for $\alpha + \beta > 1$, $X \in L^1$. 
Range-Value-at-Risk (RVaR)

Distortion functions of \( \text{VaR}_\alpha \) (red), \( \text{ES}_\beta \) (black) and \( \text{RVaR}_{\alpha,\beta} \) (blue) in the form of \( \int_0^1 \text{VaR}_\gamma(X)dg(\gamma) \)
Range-Value-at-Risk (RVaR)

For $\alpha, \beta > 0$ and $\alpha + \beta < 1$,

- $\text{RVaR}_{\alpha,\beta}$ is a distortion risk measure (Yaari’s dual utility functional in decision theory): monetary, comonotonic additive, positive homogeneous, ...
- $\text{RVaR}_{\alpha,\beta}$ is robust (continuous wrt weak convergence)
  - $\text{VaR}_\alpha$ and $\text{ES}_\beta$ are not continuous wrt weak convergence ($\text{VaR}_\alpha$ is “almost continuous”)

Range-Value-at-Risk (RVaR)

- Take $\mathcal{X} = L^1$ (the common domain) in this talk
- The set of risk measures

$$\mathcal{G} = \{ \text{RVaR}_{\alpha,\beta} : (\alpha, \beta) \in \mathbb{R}_+^2 \}$$

can be arranged into three categories:

- ES: $\alpha = 0$
- true VaR: $\beta = 0, \alpha > 0$
- true RVaR: $\beta > 0, \alpha > 0$
Quantile Inequalities

Theorem 1

For any $X_1, \ldots, X_n \in \mathcal{X}$ and $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{R}_+$, we have

$$\text{RVaR} \sum_{i=1}^{n} \alpha_i, \bigvee_{i=1}^{n} \beta_i \left( \sum_{i=1}^{n} X_i \right) \leq \sum_{i=1}^{n} \text{RVaR}_{\alpha_i, \beta_i}(X_i).$$

- $\bigvee_{i=1}^{n} \beta_i = \max\{\beta_1, \ldots, \beta_n\}$.
- RVaR satisfies a special form of subadditivity (+ and $\bigvee$ can both be viewed as additive operations on $\mathbb{R}$).
Quantile Inequalities

Corollary: Taking $\beta_1 = \cdots = \beta_n = 0$,

$$\text{VaR}_{\sum_{i=1}^{n} \alpha_i} \left( \sum_{i=1}^{n} X_i \right) \leq \sum_{i=1}^{n} \text{VaR}_{\alpha_i}(X_i).$$

(Also valid for $X = L^0$)

"Corollary": Taking $\alpha_1 = \cdots = \alpha_n = 0$ and $\beta_1 = \cdots = \beta_n = \beta$,

$$\text{ES}_{\beta} \left( \sum_{i=1}^{n} X_i \right) \leq \sum_{i=1}^{n} \text{ES}_{\beta}(X_i).$$

(Classic subadditivity of ES)

Seven proofs of subadditivity of ES: Embrechts-Wang 2015

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Pareto-optimal allocation

An allocation \((X_1, \ldots, X_n) \in \mathcal{A}_n(X)\) is Pareto-optimal if for any \((Y_1, \ldots, Y_n) \in \mathcal{A}_n(X)\), \(\rho_i(X_i) \geq \rho_i(Y_i), \ i = 1, \ldots, n\) implies equality.

