SOLVENCY CAPITAL ESTIMATION, RESERVING CYCLE AND ULTIMATE RISK

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Abstract. In this paper we propose a stochastic model for the evolution of the reserves for a non-life insurance run-off portfolio that captures the dynamic of the reserving cycle, which consists in years of prudent reserves releases followed by sudden reserves strengthening.

In our model we assume that the relative loss developments over time follow a stochastic process with dependent increments, and that the consequently estimated reserves evolve as a stochastic process with discontinuous paths, which all together could be mathematically described as a geometric fractional Brownian motion with random jumps.

The dependence between increments reflects the first phase of the reserving cycle, i.e. prudent reserve releases, whereas the second phase of the cycle is captured by the jumps. Remarkably in our model a jump in the reserves occurs after a period of systematic under-estimation of the losses, as happens in reality.

As a product of our model we propose practical estimators for the Solvency Capital Requirement and the Risk Margin as defined in the European regulation (Solvency II, [1]), and analogously in the Swiss regulation (SST, [4]), as functions of the ultimate risk.

1. Introduction

In the Solvency II European regulation [1], which will take effect at the beginning of 2016, the risk and thus the capital for the undertaking reinsurers and insurers will need to be evaluated on a one-year time horizon. This represents a major difference from what requested previously and in other regulations where the risk is measured on an ultimate time horizon, that is, until the end of the run-off.

Within the new European regulation the one-year risk for the insurance liabilities is the risk that the technical provisions in one year will exceed materially the technical provisions today, whereas the ultimate risk is the risk that the technical provisions at the end of the run-off will exceed materially the technical provisions today.

For a non-life insurance portfolio the most widely used actuarial method for the one-year risk estimation is the Merz-Wüthrich formula [6]. Explained in a nutshell the Merz-Wüthrich formula, which is similar to the Mack formula [5], consists in estimating the mean square error of the reserves developments over one year based on claims triangles and reserves computed with the chain-ladder method.

The Merz-Wüthrich formula has became a standard in the insurance industry. Indeed it is used as benchmark by regulators and auditors for the one-year risk estimation of any non-life insurance portfolio, regardless whether its assumptions

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are fulfilled or not. This is mainly due to the simplicity of the Merz-Wüthrich formula, that is a closed formula, and as well to the shortage of equally simple alternative methods. Nonetheless, as shown in [2], the use of the Merz-Wüthrich formula could be extremely misleading as it provides completely off estimations when used outside its applicability perimeter.

In this paper we propose a different and original approach for the one-year risk estimation. Our scope is to define a stochastic model for the evolution of the losses over time, and consequently a stochastic model for the evolution of the best estimates of the ultimate loss over time, which replicates the reserving actuary behaviour in the re-estimation of the reserves.

To achieve our scope we assume that the relative loss developments of the attritional losses, which materialise every year, behave as a stochastic process with possibly dependent increments as in a fractional Brownian motion. We then model the evolution of the reserves, which are estimated based on the realised losses, replicating the estimations done by the reserving actuary every quarter, which consists in prudent reserves release as payments are made* and severe reserves increase, triggered by an adverse event, in case the past realized losses systematically exceed the expectation. Those two phases contribute to form what is known as the reserving cycle.

We eventually deduce from our model simple relations between the one-year risks, for the first year and for all the future years until the end of the run-off, and the ultimate risk. Thus knowing the ultimate risk these relations can be used as an alternative to the Merz-Wüthrich formula for the one-year risk estimation. Moreover, they can be used to estimate the risk margin.

Our model, which is detailed in Section 2, has the advantage to be simple, to be widely applicable and to be realistic because it captures the dependency between losses over time, the reserving actuary behaviour and the reserving cycle, none of which is reflected in the Merz-Wüthrich formula. As shown by some relevant examples in [2], our method provides far better one-year risk estimations than the Merz-Wüthrich formula. The one-year risk and the risk margin estimators are presented in Section 3. Finally in Section 4 we provide a brief guide to the use of the model in practice.

2. Our Model

In this section we present our model for the evolution of the best estimate of the ultimate loss, and thus the reserves, over time for a non-life insurance run-off portfolio.

Note that throughout the paper, for every functions x and y depending on a parameter ϵ, $y(\epsilon) \neq 0$, $x(\epsilon) \ll y(\epsilon)$ means that $|x(\epsilon)/y(\epsilon)|/\epsilon$ is bounded, for $\epsilon \to 0$, and $x(\epsilon) \simeq y(\epsilon)$, means that $|x(\epsilon)/y(\epsilon) - 1|/\epsilon$ is bounded, for $\epsilon \ll 1$, i.e. $\epsilon \to 0$.

Let Y_t be the cumulative attritional losses of a given portfolio at any time t in $[0, 1]$, being $Y_0 > 0$ the initial loss, which is a known quantity as of today $t = 0$ and a random variable at any future time $t > 0$. For convenience the final time is normalised to $t = 1$, which could be days, months, years or anything else, and the model is defined continuously in time. Eventually we will deduce a discrete time version more consistently with the classical actuarial framework.

*Under the pressure of rates decrease.

Roughly speaking, we assume that the cumulative attritional losses Y_t evolves accordingly to a geometric Brownian motion from the beginning $t = 0$ to the end $t = 1$, except during an unpredictable period in which the increments are positively correlated. More precisely,

$$Y_t = Y_0 e^{p_t + B_t^h(T^s, T^e)}, \quad \text{for } t \in [0, 1], \tag{1}$$

where:

- $p : [0, 1] \to \mathbb{R}$, $p_0 = 0$, is a function which defines the expected relative loss developments such that e^{p_t} is non-decreasing and concave[†];
- T^e is a r.v. in $[0, 1]$ with $\mathbb{P}(T^e \leq t) = t$, $t \in [0, 1]$; that is, T^e is uniformly distributed in $[0, 1]$;
- T^s is a r.v. in $[T^e - 1, T^e]$, which depends on T^e, with $\mathbb{P}(T^s \leq t) = a^{t-T^e}$, $t \in [T^e - 1, T^e]$, for a fixed $a > 1$; the distribution of T^s is such that T^s is more likely to assume values close to T^e; $T^e - T^s$ defines for how long the relative loss increments are dependent; we define[‡] $\bar{T}^s := T^s \vee 0$;
- $B_t^h(T^s, T^e)$ is the volatility around the expected relative loss increment given by

