Valuation of Large Variable Annuity Portfolios: Monte Carlo Simulation and Benchmark Datasets

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Abstract

Metamodeling techniques have recently been proposed to address the computational issues related to the valuation of large portfolios of variable annuity contracts. However, it is extremely difficult, if not impossible, for researchers to obtain real datasets from insurance companies in order to test their metamodeling techniques on such real datasets and publish the results in academic journals. Even if a researcher can obtain real datasets from insurance companies, it is difficult for the researcher to share the datasets with the public at large. To facilitate the development and dissemination of research related to the efficient valuation of large variable annuity portfolios, this paper creates a large synthetic portfolio of variable annuity contracts based on the properties of real portfolios of variable annuities and implements a Monte Carlo simulation engine for valuing the synthetic portfolio. In addition, this paper develops benchmark datasets of fair market values and Greeks, which are important quantities for managing the financial risks associated with variable annuities. The resulting datasets provide researchers with a common basis for testing and comparing the performance of various metamodeling techniques.

1 Introduction

A variable annuity is an insurance product created and sold by insurance companies as a tax-deferred retirement vehicle to address many people's concerns about outliving their assets (Ledlie et al., 2008; The Geneva Association Report, 2013). Essentially, a variable annuity is a deferred annuity with two phases: the accumulation phase and the payout phase. During the accumulation phase, the policyholder makes purchase payments to the insurance company. During the payout phase, the policyholder received benefit payments from the insurance company. The policyholder's money is invested in a set of investment funds provided by the insurance company. The policyholder has the option of allocating the money among this set of investment funds. A major feature of a variable annuity is that it includes guarantees or riders. Due to this attractive feature, variable annuities have grown rapidly in popularity during the past two decades. According to The Geneva Association Report (2013), the annual sales of variable annuities in the U.S. were more than $100 billion for every year from 1999 to 2011.

The guarantees embedded in variable annuities are financial guarantees that cannot be adequately addressed by traditional actuarial approaches (Boyle and Hardy, 1997; Hardy, 2003). Dynamic hedging is adopted by many insurance companies to mitigate the financial risks associated with the guarantees. Dynamic hedging requires calculating the fair market values and Greeks (i.e., sensitivities) of the guarantees. Since the guarantees embedded in variable annuities are relatively complex, their fair market values cannot be calculated in closed form except for special cases (Gerber and Shiu, 2003; Feng and Volkmer, 2012). In practice, insurance companies rely on Monte Carlo simulation to calculate the fair market values and the Greeks of the guarantees. However, using Monte Carlo simulation to value a large portfolio of variable annuity contracts is extremely time-consuming because every contract needs to be projected over many economic scenarios for a long time horizon (Dardis, 2016).

In the past few years, metamodeling techniques have been proposed to address the computational issues associated with the valuation of large variable annuity portfolios. See, for example, Gan (2013), Gan and Lin (2015), Gan (2015a), Gan and Lin (2016), Gan and Valdez (2016), and Hejazi and Jackson (2016). The main idea of metamodeling techniques is to construct a surrogate model on a set of representative variable annuity contracts in order to reduce the number of contracts that are valued by Monte Carlo simulation. This is achieved by selecting a small number of representative contracts, using Monte Carlo simulation to calculate the fair market values (or other quantities of interest) of the representative contracts, building a regression model (i.e., the metamodel) based on the representative contracts and their fair market values, and finally using the regression model to value the whole portfolio of variable annuity contracts.

However, it is difficult for researchers to obtain real datasets from insurance companies to assess the performance of those metamodeling techniques. Even if a researcher can obtain real datasets from insurance companies, it is difficult for the researcher to share the datasets with the public at large. As a result, the aforementioned papers on variable annuity portfolio valuation used synthetic datasets to test the performance of the proposed metamodeling techniques. However, there are no synthetic datasets that can serve as benchmark datasets. Different researchers created different synthetic datasets to test various proposed methods. For example, the synthetic datasets used by Gan (2013) and Hejazi and Jackson (2016) are different in portfolio composition.

In this paper, we create some benchmark datasets to facilitate the development and dissemination of research related to the efficient valuation of large variable annuity portfolios. In particular, we create a large synthetic portfolio of variable annuity contracts based on the properties of real portfolios of variable annuities and implement a Monte Carlo simulation engine that is used to calculate the fair market values and the Greeks of the guarantees embedded in those synthetic variable annuity contracts.

The remaining part of this paper is organized as follows. Section 2 describes how the synthetic portfolio of variable annuity contracts is created. Section 3 presents a Monte Carlo simulation engine for valuing the guarantees embedded in variable annuities. In Section 4, we present some benchmark datasets that can be used to test the performance of metamodeling techniques. Section 5 concludes the paper with some remarks. The software that is used to generate the synthetic portfolio and implement the Monte Carlo simulation engine is described in Appendix A.

2 Synthetic Portfolio of Variable Annuity Contracts

In this section, we describe how to create a synthetic portfolio of variable annuity contracts to mimic a real portfolio of variable annuity contracts. In particular, we create a synthetic portfolio of variable annuity contracts based on the following major properties typically observed on real portfolios of variable annuity contracts:

- Different contracts may contain different types of guarantees.
- The contract holder has the option to allocate the money among multiple investment funds.
- Real variable annuity contracts are issued at different dates and have different times to maturity.