For monetary (monotone and cash-invariant) risk measures:

A Pareto-optimal allocation \(\iff\) an optimal allocation

which is defined as one that minimizes

\[
\sum_{i=1}^{n} \rho_i(X_i) \quad \text{subject to} \quad X_1 + \cdots + X_n = X.
\]
The inf-convolution of \( n \) risk measures is a risk measure \( \square_{i=1}^{n} \rho_i \) mapping \( \mathcal{X} \) to \([-\infty, \infty]\):

\[
\inf_{i=1}^{n} \rho_i(X) = \inf \left\{ \sum_{i=1}^{n} \rho_i(X_i) : (X_1, \ldots, X_n) \in \mathbb{A}_n(X) \right\}.
\]

An optimal allocation \((X_1^*, \ldots, X_n^*)\) of \(X\) satisfies

\[
\sum_{i=1}^{n} \rho_i(X_i^*) = \inf_{i=1}^{n} \rho_i(X).
\]

From now on, \( \rho_i = \text{RVaR}_{\alpha_i, \beta_i}, \ i = 1, \ldots, n. \)

Solution to the Risk Sharing Problem

Theorem 2

For $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{R}_+$ and $X \in \mathcal{X}$, we have

$$\bigwedge_{i=1}^n \text{RVaR}_{\alpha_i, \beta_i}(X) = \text{RVaR}_{\sum_{i=1}^n \alpha_i, \bigvee_{i=1}^n \beta_i}(X).$$

Proof of Theorem:

- “≤”: by construction
- “≥”: by the previous RVaR inequality

Remark:

- $(\mathcal{G}, \bigwedge)$ is a commutative monoid (semi-group), isomorphic to the monoid $(\mathbb{R}_+^2, (+, \lor))$. 
Solution to the Risk Sharing Problem

Assumptions

(a) \( \sum_{i=1}^{n} \alpha_i + \bigvee_{i=1}^{n} \beta_i < 1 \): existence of an optimal allocation

(b) \( \beta_n = \bigvee_{i=1}^{n} \beta_i \): agent \( n \) is the remaining-risk bearer (most tolerant to risk beyond \( \alpha \)-level)

(c) \( X \geq 0 \) and \( \mathbb{P}(X > \text{VaR} \sum_{i=1}^{n} \alpha_i(X)) = \sum_{i=1}^{n} \alpha_i \) (not essential)

Optimal allocation

Under (a)-(c), an optimal allocation \((X_1^*, \ldots, X_n^*)\) of \( X \) is

\[
X_i^* = X I_{A_i}, \quad i = 1, \ldots, n.
\]

(\( \star \))

where \( A_1, \ldots, A_n \) form a partition of \( \Omega \) with \( \mathbb{P}(A_i) = \alpha_i \) for \( i = 1, \ldots, n - 1 \) and \( \bigcup_{i=1}^{n-1} A_i \subset \{X > \text{VaR} \sum_{i=1}^{n} \alpha_i(X)\} \).
Solution to the Risk Sharing Problem

\[ 1 - \sum_{i=1}^{n-1} \alpha_i \]

- \( X_i^* = X I_{A_i} \)
- For \( i = 1, \ldots, n - 1 \), \( \mathbb{P}(X_i^* > 0) = \alpha_i \Rightarrow \) \( \text{RVaR}_{\alpha_i, \beta_i}(X_i^*) = 0; \) "Agent \( i \) walks away thinking the risk is free"
- All the remaining risk is taken by agent \( n \) (most tolerant)
- "Neglecting the tail risk" (\( \alpha_i > 0 \)) vs "capturing the tail risk" (\( \alpha_i = 0 \) )
Case of VaR

Corollary: For $\alpha_1, \ldots, \alpha_n \geq 0$ and $X \in \mathcal{X}$,

$$\prod_{i=1}^{n} \text{VaR}_{\alpha_i}(X) = \text{VaR} \sum_{i=1}^{n} \alpha_i(X).$$

Moreover, if $\sum_{i=1}^{n} \alpha_i < 1$ and (c) holds, an optimal allocation of $X$ is given by (⋆).
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Competitive Equilibria

Question: Can the optimal allocation (⋆) be achieved in a competitive market? Our setup:

- Agent $i$ has an initial risk $\xi_i \in \mathcal{X}$. Assume $X = \sum_{i=1}^{n} \xi_i \geq 0$.
- An rv $\psi \geq 0$: the pricing rule. Each agent $i$ may trade $\xi_i$ for $X_i \in \mathcal{X}$ and receive the monetary amount $\mathbb{E}[\psi(X_i - \xi_i)]$.
- For a given $\psi$, agent $i$ minimizes

$$ \text{RVaR}_{\alpha_i, \beta_i} \left( \underbrace{X_i - \mathbb{E}[\psi(X_i - \xi_i)]}_{\text{risk held}} \right). $$

($\xi_i$ is irrelevant in this optimization.)