$$B_t^h(T^s, T^e) := \begin{cases} \sigma_h B_{qc_t} - \sigma_h^2 q c_t/2, & \text{if } t \in [0, \bar{T}^s], \\ \sigma_h B^h_{q(c_t - c_{\bar{T}^s})} - \sigma_h^2 q^{2h}(c_t - c_{\bar{T}^s})^{2h}/2 + B^h_{\bar{T}^s}(T^s, T^e), & \text{if } t \in (\bar{T}^s, T^e], \\ \sigma_h B_{q(c_t - c_{T^e})} - \sigma_h^2 q(c_t - c_{T^e})/2 + B^h_{T^e}(T^s, T^e), & \text{if } t \in (T^e, 1], \end{cases}$$

  where $\{B_c : c \geq 0\}$, $B_0 = 0$, which appears in the first and third pieces in the definition are different Brownian motions for which we use the same notation for simplicity, $\{B_c^h : c \geq q(c_{1-|T^s \wedge 0|} - 1)\}$, $B_0^h = 0$, is a fractional Brownian motion[§] with Hurst exponent $1/2 < h < 1$, which defines the dependency between increments in $(T^s, T^e]$, $c_t := (e^{p_t} - 1)/(e^{p_1} - 1)$ is the expected future losses in $(0, t]$ relative to the total future losses and $\sigma_h := \sigma_0/q^h$, with $\sigma_0 > 0$, $q \in \mathbb{N}$, is the ultimate volatility parameter.

By definition, the standard deviation of $B_t^h(T^s, T^e) - B_s^h(T^s, T^e)$ is equal to σ_h, for every $s, t \in [0, 1]$ with $c_t - c_s = 1/q$. In the discrete model which will be presented in Section 3, $1/q$ represents the smallest possible relative reserves increment. So σ_h can be regarded as the infinitesimal volatility parameter. If $c_t - c_s = n/q$, $n \in \mathbb{N}$, then the standard deviation of $B_t^h(T^s, T^e) - B_s^h(T^s, T^e)$ is equal to $\sigma_h n^{1/2}$, for $s, t \in [0, \bar{T}^s]$, or $s, t \in [T^e, 1]$, and equal to $\sigma_h n^h$, for $s, t \in [\bar{T}^s, T^e]$.

The process $\{Y_t\}$ is nothing but a geometric fractional Brownian motion in $(\bar{T}^s, T^e]$ and a geometric Brownian motion anywhere else. The drifts in the definition of $B_t^h(T^s, T^e)$ are such that $\mathbb{E}(\exp[B_t^h(T^s, T^e)]) = 1$, i.e.

$$\mathbb{E}(Y_t) = Y_0 e^{p_t}, \quad \text{for } t \in [0, 1].$$

The relation in (1) translates in mathematical terms the assumption that the losses develop around their expected value with a volatility that is log-normal distributed and proportional to the expected future losses $Y_0(e^{p_t} - 1)$, $t \in (0, 1]$, and that the relative loss increments are positively correlated at different times in $(\bar{T}^s, T^e]$.

[†]For our purpose it would be actually enough that $e^{p_t} \geq 1 + t(e^{p_1} - 1)$, for $t \in [0, 1]$.

[‡]Recall that $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$.

[§]Note that if $T^s \geq 0$, then $q(c_{1-|T^s \wedge 0|} - 1) = 0$, otherwise $q(c_{1-|T^s \wedge 0|} - 1) < 0$.

The random time T^e is when a sudden increase of the reserves may occur as a result of a period $(\bar{T}^s, T^e]$ of systematic under-estimation of the losses and a triggering adverse event occurring at T^e. If γ is the relative size of the reserves jump, we then model the evolution of the best estimate of the ultimate loss at time t, which we denote with BE_t, by the stochastic differential equation

$$d\mathrm{BE}_t = d\mathbb{E}(Y_1|\mathfrak{F}_t) + \gamma(\mathrm{BE}_t - Y_t)dJ_t(T^e), \qquad \text{for } t \in [0,1], \tag{2}$$

with initial value $\mathrm{BE}_0 = \mathbb{E}(Y_1)$, where $\mathfrak{F}_t$ is the σ-algebra generated by $\{\{Y_\tau : \tau \in [0,t]\}, \{T^e \leq \tau : \tau \in [0,t]\}, \{T^s : T^e \leq \tau\}\}$[¶], and

$$J_t(T^e) := -k_t + \begin{cases} 1, & \text{if } t \in (T^e, 1], f_\alpha(T^e) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $f_\alpha(T^e)$ is a function which represents the reserving actuary decision at time T^e defined by

$$f_\alpha(T^e) := \begin{cases} 1, & \text{if } Y_{T^e}/(Y_{\bar{T}^s} e^{p_{T^e} - p_{\bar{T}^s}}) \geq 1 + \xi_\alpha \mathrm{Std}[Y_{T^e}/(Y_{\bar{T}^s} e^{p_{T^e} - p_{\bar{T}^s}})], \\ 0, & \text{otherwise,} \end{cases}$$

$\xi_\alpha \geq 0$ is such that $\mathbb{P}(f_\alpha(T^e) = 1|T^e = 1) = \alpha$[‖], and

$$k_t := \int_0^{t \wedge T^e} \frac{\mathbb{P}(f_\alpha(T^e) = 1|T^e = s)}{1-s} ds.$$

Observe that $\mathbb{P}(f_\alpha(T^e) = 1|T^e = t) \leq \alpha$ and, if $\sigma_0 \ll 1$, $\mathbb{P}(f_\alpha(T^e) = 1|T^e = t) \simeq \alpha$, for any $t \in (0,1]$.

The term $\mathrm{BE}_t - Y_t$ in (2) represents the reserve if Y_t is the cumulative paid loss, or the IBNR if Y_t is the cumulative incurred loss. As the reserves for a run-off portfolio tend to decrease over time, the reserve jump size $\gamma(\mathrm{BE}_t - Y_t)$ also tends to decrease as t increases.

The term $-k_t$ in the definition of $J_t(T^e)$ ensures that the process $\{\mathrm{BE}_t\}$ is a martingale, as shown in Proposition 5.2. Note that BE_t represents the best estimate at time t of the ultimate loss. That means that, if we expect at time t that BE_{t+1} is going to be bigger, or smaller, than BE_t, then BE_t would not be the best estimate of the ultimate loss, and thus the estimation would not be complaint with the regulation. Thus $\{\mathrm{BE}_t\}$ needs to be a martingale, i.e. $\mathbb{E}(\mathrm{BE}_{t+1}|\mathfrak{F}_t) = \mathrm{BE}_t$.

The function $f_\alpha(T^e)$ models the actuarial behaviour that consists in prudently react to a period $(\bar{T}^s, T^e]$ in which the realised losses exceed the expected by ξ_α-times the standard deviation, after a triggering adverse event occurs at time T^e. This is indeed what often happens in reality when a severe reserve strengthening occurs as a result of a period of systematic under-estimation of the losses.