2.1 Guarantee Types

Guarantees embedded in variable annuities can be divided into two broad categories: the guaranteed minimum death benefit (GMDB) and guaranteed minimum living benefits (GMLB). The GMDB rider guarantees the policyholder a specific amount upon death during the term of the contract (Hardy, 2003). The death benefit is paid to the designated beneficiary of the policyholder upon the death of the policyholder. The death benefit comes in several forms (Bauer et al., 2008): return of premium death benefit, annual roll-up death benefit, and annual ratchet death benefit. The return of premium death benefit is the most basic form of the death benefit. Under this form, the death benefit paid is equal to the maximum of the account value at time of death and the premium. This option is usually offered without additional charges. Under the annual roll-up death benefit option, the death benefit increases at a specified interest rate. Under the annual ratchet death benefit option, the death benefit is reset to the account value if it is higher than the current death benefit.

There are several types of GMLBs: guaranteed minimum accumulation benefits (GMAB), guaranteed minimum income benefits (GMIB), guaranteed minimum maturity benefits (GMMB), and guaranteed minimum withdrawal benefits (GMWB). The GMAB rider guarantees that the policyholder has the option to renew the contract during a specified window after a specified waiting period, which is usually 10 years. The specified widows typically begins on an anniversary date and remains open for 30 days (Brown et al., 2002). The GMIB rider guarantees that the policyholder can convert the lump sum accumulated during the term of the contract to an annuity at a guaranteed rate (Hardy, 2003). The GMMB rider guarantees the policyholder a specific amount at the maturity of the contract (Hardy, 2003). The GMWB rider gives a policyholder the right to withdraw a specified amount during the life of the contract until the initial investment is recovered. Similar to the death benefit, the living benefit can be the original premium or subject to regular or equity-dependent increases.

The riders can be purchased individually or in combination. For example, the GMDB and the GMWB riders can be purchased simultaneously. To create a synthetic portfolio of variable annuity contracts, we consider 19 products shown in Table 1. For the synthetic variable annuity policies, we set the rider fees of individual riders in the range of 0.25% to 0.75% according to the ranges given in Bauer et al. (2008). The rider fee of the combined guarantees is set equal to the sum of the fees of the individual guarantees minus 0.20%.

2.2 Investment Funds

In practice, the policyholder's money is invested in one or more investment funds provided by the insurance company. The policyholder is allowed to

Table 1: Variable annuity contracts in the synthetic portfolio.

Product	Description	Rider Fee
DBRP	GMDB with return of premium	0.25%
DBRU	GMDB with annual roll-up	0.35%
DBSU	GMDB with annual ratchet	0.35%
ABRP	GMAB with return of premium	0.50%
ABRU	GMAB with annual roll-up	0.60%
ABSU	GMAB with annual ratchet	0.60%
IBRP	GMIB with return of premium	0.60%
IBRU	GMIB with annual roll-up	0.70%
IBSU	GMIB with annual ratchet	0.70%
MBRP	GMMB with return of premium	0.50%
MBRU	GMMB with annual roll-up	0.60%
MBSU	GMMB with annual ratchet	0.60%
WBRP	GMWB with return of premium	0.65%
WBRU	GMWB with annual roll-up	0.75%
WBSU	GMWB with annual ratchet	0.75%
DBAB	GMDB + GMAB with annual ratchet	0.75%
DBIB	GMDB + GMIB with annual ratchet	0.85%
DBMB	GMDB + GMMB with annual ratchet	0.75%
DBWB	GMDB + GMWB with annual ratchet	0.90%

select the investment funds. In dynamic hedging, a fund mapping is used to map an investment fund to a combination of tradable and liquid indices such as the S&P500 index. A fund mapping is used for the following reasons. First, different policyholders may invest their money in different combinations of investment funds. Second, most of the investment funds are not tradable and guarantees need to be hedged by derivatives on tradable indices such as S&P500. Third, using tradable and liquid indices in the asset model is also convenient in terms of calibrating the asset model parameters from the market.

A fund mapping that maps an investment fund to k indices is denoted by a vector of k weights $(w_1, w_2, \ldots, w_k)$ such that

$$\sum_{j=1}^{k} w_j = 1.$$

The rate of return r_f of the investment fund at a period is calculated as

$$r_f = \sum_{j=1}^{k} w_j r_{I_j}, \tag{1}$$

where r_{I_j} is the rate of return of index I_j at the same period for $j = 1, 2, \ldots, k$.

The weights of an investment fund can be estimated by the method of least squares from the historical returns of the investment fund and the indices.

Table 2 shows the fund mappings of ten investment funds. Funds 1 to 5 are the index funds that replicate US large-cap equity, US small-cap equity, international equity, fixed income, and money market fund, respectively. Fund 6 is a balanced mix of US large-cap equity and US small-cap equity. Other funds are different combinations of the indices.

Table 2: Ten investment funds. Each row is a mapping from an investment fund to a combination of five indices.