- Assume one is not allowed to take a risk more than the total risk $X$ or less than zero (no short selling).
Under our setting, the objective of agent $i$ is

$$V_i(X_i) = \text{RVaR}_{\alpha_i, \beta_i}(X_i - E[\psi X_i])$$

(E)

over $X_i \in \mathcal{X}$ satisfying $0 \leq X_i \leq X$. To reach an equilibrium, the market clearing equation

$$\sum_{i=1}^{n} X_i^* = X = \sum_{i=1}^{n} \xi_i$$

needs to be satisfied, where $X_i^*$ solves (E), $i = 1, \ldots, n$.  

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e.g. Föllmer-Schied 2016
Arrow-Debreu Equilibria

Arrow-Debreu equilibrium

A pair \((\psi, (X_1^*, \ldots, X_n^*))\) is an Arrow-Debreu (AD) equilibrium if

\[
X_i^* \in \arg\min \{V_i(X_i) : X_i \in \mathcal{X}, \ 0 \leq X_i \leq X\}, \ i = 1, \ldots, n
\]

and \(X_1^* + \cdots + X_n^* = X\) (i.e. \((X_1^*, \ldots, X_n^*) \in \mathbb{A}_n(X)\)).

Proposition 3

Assume (a) and \((\psi, (X_1^*, \ldots, X_n^*))\) is an AD equilibrium. Then \((X_1^*, \ldots, X_n^*)\) is necessarily an optimal allocation.

- a version of the First Welfare Economics Theorem (an equilibrium allocation is Pareto-optimal under some conditions)
Theorem 4

Assume $X \geq 0$ satisfies $\mathbb{P}(X > 0) \leq \max\{\sum_{i=1}^{n} \alpha_i + \beta_n, \sum_{i=1}^{n} \alpha_i\}$ and (a)-(b). Let $(X_1^*, \ldots, X_n^*)$ be given by ($\star$), and

$$\psi = \min \left\{ \frac{x}{X \beta_n}, \frac{1}{\beta_n} \right\} \mathbb{I}\{X \beta_n > 0\} \quad \text{where} \quad x = \text{VaR}_{\sum_{i=1}^{n} \alpha_i}(X).$$

Then $(\psi, (X_1^*, \ldots, X_n^*))$ is an AD equilibrium.

- We assumed $\mathbb{P}(X > 0)$ is not too large - e.g. credit portfolio
- Results for some more general settings are available
- The pricing rule $\psi$ is a reciprocal function of $X$ pasted to a constant
The equilibrium pricing rule \( \psi \) as a function of \( X \)

\[
\psi = \min \left\{ \frac{x}{X \beta_n}, \frac{1}{\beta_n} \right\} I\{X \beta_n > 0\}, \text{ where } x = \text{VaR} \sum_{i=1}^{n} \alpha_i(X)
\]
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Robustness

How does one know about a risk?

- Typically through a model/distribution: simulation, parametric models, expert opinion, ...
- Information asymmetry, model misspecification, data sparsity, random errors ...
- Model uncertainty!
Robustness

\( F_n(X) \): the set of sharing principles \((f_1, \ldots, f_n)\) where each 
\( f_i: \mathbb{R} \to \mathbb{R} \) has at most finitely many discontinuous points and 
\((f_1(X), \ldots, f_n(X)) \in A_n(X)\).