Note that the pieces of information available at time t are only the realised losses up to t together with T^e. As is the case in reality, we do not know T^s a priori but only a posteriori.

The integral in $d\mathrm{BE}_t$ and in $d\mathbb{E}(Y_1|\mathfrak{F}_t)$ are Itô stochastic integrals, and the integral in $dJ_t(T, Y_T)$ is the standard Riemann-Stieltjes integral. However, one may

[¶]Note that $\mathfrak{F}_0 = \sigma(\{T^e = 0\})$ and, since $\mathbb{P}(T^e = 0) = 0$, conditioning on $\mathfrak{F}_0$ is the same as conditioning on the trivial σ-algebra. Clearly $\{\mathfrak{F}_t\}$ is a filtration.

[‖]$Y_{T^e}/(Y_{\bar{T}^s} e^{p_{T^e} - p_{\bar{T}^s}})$ is log-normal with mean equal to 1 and standard deviation equal to $\{\exp[\sigma_0^2 \mathrm{Std}(B^h_{c_{T^e} - c_{\bar{T}^s}})^2] - 1\}^{1/2}$. For $\sigma_0 \ll 1$, if ξ_α is about 2, then α is about 1/50.

ignore the formulation in (2) and replace it by the following simpler and equivalent relation which does not involve stochastic integrals (see Proposition 5.2):

$$\begin{aligned} \mathrm{BE}_t &= \mathbb{E}(Y_1|\mathfrak{F}_t) - A_t \\ &\quad + \begin{cases} \gamma[\mathbb{E}(Y_1 - Y_{T^e}|\mathfrak{F}_{T^e}) - A_{T^e}], & \text{if } t \in (T^e, 1], f_\alpha(T^e) = 1, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{3}$$

with $A_t := e^{-\gamma k_{t\wedge T^e}} \int_0^{t\wedge T^e} [\mathbb{E}(Y_1|\mathfrak{F}_s) - Y_s] de^{\gamma k_s}$.

This relation, or equivalently (2), translates in mathematical terms the assumption that the best estimates, and hence the reserves, evolve smoothly over time like $\mathbb{E}(Y_1|\mathfrak{F}_t)$ until they suffer a sudden material increase after a period $(\bar{T}^s, T^e]$ in which the mean loss developments have been above expectation, i.e. $f_\alpha(T^e) = 1$. Since the relative loss increments in $(\bar{T}^s, T^e]$ are dependent, this means that they very likely are systematically above expectation.

Note that BE_1 differs from Y_1, especially when a reserves jump occur. This is because Y_1 is the ultimate attritional losses and thus does not include the unmodelled losses behind the reserves jump. In reality that type of losses are difficult to explicitly model because those are, by their nature, rarely observable. Our model has the capability to reflect the effect of those losses together with their dynamic.

The process in (2) may look similar to a Lévy process, characterised by a smooth component plus a jump component, but clearly it is not because the smooth part is not a Brownian motion and the jump part has not independent increments.

The three processes p_t, $B_t^h(T^s, T^e)$ and $J_t(T^e)$ in (1) and (2) model a reserving cycle: p_t determines the expected cumulative losses at time t, $B_t^h(T^s, T^e)$ captures the first phase of the cycle in which systematic under-estimation of the losses may occur and $J_t(T^e)$ captures the second phase in which a severe deterioration of the reserves occurs as a result of the preceding systematic under-estimation and a triggering adverse event occurring at T^e. Note that only a systematic under-estimation of the losses trigger a sudden deterioration of the reserves, whereas systematic over-estimations certainly do not result in a sudden deterioration, and not even in a sudden improvement because this situation is treated prudently in reality.

Our model depends upon three parameters: the reserves jump probability α, the reserves jump size γ and the loss developments dependency exponent h.

The further parameters a, q and σ_0 used in the definition of the distribution of $B_t^h(T^s, T^e)$ will be fixed in the following Section 3.

3. SCR and Risk Margin Estimation

Observe that, if $\gamma\alpha, \sigma_0 \ll 1$, then $A_t \ll \mathbb{E}(Y_1|\mathfrak{F}_t)$ with arbitrarily high probability. In fact,

$$|A_t| \le e^{-\gamma k_{t\wedge T^e}} \int_0^{t\wedge T^e} |\mathbb{E}(Y_1|\mathfrak{F}_s) - Y_s| e^{\gamma k_{t\wedge T^e}} \gamma dk_s$$

and, from the definition of k_t and ξ_α, $dk_s = \mathbb{P}(f_\alpha(T^e) = 1|T^e = s)/(1-s)ds$, $\mathbb{P}(f_\alpha(T^e) = 1|T^e = s) \le \alpha$, for $s \in (0, 1]$,

$$\begin{aligned} & e^{-\gamma k_{t\wedge T^e}} \int_0^{t\wedge T^e} |\mathbb{E}(Y_1|\mathfrak{F}_s) - Y_s| e^{\gamma k_{t\wedge T^e}} \gamma dk_s \\ & \le \gamma\alpha \mathbb{E}(Y_1|\mathfrak{F}_t) \int_0^{t\wedge T^e} \frac{1}{\mathbb{E}(Y_1|\mathfrak{F}_t)} \frac{|\mathbb{E}(Y_1|\mathfrak{F}_s) - Y_s|}{1-s} ds. \end{aligned}$$

Moreover, $|\mathbb{E}(Y_1|\mathfrak{F}_s)-Y_s| \le Y_0(e^{p_1}-e^{p_s})(1+k\sigma_0)$, $\mathbb{E}(Y_1|\mathfrak{F}_t) \ge Y_0 e^{p_1}(1-k\sigma_0)$, with probability depending on an arbitrary $k \in \mathbb{N}^{**}$, and hence the integrand is bounded by $[(e^{p_1}-e^{p_s})/(e^{p_1}(1-s))][(1+k\sigma_0)/(1-k\sigma_0)]$. This implies that

$$|A_t| \le \gamma\alpha\mathbb{E}(Y_1|\mathfrak{F}_t)t \wedge T^e(1+2k\sigma_0).$$

In addition, since $Y_t = Y_{T^e}e^{p_t-p_{T^e}+B_t^h(T^s,T^e)-B_{T^e}^h(T^s,T^e)}$ and $\mathbb{E}(Y_1|\mathfrak{F}_t) = Y_t e^{p_1-p_t}$, for $t \ge T^e$ (see Section 5.3), we have that

$$\begin{aligned}
&\mathbb{E}(Y_1|\mathfrak{F}_t) - \mathbb{E}(Y_1|\mathfrak{F}_{T^e}) \\
&= Y_{T^e}e^{p_1-p_{T^e}}[e^{B_t^h(T^s,T^e)-B_{T^e}^h(T^s,T^e)}-1] \\
&\quad \le Y_{T^e}e^{p_1-p_{T^e}}k\sigma_0 q^{1/2-h}(c_t-c_{T^e})^{1/2} \\
&\quad \ll Y_{T^e}e^{p_1-p_{T^e}} + \gamma Y_{T^e}(e^{p_1-p_{T^e}}-1) = \mathbb{E}(Y_1|\mathfrak{F}_{T^e}) + \gamma\mathbb{E}(Y_1-Y_{T^e}|\mathfrak{F}_{T^e}),
\end{aligned}$$

for $t \ge T^e$, with probability which depends on an arbitrary $k \in \mathbb{N}$.