Fund	US Large	US Small	Intl Equity	Fixed Income	Money Market
1	1	0	0	0	0
2	0	1	0	0	0
3	0	0	1	0	0
4	0	0	0	1	0
5	0	0	0	0	1
6	0.6	0.4	0	0	0
7	0.5	0	0.5	0	0
8	0.5	0	0	0.5	0
9	0	0.3	0.7	0	0
10	0.2	0.2	0.2	0.2	0.2

In the synthetic portfolio, we generate the account values of the investment funds of a policy as follows. First, we generate randomly the total account value AV from a specified range. Second, we generate a random integer l between 1 and 10, inclusive. Third, we select randomly l investment funds from the ten investment funds. Finally, we set the account values of those l selected investment funds to be AV/l, that is, the total account values are allocated to the l investment funds equally.

2.3 Aging

Aging refers to the process of adjusting a variable annuity contract from an old date to a new date to reflect the changes of the account values and other relevant items (e.g., withdrawals, benefit base). In practice, variable annuity policies in a portfolio are issued at different dates. To value the policies at the valuation date, the policies are aged from the issue dates to the valuation date.

To create the synthetic portfolio of variable annuity policies, we make some assumptions for the sake of simplicity. In particular, we assume that all policies are issued on the first day of a month and the policyholders' birth dates are also on the first day of a month. The birth dates of policyholders are randomly generated from an interval of dates and the issue dates of the policies are randomly generated from another interval of dates. Table 3 shows some parameters used to create synthetic policies. Once we generate a variable annuity policy, we age it to the specified valuation date. In practice, the aging process reflect what happens actually to the policies. To generate the synthetic portfolio, aging a policy is just projecting the policy from the issue date to the valuation date based on one economic scenario of the investment funds. Details of the liability cash flow projection are discussed in Section 3.2.

Table 3: Parameter values used to generate the synthetic portfolio.

Feature	**Value**
Policyholder birth date	[1/1/1950, 1/1/1980]
Issue date	[1/1/2000, 1/1/2014]
Valuation date	1/6/2014
Maturity	[15, 30] years
Account value	[50000, 500000]
Female percent	40%
	(20% of each type)
Fund fee	30, 50, 60, 80, 10, 38, 45, 55, 57, 46bps
	for Funds 1 to 10, respectively
M&E fee	200 bps

3 Monte Carlo Valuation

In this section, we present a simple Monte Carlo simulation engine for valuing guarantees of the synthetic portfolio of variable annuities. In particular, we present a risk-neutral scenario generator, liability cash flow modeling, and fair market value and Greek calculation. This Monte Carlo simulation model was presented at a conference by one of the authors (Gan, 2015b).

3.1 Risk-Neutral Scenario Generator

Economic scenario generators are used to simulate movement scenarios of the indices according to an asset model. There are two types of scenarios: risk-neutral and real-world. Risk-neutral scenarios are simulated under the risk-neutral measure; while real-world scenarios are simulated under the real-world measure. Risk-neutral scenarios are used to calculate the fair market values of financial derivatives such as the guarantees embedded in variable annuities. Real-world scenarios are used to calculate solvency capitals or evaluate hedging strategies.

Most economic scenario generators remain proprietary, but two economic scenario generators are in the public domain: the one developed by the CAS (Casualty Actuarial Society) and the SOA (Society of Actuaries) and the one developed by the AAA (American Academy of Actuaries) and the SOA (Ahlgrim et al., 2008). The CAS-SOA scenario generator is used to generate economic scenarios for asset-liability analysis for property-liability insurers (D'Arcy and Gorvett, 2000; Ahlgrim et al., 2005).

The AAA and the SOA have created an economic scenario generator, named Academy's Interest Rate Generator (AIRG), for regulatory reserve and capital calculations (Group, 2010)[1]. It is a real-world economic scenario generator and can be used to generate both interest rate and equity scenarios.

Both the CAS-SOA generator and the AAA-SOA generator can generate interest rate and equity scenarios. However, the resulting scenarios generated by these two generators differ significantly. In particular, the interest rates generated by the CAS-SOA generator have a wider distributions than those generated by the AAA-SOA generator. For a detailed comparison of the two economic scenario generators, readers are referred to Ahlgrim et al. (2008).

Although using economic scenario generators is the only practical way to value many life insurance contracts, it has received little attention in the academic literature. The paper by Varnell (2011) is among the few papers devoted to this subject. Varnell (2011) gives a brief background of the Solvency II and discusses the use of economic scenario generators in the context of Solvency II.

In this paper, we present a simple economic scenario generator to generate risk-neutral scenarios. In this simple generator, we model fixed income indices directly rather than use an interest rate model. The inputs to the generator consists of the yield curve, the correlation matrix, and the volatilities.

Table 4: The US swap rates at various tenors as of June 11, 2014.

Tenor	Swap Rate
1 year	0.28%
2 year	0.58%
3 year	1.01%
4 year	1.42%
5 year	1.76%
7 year	2.27%
10 year	2.73%
30 year	3.42%

Let Δ be the time step and m be the number of time steps. For example,

[1]The latest version of the economic scenario generator can be obtained from `https://www.soa.org/tables-calcs-tools/research-scenario/`.