**Definition 5**

For \( n \) risk measures \( \rho_1, \ldots, \rho_n \), a (pseudo-)metric \( \pi \) on \( \mathcal{X} \) and 
\( X \in \mathcal{X} \), an allocation \((f_1(X), \ldots, f_n(X)) \in A_n(X)\) with 
\((f_1, \ldots, f_n) \in F_n(X)\) is \( \pi \)-robust if \( \sum_{i=1}^{n}(\rho_i \circ f_i) \) at \( X \) is continuous 
with respect to \( \pi \).

- Robustness \( \Rightarrow \) small model misspecification does not ruin the 
  optimality of a sharing principle
- metrics: e.g. \( L^1, L^\infty, \) Wasserstein, \( \pi_W = \) Lévy, ...
Robustness

Theorem 6

Assume (a) and $X \in \mathcal{X}$ has continuous cdf and inverse cdf.

(i) There exists an $L^1$-robust optimal allocation of $X$ if and only if $\beta_1, \ldots, \beta_n > 0$ (all ES or true RVaR).

(ii) If $X$ is bounded, then there exists an $L^\infty$-robust optimal allocation of $X$ if and only if $\beta_1, \ldots, \beta_n > 0$ (all ES or true RVaR).

(iii) There exists a $\pi_W$-robust optimal allocation of $X$ if and only if $\beta_1, \ldots, \beta_n > 0$ and $\alpha_i > 0$ for some $i = 1, \ldots, n$ (all ES or true RVaR, and at least one true RVaR).

- No true VaR is allowed for robust optimal allocations
- True RVaR is the most robust, and ES is in between
Comonotonicity

Theorem 7

Assume (a) and $X \in \mathcal{X}$ has continuous cdf. There exists a comonotonic optimal allocation of $X$ if and only if there exists $i = 1, \ldots, n$, such that for all $j = 1, \ldots, n$, $j \neq i$, $\alpha_j = 0$ and $\beta_i \geq \beta_j$.

- To have a comonotonic optimal allocation, at most one true RVaR or true VaR is allowed.
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Implications

Is risk sharing of type (⋆) realistic?

“Starting in 2006, the CDO group at UBS noticed that their risk-management systems treated AAA securities as essentially riskless even though they yielded a premium (the proverbial free lunch). So they decided to hold onto them rather than sell them.”

- From Feb 06 to Sep 07, UBS increased investment in AAA-rated CDOs by more than 10 times; many large banks did the same.
  - Take a risk of big loss with small probability, $X_i = X_{1_A}$
  - Treat it as free money - profit
  - Financial crisis?

quoted from Acharya-Cooley-Richardson-Walter 2010
Implications

Some issues with VaR as the regulatory risk measure, assuming arbitrary risk sharing is allowed in the market.

Recall that $\prod_{i=1}^{n} \text{VaR}_\alpha(X) = \text{VaR}_{n\alpha}(X)$.

- A firm has incentives to split its risk: regulatory arbitrage
- Sharing among firms is not comonotonic: moral hazard
- Sharing is not robust: insolvency under model uncertainty
- Total regulatory capital after sharing is much smaller than $\text{VaR}_\alpha(X)$: insufficient capital for the whole economy
- A firm takes big losses with small probability as free money: problematic risk management
Implications

Some partial solutions

- Regulate **against** (particular forms of) risk sharing
  - risk sharing **does not reduce** the total risk in the economy, but **reduces** the total regulatory capital

- Introduce costs for splitting the business of a firm

- Use ES (consistent with the Basel III proposals for Market Risk)

**VaR** should be taken with extra caution as a regulatory risk measure

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Discussion paper: Embrechts et al. 2014
Conclusions

Mathematical contributions

- RVaR inequalities
- Inf-convolution of VaRs/RVaRs
- Pareto-optimal allocations
- Arrow-Debreu equilibria
- Robustness in risk allocation
References I


References II


Thank you

The manuscript can be downloaded at
http://ssrn.com/abstract=2744142