Hence, if $\gamma\alpha, \sigma_0 \ll 1$, we can approximate (3) by

$$\mathrm{BE}_t \simeq \begin{cases} \mathbb{E}(Y_1|\mathfrak{F}_{T^e}) + \gamma\mathbb{E}(Y_1-Y_{T^e}|\mathfrak{F}_{T^e}), & \text{if } t \in (T^e,1], f_\alpha(T^e)=1, \\ \mathbb{E}(Y_1|\mathfrak{F}_t) & \text{otherwise.} \end{cases} \tag{4}$$

As last remark, note that (4) for $t=1$ gives a model for the ultimate BE_1, which is composed by the sum of a smooth-type part given by Y_1 and a jump-type part given by the reserves jump. As can be computed out of (4), the $\mathrm{Var}(\mathrm{BE}_1)$ is approximately equal to

$$\begin{aligned}
&\mathrm{Var}(Y_1) + \int_0^1 \gamma^2\mathbb{E}(Y_1-Y_{T^e}|\mathfrak{F}_{T^e})^2\mathbb{P}(f_\alpha(T^e)=1|T^e=t)dt \\
&\simeq \mathrm{Var}(Y_1) + \gamma^2\alpha\int_0^1 Y_0^2(e^{p_1}-e^{p_t})^2dt,
\end{aligned}$$

i.e. the coefficient of variation of BE_1 is

$$\mathrm{CoV}(\mathrm{BE}_1) = \frac{\mathrm{Std}(\mathrm{BE}_1)}{\mathbb{E}(\mathrm{BE}_1)} \simeq [\mathrm{CoV}(\mathrm{Y}_1)^2 + \gamma^2\alpha\int_0^1(1-e^{p_t-p_1})^2dt]^{1/2}, \tag{5}$$

and, since $\mathrm{Var}(e^X) = \mathbb{E}(e^X)^2[\exp(\mathrm{Var}(X))-1]$ for a normally distributed r.v. X,

$$\begin{aligned}
&\mathrm{CoV}(Y_1)^2 = \exp(\mathrm{Var}(B_1^h(T^s,T^e))) - 1 \\
&= \exp\{\int_0^1\int_0^t \sigma_h^2[\mathrm{Var}(B_{qc_s}) + \mathrm{Var}(B^h_{q(c_t-c_s)}) + \mathrm{Var}(B_{q(c_1-c_t)})]a^{s-t}\ln a\,ds\,dt\} - 1 \\
&= \exp\{\sigma_0^2\int_0^1\int_0^t\{q^{1-2h}[c_s+(1-c_t)] + (c_t-c_s)^{2h}\}a^{s-t}\ln a\,ds\,dt\} - 1.
\end{aligned}$$

As our scope is to model the evolution of the reserves for a reinsurer or insurer, we adapt our model to a discrete time version which is the usual actuarial modelling framework.

Suppose that $n = 1,2,\ldots,m$, with $m \in \mathbb{N}$, represents the calendar years ends which are in the future, and $n=0$ in the past is the beginning of the first calendar year. Clearly n may also represent half years, quarters or months.

Suppose that $f_1,\ldots,f_m$, represents the development factors of the incurred or paid losses, as in the chain ladder method, with $f_0 := 1$ and $f_0\cdots f_n(f_{n+1}-1)/(f_0\cdots f_m-1) \ge 1/q$, for all $n=0,\ldots,m-1$. We can relate e^{p_t} in our continuous time model to f_n by

$$e^{p_t} := f_0\cdots f_{n-1}\cdots f_n^{1-m(t-t_n)}\cdots f_{n+1}^{m(t-t_n)}, \quad \text{for } t \in [t_n,t_{n+1}], n \in \{0,\ldots,m-1\},$$

**For $k=5$ we already have a probability greater than 99.99%.

where $t_n := n/m$. Thus, p_t is piecewise linear. We assume that the smallest possible relative reserves increment $1/q$ is equal to 0.1%, i.e. $q = 1000$.

We replace T^e and T^s by $T^e_* := \min\{t_n : t_n \geq T^e, n = 1, \dots, m\}$ and $T^s_* := \max\{t_n : t_n \leq T^s, n = -m, \dots, m\}$ where $t_{-n} := -n/m$, respectively. As in the continuous case, $\bar{T}^s_* := T^s_* \vee 0$. For convenience we assume that $a > 1$ is such that

$$\mathbb{P}(T^s_* = 0, T^e_* = 1) = \mathbb{P}(T^s_* = t_n, T^e_* = t_{n+1}), \text{ for } n = 0, \dots, m-1. \tag{6}$$

Such a exists because $\mathbb{P}(T^s_* = 0, T^e_* = 1) = (a^{1/m} - 1)/(a \ln a)$ and $\mathbb{P}(T^s_* = t_n, T^e_* = t_{n+1}) = 1/m - (1 - a^{-1/m})/\ln a$. For such an a the probability in (6) is of order $\ln m/m^2$, and a of order $m/\ln m$.

We rewrite the continuous-time model (3) in its discrete-time version, for $n \geq 1$,

$$\begin{aligned} \mathrm{BE}_n &= \mathbb{E}(Y_m|\mathfrak{F}_n) - A_n \\ &\quad + \begin{cases} \gamma[\mathbb{E}(Y_m - Y_{T^e_*-1/m}|\mathfrak{F}_{T^e_*-1/m}) - A_{T^e_*-1/m}], & \text{if } t_n \geq T^e_*, f_\alpha(T^e_*) = 1 \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{7}$$

and $\mathrm{BE}_0 = \mathbb{E}(Y_m)$, with $\mathrm{BE}_n := \mathrm{BE}_{t_n}$, $Y_n := Y_{t_n}$, $A_n := A_{t_n}$, $\mathfrak{F}_n := \mathfrak{F}_{t_n}$.