$\Delta = \frac{1}{12}$ and $m = 360$ if we use a monthly time step and a horizon of 30 years. The yield curve can be bootstrapped from swap rates (Hagan and West, 2006; Gan, 2017). For example, Table 4 gives 8 swap rates of different tenors from the US market. We can bootstrap the 8 swap rates to get 8 discount factors at the maturity dates of corresponding swaps. Then we can interpolate the discount factors to get the discount factors at all months. Figure 1 shows the monthly forward rates interpolated by the loglinear method (Hagan and West, 2006).

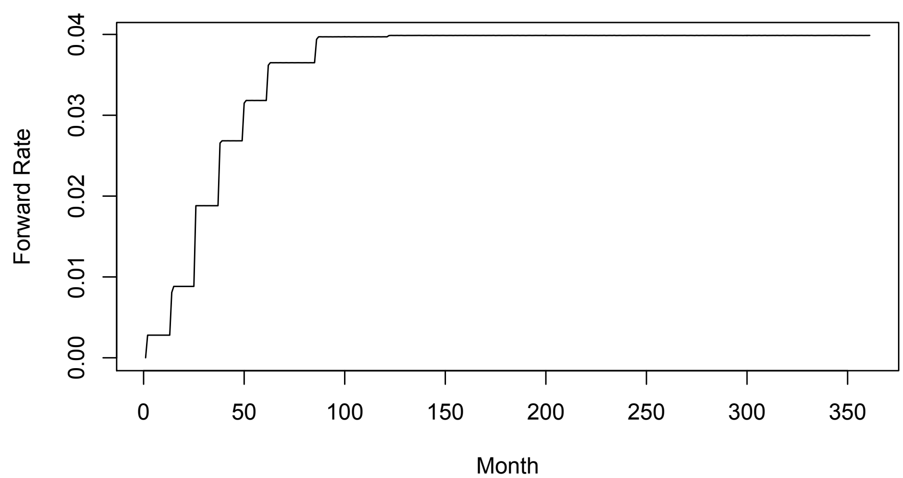

Figure 1: The monthly forward rates bootstrapped from the swap rates given in Table 4.

Now let us introduce a multivariate Black-Scholes model. Suppose that there are k indices $S^{(1)}$, $S^{(2)}$, ..., $S^{(k)}$ in the financial market and their risk-neutral dynamics are given by (Carmona and Durrelman, 2006):

$$\frac{\mathrm{d}\, S_t^{(h)}}{S_t^{(h)}} = r_t \,\mathrm{d}\, t + \sum_{l=1}^{k} \sigma_{hl} \,\mathrm{d}\, B_t^{(l)}, \quad S_0^{(h)} = 1,\ h = 1, 2, \ldots, k \tag{2}$$

where $B_t^{(1)}$, $B_t^{(2)}$, ..., $B_t^{(k)}$ are independent standard Brownian motions, r_t is the short rate of interest, and the matrix (σ_{hl}) is used to capture the correlation among the indices. The stochastic differential equations given in Equation (2) have the following solutions (Carmona and Durrelman, 2006):

$$S_t^{(h)} = \exp\left[\left(\int_0^t r_s \,\mathrm{d}\, s - \frac{t}{2}\sum_{l=1}^{k} \sigma_{hl}^2\right) + \sum_{l=1}^{k} \sigma_{hl} B_t^{(l)}\right], \quad h = 1, 2, \ldots, k. \tag{3}$$

Let $t_0 = 0, t_1 = \Delta, \ldots, t_m = m\Delta$ be time steps with equal space Δ. For $j = 1, 2, \ldots, m$, let $A_j^{(h)}$ be the accumulation factor of the hth index for the period (t_{j-1}, t_j), that is,

$$A_j^{(h)} = \frac{S_{j\Delta}^{(h)}}{S_{(j-1)\Delta}^{(h)}}. \tag{4}$$

Suppose that the continuous forward rate is constant within each period. Then we have

$$\exp\left(\Delta(f_1 + f_2 + \cdots + f_j)\right) = \exp\left(\int_0^{t_j} r_s \,\mathrm{d}\, s\right), \quad j = 1, 2, \ldots, m,$$

where f_j is the annualized continuous forward rate for period (t_{j-1}, t_j). The above equation leads to

$$f_j = \frac{1}{\Delta}\int_{t_{j-1}}^{t_j} r_s \,\mathrm{d}\, s, \quad j = 1, 2, \ldots, m.$$

Combining Equations (3) and (4), we get

$$A_j^{(h)} = \exp\left[\left(f_j - \frac{1}{2}\sum_{l=1}^{k}\sigma_{hl}^2\right)\Delta + \sum_{l=1}^{k}\sigma_{hl}\sqrt{\Delta}Z_j^{(l)}\right], \tag{5}$$

where

$$Z_j^{(l)} = \frac{B_{j\Delta}^{(l)} - B_{(j-1)\Delta}^{(l)}}{\sqrt{\Delta}}.$$

By the property of Brownian motion, we know that $Z_1^{(l)}$, $Z_2^{(l)}$, $\ldots$, $Z_m^{(l)}$ are independent random variables with a standard normal distribution.

From Equation (5), we can calculate the continuous return for the period (t_{j-1}, t_j) as

$$R_j^{(h)} = \ln A_j^{(h)} = \left(f_j - \frac{1}{2}\sum_{l=1}^{k}\sigma_{hl}^2\right)\Delta + \sum_{l=1}^{k}\sigma_{hl}\sqrt{\Delta}Z_j^{(l)}. \tag{6}$$

The mean and covariance matrix of the returns are given by

$$E\left[R_j^{(h)}\right] = \left(f_j - \frac{1}{2}\sum_{l=1}^{k}\sigma_{hl}^2\right)\Delta \tag{7}$$

and

$$\begin{aligned}
\mathrm{Cov}\left(R_j^{(h)}, R_j^{(s)}\right) &= E\left[\left(R_j^{(h)} - E[R_j^{(h)}]\right)\left(R_j^{(s)} - E[R_j^{(s)}]\right)\right] \\
&= E\left[\left(\sum_{l=1}^{k}\sigma_{hl}\sqrt{\Delta}Z_j^{(l)}\right)\left(\sum_{l=1}^{k}\sigma_{sl}\sqrt{\Delta}Z_j^{(l)}\right)\right] \\
&= \sum_{l=1}^{k}\sigma_{hl}\sigma_{sl}\Delta, \quad h, s = 1, 2, \ldots, k. \qquad (8)
\end{aligned}$$

Let Σ be the covariance matrix of the annualized continuous returns of the k indices and let

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1k} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{k1} & \sigma_{k2} & \cdots & \sigma_{kk} \end{pmatrix}.$$

Then we have

$$\sigma \cdot \sigma' = \Sigma, \tag{9}$$

where σ' is the transpose of σ. From Equation (9), we see that σ is the Cholesky decomposition of the covariance matrix Σ.

The simple scenario generator described above requires two inputs: the forward curve and the covariance matrix. In this generator, the bond index and the equity index are simulated in the same way by considering their covariance structure.

Once we have index scenarios simulated from Equation (5), we can obtain the fund scenarios by blending these index scenarios. Let n be the number of risk-neutral paths. For $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and $h = 1, 2, \ldots, k$, let $A_{ij}^{(h)}$ be the accumulation factor of the hth index at time t_j along the ith path. Suppose that there are g investment funds in the pool and the fund mappings are given by

$$W = \begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1k} \\ w_{21} & w_{22} & \cdots & w_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ w_{g1} & w_{g2} & \cdots & w_{gk} \end{pmatrix}.$$

Then the simple returns of the hth investment fund can be blended as

$$F_{ij}^{(h)} - 1 = \sum_{l=1}^{k} w_{hl} \left[A_{ij}^{(l)} - 1 \right], \quad h = 1, 2, \ldots, g,$$

where $F_{ij}^{(h)}$ is the accumulation factor of the hth fund for the period (t_{j-1}, t_j) along the ith path. Since the sum of weights is equal to 1, we have

$$F_{ij}^{(h)} = \sum_{l=1}^{k} w_{hl} A_{ij}^{(l)}, \quad h = 1, 2, \ldots, g.$$

3.2 Liability Cash Flow Projection

Once we have the risk-neutral scenarios for all the investment funds $F_{ij}^{(l)}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, $l = 1, 2, \ldots, g$, we can project the cash flows of the contract according to contract specifications and the purpose of valuation. If we are interested in the value of the whole contract, we can project the cash flows of the whole contract. For example, the valuation method proposed by Bauer et al. (2008) is based on the whole contract. In this paper, we are interested in the market-consistent value (or fair market value) of the guarantees embedded in variable annuity contracts. To do so, we only project the cash flows arising from the guarantees.

Without loss of generality, we assume that there are four types of cash flows: death benefit, guaranteed benefits, and risk charges for providing such guaranteed benefits. For a general variable annuity contract, we use the following notation to denote these cash flows that occur within the period $(t_{j-1}, t_j]$ along the ith risk-neutral path:

GB_{ij} denotes the guaranteed death or living benefit.

DA_{ij} denotes payoff of the guaranteed death benefit.

LA_{ij} denotes payoff of the guaranteed living benefit.

RC_{ij} denotes the risk charge for providing the guarantees;

$PA_{ij}^{(h)}$ denotes the partial account value of the hth investment fund, for $h = 1, 2, \ldots, g$.

TA_{ij} denotes the total account value. In general, we have

$$TA_{ij} = \sum_{l=1}^{g} PA_{ij}^{(l)}.$$

We use the following notation to denote various fees:

ϕ_{ME} denotes the annualized M&E fee of the contract;

ϕ_G denotes the annualized guarantee fee for the riders selected by the policyholder;

$\phi_F^{(h)}$ denotes the annualized fund management fee of the hth investment fund. Usually this fee goes to the fund managers rather than the insurance company.

Then we can project the cash flows in a way that is similar to the way used by Bauer et al. (2008). For the sake of simplicity, we assume that events occur in the following order during the term of the contract:

- fund management fees are first deducted;
- then M&E and rider fees are deducted;
- then death benefit is paid if the policyholder dies;

- then living benefit is paid if the policyholder is alive.