The risk measures we are interested in are the one-year risks at each time $n = 0, \dots, m-1$

$$\mathrm{SCR}_n := \mathrm{tVaR}_{99\%}(\mathrm{TP}_{n+1} - \mathrm{TP}_n|\mathfrak{F}_n),$$

where TP_n is the technical provision at time n, which is the best estimate of the ultimate loss plus the risk margin, i.e. $\mathrm{TP}_n = \mathrm{BE}_n + \mathrm{RM}_n$. This choice is in the spirit of the Solvency II European regulation where the Solvency Capital Requirement for the insurance liabilities at time 0 is

$$\mathrm{SCR}_0 = \mathrm{VaR}_{99.5\%}(\mathrm{TP}_1 - \mathrm{TP}_0),$$

and the risk margin without discounting at time n

$$\mathrm{RM}_n = 6\% \sum_{k=n}^{m-1} \mathbb{E}(\mathrm{SCR}_k|\mathfrak{F}_n) = 6\%\mathrm{SCR}_n + \mathbb{E}(\mathrm{RM}_{n+1}|\mathfrak{F}_n).$$

Here we choose to look at the tVaR at 99% instead of the VaR at 99.5%, which are however both tail measures with normally comparable values, because the VaR is not robust and not coherent whereas tVaR is. The tVaR at 99% is the risk measured prescribed in the Swiss regulation (SST)[††].

For sake of simplicity we make the widely accepted assumption that $\mathrm{TP}_{n+1} - \mathrm{TP}_n$ can be replaced by $\mathrm{BE}_{n+1} - \mathrm{BE}_n$, which is justified by the fact that $\mathrm{RM}_{n+1} - \mathrm{RM}_n$ is relatively small. Without this simplification the SCR_n formula would be more complex, and recursive [7].

In addition, we estimate the future SCR_n, $n \geq 1$, which appear in RM_0, with $\mathrm{tVaR}_{99\%}(\mathrm{BE}_{n+1} - \mathrm{BE}_n)$ instead of $\mathbb{E}(\mathrm{SCR}_n) = \mathbb{E}[\mathrm{tVaR}_{99\%}(\mathrm{BE}_{n+1} - \mathrm{BE}_n|\mathfrak{F}_n)]$. The first quantity has the advantage to be easy to compute in practice because it does not require the use of nested stochastic processes, whereas the second quantity on the contrary is very cumbersome to compute and not worth the effort since the risk margin is a second order quantity with respect to the one-year risk. However,

[††]In the SST the Risk Margin is replaced by the Market Value Margin which is equal to the Risk Margin excluding the first term in the sum, that is, in the SST the eventual run-off is assumed to take place at the end of the year, whereas in the Solvency II the run-off may happen in the middle of the year.

note that the proposed quantity provides a more conservative estimation because $\mathrm{tVaR}_{99\%}(\mathrm{BE}_{n+1}-\mathrm{BE}_n) \geq \mathbb{E}[\mathrm{tVaR}_{99\%}(\mathrm{BE}_{n+1}-\mathrm{BE}_n|\mathfrak{F}_n)]$.

Let us suppose that $\gamma\alpha, \sigma_0 \ll 1$. This is realistic as illustrated in Section 4.

We then have that, using (4) and (7),

$$\begin{aligned}&\mathrm{BE}_{n+1}-\mathrm{BE}_n\\ &\simeq X_n := \begin{cases} \gamma\mathbb{E}(Y_m - Y_n|\mathfrak{F}_n), & \text{if } t_{n+1}=T_*^e, f_\alpha(T_*^e)=1,\\ \mathbb{E}(Y_m|\mathfrak{F}_{n+1}) - \mathbb{E}(Y_m|\mathfrak{F}_n), & \text{otherwise,}\end{cases}\end{aligned}$$

and

$$\begin{aligned}&\mathrm{BE}_m-\mathrm{BE}_0\\ &\simeq X := \begin{cases} \mathbb{E}(Y_m|\mathfrak{F}_{T_*^e-1/m}) + \gamma\mathbb{E}(Y_m - Y_{T_*^e-1/m}|\mathfrak{F}_{T_*^e-1/m}) - \mathbb{E}(Y_m), & \text{if } f_\alpha(T_*^e)=1,\\ Y_m - \mathbb{E}(Y_m), & \text{otherwise.}\end{cases}\end{aligned}$$

Let α be such that $\alpha/m < 1\%$ and γ be such that $\gamma(\mathrm{BE}_0 - Y_0)$ is bigger than $\mathrm{tVaR}_{99.5\%-\delta}[Y_m - \mathbb{E}(Y_m)]$ and smaller than $\mathrm{tVaR}_{99.5\%+\delta}[Y_m - \mathbb{E}(Y_m)]$, with $\delta := |\alpha/m - 0.5\%|$.

Set $\lambda := \alpha/(m1\%)$. We then have

$$\begin{aligned}\mathrm{tVaR}_{99\%}(X) &\simeq \mathrm{tVaR}_{99\%+\alpha/m}[Y_m - \mathbb{E}(Y_m)](1-\lambda) + \gamma Y_0(e^{p_m}-1)\lambda\\ &= \mathrm{tVaR}_{99\%+\alpha/m}[Y_m - \mathbb{E}(Y_m)](1-\lambda) + \gamma(\mathrm{BE}_0 - Y_0)\lambda\\ &\simeq \gamma(\mathrm{BE}_0 - Y_0),\end{aligned}$$

and

$$\mathrm{tVaR}_{99\%}(X_0) = \mathrm{tVaR}_{99\%+\alpha/m}[\mathbb{E}(Y_m|\mathfrak{F}_1) - \mathbb{E}(Y_m)](1-\lambda) + \gamma Y_0(e^{p_m}-1)\lambda.$$

For the future one-year periods $n = 1, \ldots, m-1$,

$$\begin{aligned}&\mathrm{tVaR}_{99\%}(X_n)\\ &\simeq \mathrm{tVaR}_{99\%+\alpha/m}[\mathbb{E}(Y_m|\mathfrak{F}_{n+1}) - \mathbb{E}(Y_m|\mathfrak{F}_n)](1-\lambda) + \gamma Y_0(e^{p_m} - e^{p_n})\lambda.\end{aligned}$$

On the other hand, since the worst cases for $\mathbb{E}(Y_m|\mathfrak{F}_j) - \mathbb{E}(Y_m|\mathfrak{F}_i)$, $j > i$, are the ones with $T_*^s = t_i, T_*^e = t_j$, $\mathbb{E}(Y_m|\mathfrak{F}_{n+1}) - \mathbb{E}(Y_m|\mathfrak{F}_n)$ and $Y_m - \mathbb{E}(Y_m)$ coincide with a geometric fractional Brownian motion over $(t_n, t_{n+1}]$ and $(0,1]$, respectively. Hence, by (6) and (8),

$$\mathrm{tVaR}_{99\%+\alpha/m}[\mathbb{E}(Y_m|\mathfrak{F}_{n+1}) - \mathbb{E}(Y_m|\mathfrak{F}_n)] \simeq (c_{n+1}-c_n)^h\,\mathrm{tVaR}_{99\%+\alpha/m}[Y_m - \mathbb{E}(Y_m)],$$

where $c_n := c_{t_n}$. We therefore deduce the estimator for the SCR_0 which is

$$\mathrm{SCR}_0 \simeq [c_1^h(1-\lambda)+\lambda]\mathrm{tVaR}_{99\%}(\mathrm{BE}_m - \mathrm{BE}_0),$$

where $\mathrm{tVaR}_{99\%}(\mathrm{BE}_m - \mathrm{BE}_0)$ is the ultimate risk.