We also assume that the fees are charged from the account values at the end of every month and the the policyholder takes withdrawal at anniversaries of the contracts.

Once we have all the cash flows, we can calculate the fair market values of the riders as follows:

$$
\begin{aligned}
V_0 &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m} {}_{(j-1)\Delta}p_{x_0} \cdot {}_{\Delta}q_{x_0+(j-1)\Delta} DA_{i,j} d_j \\
&+ \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m} {}_{j\Delta}p_{x_0} LA_{i,j} d_j, \qquad (10)
\end{aligned}
$$

where x_0 is the age of the policyholder, p is the survival probability, q is the probability of death, and d_j is the discount factor defined as

$$
d_j = \exp\left(-\Delta \sum_{l=1}^{j} f_l\right).
$$

The risk charge value can be calculated as

$$
RC_0 = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m} {}_{j\Delta}p_{x_0} RC_{i,j} d_j. \qquad (11)
$$

In the following subsections, we describe how the cash flows of various guarantees are projected.

3.3 GMDB Projection

For $j = 0, 1, \ldots, m-1$, the cash flows of the GMDB from t_j to t_{j+1} are projected as follows:

- The partial account values evolve as follows:

$$
PA^{(h)}_{i,j+1} = PA^{(h)}_{ij} F^{(h)}_{i,j+1}\left(1 - \Delta\phi^{(h)}_F\right)\left(1 - \Delta[\phi_{ME} + \phi_G]\right) \qquad (12)
$$

  for $h = 1, 2, \ldots, g$, where Δ is the time step. Here we assume that the fees are deducted at the end of each period and the fund management fees are deducted before the insurance fees and withdrawal.

- The risk charges are projected as

$$
RC_{i,j+1} = \sum_{h=1}^{k} PA^{(h)}_{ij} F^{(h)}_{i,j+1}\left(1 - \Delta\phi^{(h)}_F\right)\Delta\phi_G. \qquad (13)
$$

  Note that the risk charge does not include the basic insurance fees.

- If the guaranteed death benefit is evolves as follows:

$$GB_{i,j+1} = \begin{cases} GB_{i,j}, & \text{if } t_{j+1} \text{ is not an anniversary,} \\ GB_{i,j}, & \text{if } t_{j+1} \text{ is an anniversary and} \\ & \text{the benefit is return of premium,} \\ GB_{i,j}(1+r), & \text{if } t_{j+1} \text{ is an anniversary and} \\ & \text{the benefit is annul roll-up,} \\ \max\{TA_{i,j+1}, GB_{i,j}\}, & \text{if } t_{j+1} \text{ is an anniversary and} \\ & \text{the benefit is annul ratchet,} \end{cases} \tag{14}$$

  with $GB_{i,0} = TA_{i,0}$.

- If the policyholder dies within the period $(t_j, t_{j+1}]$, then the payoff of the death benefit is projected as

$$DA_{i,j+1} = \max\{0, GB_{i,j+1} - TA_{i,j+1}\}. \tag{15}$$

- The payoff of the living benefit is zero, i.e., $LA_{i,j+1} = 0$.

- After the maturity of the contract, all the state variables are set to zero.

3.4 GMAB and DBAB Projection

Different specifications for the GMAB rider exist. See Hardy (2003) and Shevchenko and Luo (2016) for examples. Here we follow the specification given in Hardy (2003) and consider GMAB riders that give policyholders to renew the policy at the maturity date. As a result, a policy with the GMAB rider may have multiple maturity dates.

At the maturity dates, if the guaranteed benefit is higher than the fund value, then the insurance company has to pay out the difference and the policy is renewed by reseting the fund value to the guaranteed benefit. If the guaranteed benefit is lower than the fund values, then the policy is renewed by reseting the guaranteed benefit to the fund value. Let $T_1 = T$ be the first renewal date. Let T_2, T_3, …, T_J be the subsequent renewal dates. Under such a GMAB rider, the guaranteed benefit evolves as follows:

$$GB_{i,j+1} = \begin{cases} \max\{GB_{i,j}, TA_{i,j+1}\} & \text{if } t_{j+1} \in \mathcal{T}, \\ GB^*_{i,j+1}, & \text{if otherwise,} \end{cases} \tag{16}$$

where $GB^*_{i,j+1}$ is defined in Equation (24) and $\mathcal{T} = \{T_1, T_2, \ldots, T_J\}$ is the set of renewal dates. We assume that the policyholder renews the policy only when the account value at a maturity date is higher than the guaranteed benefit.

The payoff of the living benefit is calculated as follows:

$$LA_{i,j+1} = \begin{cases} 0, & \text{if } t_{j+1} \notin \mathcal{T}, \\ \max\{0, GB_{i,j+1} - TA_{i,j+1}\}, & \text{if } t_{j+1} \in \mathcal{T}. \end{cases} \tag{17}$$

The payoff of the death benefit is zero if the policy contains only the GMAB rider. For the DBAB policy, the death benefit is calculated according to Equation (15).