For SCR_n, $n \geq 1$, which are random variables equal to $\mathrm{tVaR}_{99\%}(\mathrm{BE}_{n+1} - \mathrm{BE}_n|\mathfrak{F}_n)$, we have that

$$\mathrm{tVaR}_{99\%}(\mathrm{BE}_{n+1} - \mathrm{BE}_n) \simeq [(c_{n+1}-c_n)^h(1-\lambda) + (1-c_n)\lambda]\mathrm{tVaR}_{99\%}(\mathrm{BE}_m - \mathrm{BE}_0).$$

We therefore deduce the estimator for RM_0 without discounting[‡‡]

$$\mathrm{RM}_0 \simeq 6\%\{c_1^h(1-\lambda)+\lambda+\sum_{n=1}^{m-1}[(c_{n+1}-c_n)^h(1-\lambda)+(1-c_n)\lambda]\}\mathrm{tVaR}_{99\%}(\mathrm{BE}_m-\mathrm{BE}_0).$$

These formulas provide simple way to estimate the SCR_n and the RM_n, for $n \geq 0$. In the following Section 4 we show how to use them in practice.

[‡‡]Clearly the risk margin with discounting can be obtained by just multiplying any term in the sum by its discounting factor. In the paper we used the risk margin without discounting to make the important parts of our formula as clear as possible.

4. User's guide

Our model is presented in full details in Section 2, and the formulas are derived in Section 3. In this section we provide a brief guide for the use of the formulas in practice.

We need to collect the following inputs at the portfolio level:

- the ultimate loss r.v. BE_m, which could either include the reserves jump component, or not (i.e. $\mathrm{BE}_m = Y_m$);
- the cumulative calendar year incurred losses pattern $\{c_n : n = 1, \ldots, m\}$.

We need to fix the following parameters at the portfolio level:

- the dependency exponent between relative loss increments h;
- the frequency α of a reserve jump;
- the relative size γ of a reserve jump; if BE_m includes the reserve jump component, then $\gamma(\mathrm{BE}_0 - Y_0)$ should be approximately equal to $\mathrm{tVaR}_{99\%}[\mathrm{BE}_m - \mathrm{BE}_0]$; if BE_m does not include the reserves jump component, then the experts quantify γ such that its value is around three times the coefficient of variation of the reserves, which is a realistic plausibility condition for γ.

The formulas for the SCR and the risk margin without discounting at time 0 are

$$\begin{aligned}
&\mathrm{SCR}_0 = [c_1^h(1-\lambda) + \lambda]\gamma(\mathrm{BE}_0 - Y_0),\\
&\mathrm{RM}_0 = 6\%\{c_1^h(1-\lambda) + \lambda + \textstyle\sum_{n=1}^{m-1}[(c_{n+1} - c_n)^h(1-\lambda) + (1-c_n)\lambda]\}\gamma(\mathrm{BE}_0 - Y_0),
\end{aligned}$$

and at the future time $n = 1, \ldots, m-1$,

$$\begin{aligned}
&\mathrm{SCR}_n = [(c_{n+1} - c_n)^h(1-\lambda) + (1-c_n)\lambda]\gamma(\mathrm{BE}_0 - Y_0),\\
&\mathrm{RM}_n = 6\% \textstyle\sum_{k=n}^{m-1}[(c_{k+1} - c_k)^h(1-\lambda) + (1-c_k)\lambda]\gamma(\mathrm{BE}_0 - Y_0).
\end{aligned}$$

In our model we assume that $\alpha/m < 1\%$ and the factor λ in the formulas above is equal to $\alpha/(m1\%)$. However, in practice $\lambda \in [0,1]$ could be quantified by the experts considering that it describes the contribution of the jump risk to the total risk.

To conclude, no model is perfect! In our we assume that:

- $\alpha/m < 1\%$; usually m is greater than 10 and α is smaller than 1/10;
- $\gamma\alpha$ is small, i.e. not greater than 0.05; usually γ is around 30% and α is smaller than 1/10;
- σ_0 is small; usually this quantity, which is related the coefficient of variation of the ultimate loss BE_m without the reserves jump component, is smaller than 5%;
- the increments of the calendar year incurred loss pattern are such that $\{c_n\}$ stays above the straight line $\{t_n c_m\}$, i.e. $c_n \geq t_n c_m$, and $c_n - c_{n-1} \geq 1/q$, for $n = 1, \ldots, m$;
- the incurred losses evolves as described in Section 2;
- the best estimate of the ultimate loss evolves as described in Section 2;
- the risk margin stays approximately constant one year to the following;
- we estimate the future SCR_n, $n \geq 1$, with $\mathrm{tVaR}_{99\%}(\mathrm{BE}_{n+1} - \mathrm{BE}_n)$ instead of $\mathbb{E}[\mathrm{tVaR}_{99\%}(\mathrm{BE}_{n+1} - \mathrm{BE}_n|\mathfrak{F}_n)]$; this is a conservative estimation since the first quantity is greater than the second.

5. Appendix

5.1. **Fractional Brownian motions.** For $0 < h < 1$ a fractional Brownian motion (fBm) is a continuous stochastic process $\{B_t^h\}$ which is Gaussian, $B_0^h = 0$, $\mathbb{E}(B_t^h) = 0$, for all $t \geq 0$, and

$$\mathbb{E}(B_t^h B_s^h) = (t^{2h} + s^{2h} - |t-s|^{2h})/2, \quad t, s \geq 0.$$

If $h > 1/2$ then increments are positively correlated, wheres for $h < 1/2$ the increments are negatively correlated. For $h = 1/2$ the increments are independent and the fractional Brownian motion is nothing but a Brownian motion $\{B_t\}$.

The fBm is self-similar, i.e. for $\sigma > 0$,

$$B_{\sigma t}^h \sim \sigma^h B_t^h, \quad t \geq 0, \tag{8}$$

which says how paths change when re-scaled, is stable, i.e. for $t > s$, $B_t^h - B_s^h \sim B_{t-s}^h$, which means that increments of equal size have equal distribution, and for $h > 1/2$ has long memory, i.e. $\sum_{n=1}^{\infty} \mathbb{E}[B_1^h(B_{n+1}^h - B_n^h)] = \infty$.

The fBm is an important stochastic process because it is the only self-similar Gaussian process.

We can represent a fBm in terms of a Brownian motion by the following Itô stochastic integral

$$B_t^h = b_h \int_{-\infty}^{t} [(t-s)^{h-1/2} - (-s)_+^{h-1/2}] dB_s, \quad t \geq 0,$$

where b_h is a constant, $b_h \geq 1$ for $1/2 < h < 1$. The constance b_h is equal to $1/\Gamma(h+1/2)$, where Γ is the meromorphic extension of the factorial function.

We can define a fBm $\{\bar{B}_t^h\}$ starting at a negative point t_0, and passing through 0 at $t = 0$, by

$$\bar{B}_t^h := B_{t-t_0}^h - B_{-t_0}^h, \qquad \text{for } t \geq t_0.$$

5.2. **The stochastic process $\{\mathrm{BE}_t\}$ is a martingale.** Let us first prove that

$$\mathrm{BE}_t = \mathbb{E}(Y_1|\mathfrak{F}_t) - A_t + \begin{cases} \gamma[\mathbb{E}(Y_1 - Y_{T^e}|\mathfrak{F}_{T^e}) - A_{T^e}], & \text{if } t \in (T^e, 1], f_\alpha(T^e) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

is solution to (2), and thus that is a martingale.

For $t \in [0, \bar{T}^e]$, multiplying by $e^{\gamma k_t}$ and summing $e^{\gamma k_t} dY_t$ to both members of (2), we have

$$e^{\gamma k_t} d\mathrm{BE}_t = e^{\gamma k_t} d\mathbb{E}(Y_1|\mathfrak{F}_t) - e^{\gamma k_t} \gamma(\mathrm{BE}_t - Y_t) dk_t,$$

that is equivalent to

$$d[e^{\gamma k_t}(\mathrm{BE}_t - Y_t)] = e^{\gamma k_t} d[\mathbb{E}(Y_1|\mathfrak{F}_t) - Y_t],$$

since $d[e^{\gamma k_t}(\mathrm{BE}_t - Y_t)] = e^{\gamma k_t} d[\mathbb{E}(Y_1|\mathfrak{F}_t) - Y_t] + (\mathrm{BE}_t - Y_t) e^{\gamma k_t} \gamma dk_t$. A similar identity holds for $d\{e^{\gamma k_t}[\mathbb{E}(Y_1|\mathfrak{F}_t) - Y_t]\}$ and hence

$$d[e^{\gamma k_t}(\mathrm{BE}_t - Y_t)] = d\{e^{\gamma k_t}[\mathbb{E}(Y_1|\mathfrak{F}_t) - Y_t]\} - [\mathbb{E}(Y_1|\mathfrak{F}_t) - Y_t] de^{\gamma k_t}.$$

Integrating in t over $[0, \bar{T}^e]$, we obtain

$$\begin{aligned} & e^{\gamma k_t}(\mathrm{BE}_t - Y_t) - (\mathrm{BE}_0 - Y_0) \\ & = e^{\gamma k_t}[\mathbb{E}(Y_1|\mathfrak{F}_t) - Y_t] - [\mathbb{E}(Y_1|\mathfrak{F}_0) - Y_0] - \int_0^t [\mathbb{E}(Y_1|\mathfrak{F}_s) - Y_s] de^{\gamma k_s}, \end{aligned}$$

that, since $\mathrm{BE}_0 = \mathbb{E}(Y_1|\mathfrak{F}_0) = \mathbb{E}(Y_1)$, implies

$$\mathrm{BE}_t = \mathbb{E}(Y_1|\mathfrak{F}_t) - e^{-\gamma k_t}\int_0^t [\mathbb{E}(Y_1|\mathfrak{F}_s) - Y_s]de^{\gamma k_s}, \ \ t \in [0, T^e].$$

For $t \in (T^e, 1]$, we have that $dJ_t(T^e) = 0$ and hence $\mathrm{BE}_t - \lim_{t\to T^e_+} \mathrm{BE}_t = \mathbb{E}(Y_1|\mathfrak{F}_t) - \mathbb{E}(Y_1|\mathfrak{F}_{T^e})$. Observing that

$$\begin{aligned}\lim_{t\to T^e_+} \mathrm{BE}_t - \mathrm{BE}_{T^e} &= \lim_{t\to T^e_+}\int_{T^e}^t [d\mathbb{E}(Y_1|\mathfrak{F}_s) + \gamma(\mathrm{BE}_s - Y_s)dJ_s(T^e)]\\ &= \begin{cases} \gamma(\mathrm{BE}_{T^e} - Y_{T^e}) = \gamma[\mathbb{E}(Y_1 - Y_{T^e}|\mathfrak{F}_{T^e}) - A_{T^e}], & \text{if } f_\alpha(T^e) = 1,\\ 0, & \text{otherwise,}\end{cases}\end{aligned}$$

we obtain, for $t \in (T^e, 1]$,

$$\mathrm{BE}_t = \mathbb{E}(Y_1|\mathfrak{F}_t) - A_{T^e} + \begin{cases} \gamma[\mathbb{E}(Y_1 - Y_{T^e}|\mathfrak{F}_{T^e}) - A_{T^e}], & \text{if } f_\alpha(T^e) = 1,\\ 0, & \text{otherwise,}\end{cases}$$

as claimed.