If the payoff is larger than zero, then the fund value is reseted to the guaranteed benefit. In other words, the payoff is deposited to the investment funds. We assume that the payoff is deposited to the investment funds proportionally. Specifically, the partial account values are reseted as follows:

$$PA_{i,j+1}^{(h)} = PA_{ij}^{(h)} F_{i,j+1}^{(h)} \left(1 - \Delta\phi_F^{(h)}\right)(1 - \Delta[\phi_{ME} + \phi_G]) + LA_{i,j+1}^{(h)} \tag{18}$$

for $h = 1, 2, \ldots, g$, where $LA_{i,j}^{(h)}$ is the amount calculated as,

$$LA_{i,j+1}^{(h)} = LA_{i,j+1} \frac{PA_{ij}^{(h)} F_{i,j+1}^{(h)} \left(1 - \Delta\phi_F^{(h)}\right)}{\sum_{l=1}^{p} PA_{ij}^{(l)} F_{i,j+1}^{(l)} \left(1 - \Delta\phi_F^{(l)}\right)}.$$

3.5 GMIB and DBIB Projection

A variable annuity policy with a GMIB rider gives the policyholder three options at the maturity date (Bauer et al., 2008; Marshall et al., 2010):

- get back the accumulated account values,
- annuitize the accumulated account values at the market annuitization rate, or
- annuitize the guaranteed benefit at a payment rate r_g per annum.

As a result, the payoff of the GMIB rider is given by

$$LA_{i,j+1} = \begin{cases} 0, & \text{if } t_{j+1} < T, \\ \max\left\{0, GB_{i,j+1}\dfrac{\ddot{a}_T}{\ddot{a}_g} - TA_{i,j+1}\right\}, & \text{if } t_{j+1} = T, \end{cases} \tag{19}$$

where $\ddot{a}_T$ and $\ddot{a}_g$ are the market price and the guaranteed price of an annuity with payments of \$1 per annum beginning at time T, respectively. In this paper, we determine $\ddot{a}_T$ by using the current yield curve. We specify $\ddot{a}_g$ by using a particular interest rate, i.e.,

$$\ddot{a}_g = \sum_{n=0}^{\infty} {}_n p_x e^{-nr},$$

where r is an interest rate set to 5%, which is about 1% higher than the 30 year forward rate shown in Figure 1.

The guaranteed benefits and risk charges are projected according to Equations (14) and (13), respectively. The payoff of the death benefit is zero if the policy contains only the GMIB. For the DBIB policy, the death benefit is projected according to Equation (15).

3.6 GMMB and DBMB Projection

For the GMMB and DBMB guarantees, account values, risk charges, and guaranteed benefits are projected according to the GMDB case specified in Equation (12), Equation (13), and Equation (14), respectively. The payoff of the living benefit is projected as

$$LA_{i,j+1} = \begin{cases} 0, & \text{if } t_{j+1} < T, \\ \max\{0, GB_{i,j+1} - TA_{i,j+1}\}, & \text{if } t_{j+1} = T. \end{cases} \tag{20}$$

For the GMMB guarantee, the payoff of the guaranteed death benefit is zero. For the DBMB guarantee, the payoff of the guaranteed death benefit is projected according to Equation (15).

3.7 GMWB and DBWB Projection

To describe the cash flow project for the GMWB, we need the following additional notation:

WA^G_{ij} denotes the guaranteed withdrawal amount per year. In general, WA^G_{ij} is a specified percentage of the guaranteed withdrawal base.

WB^G_{ij} denotes the guaranteed withdrawal balance, which is the remaining amount that the policyholder can withdrawal.

WA_{ij} denotes the actual withdrawal amount per year.

For $j = 0, 1, \ldots, m-1$, the cash flows of the GMWB from t_j to t_{j+1} are projected as follows:

- Suppose that the policyholder takes maximum withdrawals allowed by a GMWB rider at anniversaries. Then we have

$$WA_{i,j+1} = \begin{cases} \min\{WA^G_{i,j}, WB^G_{i,j}\}, & \text{if } t_{j+1} \text{ is an anniversary}, \\ 0, & \text{if otherwise}. \end{cases} \tag{21}$$

- The partial account values evolve as follows:

$$PA^{(h)}_{i,j+1} = PA^{(h)}_{ij} F^{(h)}_{i,j+1} \left(1 - \Delta\phi^{(h)}_F\right) (1 - \Delta[\phi_{ME} + \phi_G]) - WA^{(h)}_{i,j+1} \tag{22}$$

  for $h = 1, 2, \ldots, g$, where Δ is the time step and $WA^{(h)}_{i,j}$ is the amount withdrawn from the hth investment fund, i.e.,

$$WA^{(h)}_{i,j+1} = WA_{i,j+1} \frac{PA^{(h)}_{ij} F^{(h)}_{i,j+1} \left(1 - \Delta\phi^{(h)}_F\right)}{\sum_{l=1}^{p} PA^{(l)}_{ij} F^{(l)}_{i,j+1} \left(1 - \Delta\phi^{(l)}_F\right)}.$$

  If the account values from the investment funds cannot cover the withdrawal, the account values are set to zero.