The process $\{\mathrm{BE}_t\}$, as any stochastic process, is a martingale if and only if $|\mathbb{E}(\mathrm{BE}_{t+\epsilon} - \mathrm{BE}_t|\mathfrak{F}_t)| \le o(\epsilon)$, for every t, where $o(\epsilon)$ is a function such that $o(\epsilon)/\epsilon \to 0$, as $\epsilon \to 0$. Indeed, for $s < t$, $\epsilon_n := (t-s)/n$,

$$\begin{aligned}|\mathbb{E}(\mathrm{BE}_t - \mathrm{BE}_s|\mathfrak{F}_s)| &\le \mathbb{E}[\textstyle\sum_{k=1}^n |\mathbb{E}(\mathrm{BE}_{s+k\epsilon_n} - \mathrm{BE}_{s+(k-1)\epsilon_n}|\mathfrak{F}_{s+(k-1)\epsilon_n})||\mathfrak{F}_s]\\ &\le (t-s)o(\epsilon_n)/\epsilon_n \to 0, \quad \text{for } n \to \infty.\end{aligned}$$

As $\{\mathrm{BE}_t\}$ is solution to (2), then

$$\begin{aligned}&\mathbb{E}(\mathrm{BE}_{t+\epsilon} - \mathrm{BE}_t|\mathfrak{F}_t) = \mathbb{E}(\textstyle\int_t^{t+\epsilon} d\mathrm{BE}_s|\mathfrak{F}_t)\\ &= \mathbb{E}(\textstyle\int_t^{t+\epsilon} d\mathbb{E}(Y_1|\mathfrak{F}_s) + \gamma(\mathrm{BE}_s - Y_s)dJ_s(T^e)|\mathfrak{F}_t)\\ &= \mathbb{E}(\textstyle\int_t^{t+\epsilon} \gamma(\mathrm{BE}_s - Y_s)dJ_s(T^e)|\mathfrak{F}_t).\end{aligned}$$

Observe that since $\mathrm{BE}_s - Y_s$ is continuous on the compact set $[0,1]$, $|\mathrm{BE}_s - Y_s - \mathrm{BE}_t + Y_t| \le o(\epsilon)/\epsilon$, for $s \in [t, t+\epsilon]$, $t \in [0,1)$, and hence

$$\begin{aligned}&|\mathbb{E}(\textstyle\int_t^{t+\epsilon} \gamma(\mathrm{BE}_s - Y_s)dJ_s(T^e)|\mathfrak{F}_t)|\\ &\le \begin{cases} \gamma[|\mathrm{BE}_t + Y_t| + o(\epsilon)/\epsilon]\\ \quad \cdot|\mathbb{E}[\mathbb{1}_{\{1\}}(f_\alpha(T^e))\mathbb{1}_{(t,t+\epsilon]}(T^e) - (k_{t+\epsilon} - k_t)|T^e \in (t,1]]|, & \text{if } T^e \in (t,1],\\ 0, & \text{if } T^e \in [0,t],\end{cases}\\ &= \begin{cases} \gamma[|\mathrm{BE}_t + Y_t| + o(\epsilon)/\epsilon]\\ \quad \cdot|[\mathbb{P}(f_\alpha(T^e) = 1|T^e = t) + o(\epsilon)/\epsilon]\epsilon/(1-t) - (k_{t+\epsilon} - k_t)|, & \text{if } T^e \in (t,1],\\ 0, & \text{if } T^e \in [0,t],\end{cases}\\ &= o(\epsilon),\end{aligned}$$

which proves that $\{\mathrm{BE}_t\}$ is a martingale.

5.3. **Explicit formula for $\mathbb{E}(Y_1|\mathfrak{F}_t)$.** Note that, for $t \in [T^e, 1]$,

$$\begin{aligned}&\mathbb{E}(Y_1|\mathfrak{F}_t) = \mathbb{E}(Y_0 e^{p_1 + B_1^h(T^s,T^e)}|\mathfrak{F}_t)\\ &= Y_0 e^{p_1 + B_t^h(T^s,T^e)}\mathbb{E}(e^{B_1^h(T^s,T^e) - B_t^h(T^s,T^e)}|\mathfrak{F}_t)\\ &= Y_0 e^{p_1 + B_t^h(T^s,T^e)}\mathbb{E}(e^{\sigma_h B_{q(c_1-c_t)} - \sigma_h^2 q(c_1-c_t)/2}|\mathfrak{F}_t) = Y_0 e^{p_1 + B_t^h(T^s,T^e)},\end{aligned}$$

where the last equality is a consequence of that

$$\mathbb{E}(e^X|\mathfrak{F}) = \exp\{\mathbb{E}(X|\mathfrak{F}) + \mathbb{E}(|X - \mathbb{E}(X|\mathfrak{F})|^2|\mathfrak{F})/2\}$$

for any normally distributed r.v. X.

Similarly, for $t \in (0, T^e)$, we have that

$$\begin{aligned}
&\mathbb{E}(Y_1|\mathfrak{F}_t) = \mathbb{E}(\mathbb{E}(Y_1|\mathfrak{F}_{T^e})|\mathfrak{F}_t) \\
&= Y_0 e^{p_1} \mathbb{E}(e^{B^h_{T^e}(T^s,T^e)}|\mathfrak{F}_t) \\
&= Y_0 e^{p_1} \mathbb{E}[\exp(\sigma_h B^h_{q(c_{T^e}-c_{\bar{T}^s})})|\mathfrak{F}_t]\mathbb{E}[\exp(-\sigma_h^2 q^{2h}(c_{T^e}-c_{\bar{T}^s})^{2h}/2 + B^h_{\bar{T}^s}(T^s,T^e))|\mathfrak{F}_t] \\
&= Y_0 e^{p_1} \exp[\sigma_h \mathbb{E}(B^h_{q(c_{T^e}-c_{\bar{T}^s})}|\mathfrak{F}_t) + \sigma_h^2 \mathbb{E}((B^h_{q(c_{T^e}-c_{\bar{T}^s})})^2|\mathfrak{F}_t)/2] \\
&\quad \cdot \mathbb{E}[\exp(-\sigma_h^2 q^{2h}(c_{T^e}-c_{\bar{T}^s})^{2h}/2 + B^h_{\bar{T}^s}(T^s,T^e))|T^e > t].
\end{aligned}$$

The term $\mathbb{E}((B^h_{q(c_{T^e}-c_{\bar{T}^s})})^2|\mathfrak{F}_t)$ could be re-written by using, for any fBm $\{B^h_t\}$,

$$\begin{aligned}
&\mathbb{E}((B^h_1)^2|\mathfrak{F}_t) \\
&= \mathbb{E}((b_h \textstyle\int_{-\infty}^t [(1-s)^{h-1/2} - (-s)_+]dB_s)^2|\mathfrak{F}_t) + \mathbb{E}((b_h \textstyle\int_t^1 (1-s)^{h-1/2} dB_s)^2|\mathcal{F}_t) \\
&= b_h^2 (\textstyle\int_{-\infty}^t [(1-s)^{h-1/2} - (-s)_+]dB_s)^2 + b_h^2 \textstyle\int_t^1 (1-s)^{2h-1} ds \\
&= b_h^2 (\textstyle\int_{-\infty}^t [(1-s)^{h-1/2} - (-s)_+]dB_s)^2 + b_h^2 (t-s)^{2h}/(2h),
\end{aligned}$$

where the first equality is consequence of the representation formula of a fBm, and the second equality is a consequence of the Itô's isometry.

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