- The risk charges are projected according to Equation (13).
- If the guaranteed benefit is evolves as follows:

$$GB_{i,j+1} = GB^*_{i,j+1} - WA_{i,j+1}, \tag{23}$$

where

$$GB^*_{i,j+1} = \begin{cases} GB_{i,j}, & \text{if } t_{j+1} \text{ is not an anniversary,} \\ GB_{i,j}, & \text{if } t_{j+1} \text{ is an anniversary and} \\ & \text{the benefit is return of premium,} \\ GB_{i,j}(1+r), & \text{if } t_{j+1} \text{ is an anniversary and} \\ & \text{the benefit is annul roll-up,} \\ \max\{TA_{i,j+1}, GB_{i,j}\}, & \text{if } t_{j+1} \text{ is an anniversary and} \\ & \text{the benefit is annul ratchet,} \end{cases} \tag{24}$$

with $GB_{i,0} = TA_{i,0}$. The guaranteed benefit is reduced by the amount withdrawn.

- The guaranteed withdrawal balance and the guaranteed withdrawal amount evolve as follows:

$$WB^G_{i,j+1} = WB^G_{i,j} - WA_{i,j+1}, \quad WA^G_{i,j+1} = WA^G_{i,j} \tag{25}$$

with $WB^G_{i,0} = TA_{i,0}$ and $WA^G_{i,0} = x_W TA_{i,0}$. Here x_W is the withdrawal rate. The guaranteed base is adjusted for the withdrawals.

- The payoff of the guaranteed withdrawal benefit is projected as

$$LA_{i,j+1} = \begin{cases} \max\{0, WA_{i,j+1} - TA_{i,j+1}\}, & \text{if } t_{j+1} < T, \\ \max\{0, WB^G_{i,j+1} - TA_{i,j+1}\}, & \text{if } t_{j+1} = T. \end{cases} \tag{26}$$

It is the amount that the insurance company has to pay by its own money to cover the withdrawal guarantee. At maturity, the remaining withdrawal balance is returned to the policyholder.

- The payoff of the guaranteed death benefit for the GMWB is zero, i.e., $DA_{i,j+1} = 0$. For the DBWB, the payoff is projected according to Equation (15).
- After the maturity of the contract, all the state variables are set to zero.

3.8 Fair Market Value and Greek Calculation

We use the bump approach (Cathcart et al., 2015) to calculate the Greeks. Specifically, we calculate the partial dollar deltas of the guarantees as follows:

$$
\begin{aligned}
& Delta^{(l)} \\
= \; & \frac{V_0\left(PA_0^{(1)}, \ldots, PA_0^{(l-1)}, (1+s)PA_0^{(l)}, PA_0^{(l+1)}, \ldots, PA_0^{(k)}\right)}{2s} - \\
& \frac{V_0\left(PA_0^{(1)}, \ldots, PA_0^{(l-1)}, (1-s)PA_0^{(l)}, PA_0^{(l+1)}, \ldots, PA_0^{(k)}\right)}{2s}, \qquad (27)
\end{aligned}
$$

for $l = 1, 2, \ldots, k$, where s is the shock amount applied to the partial account value and $V_0(\cdots)$ is the fair market value written as a function of partial account values. Usually, we use $s = 0.01$ to calculate the dollar deltas. The partial dollar delta measures the sensitivity of the guarantee value to an index and can be used to determine the hedge position with respect to the index.

We calculate the partial dollar rhos in a similar way. In particular, we calculate the lth partial dollar rho as follows:

$$Rho^{(l)} = \frac{V_0(r_l + s) - V_0(r_l - s)}{2s}, \qquad (28)$$

where $V_0(r_l + s)$ is the fair market value calculated based on the yield curve bootstrapped with the lth input rate r_l being shocked up s bps (basis points) and $V_0(r_l - s)$ is defined similarly. A common choice for s is 10 bps.

4 Benchmark Datasets

In this section, we present the synthetic portfolio and the corresponding fair market values and greeks calculated by the Monte Carlo simulation method described in the previous section. The datasets can be downloaded from `http://www.math.uconn.edu/~gan/software.html`.

4.1 Synthetic Portfolio

We generated 10,000 synthetic variable annuity policies for each of the guarantee types given in Table 1. The synthetic portfolio contains 190,000 policies. The fields of the synthetic variable annuity policies are described in Table 5. There are 45 fields in total, including 10 fund values, 10 fund numbers, and 10 fund fees.

The synthetic portfolio contains about 40% policies with female policyholders. The distribution of gender by product type is shown in Table 6. Table 7 shows the summary statistics of the age, the time to maturity, and the dollar fields. The age is the years between the birth date and the current date. The

Table 5: Description of the policy fields.

Field	Description
`recordID`	Unique identifier of the policy
`survivorShip`	Positive weighting number
`gender`	Gender of the policyholder
`productType`	Product type
`issueDate`	Issue date
`matDate`	Maturity date
`birthDate`	Birth date of the policyholder
`currentDate`	Current date
`baseFee`	M&E fee
`riderFee`	Rider fee
`rollUpRate`	Rull-up rate
`gbAmt`	Guaranteed benefit
`gmwbBalance`	GMWB balance
`wbWithdrawalRate`	Guaranteed withdrawal rate
`withdrawal`	Withdrawal so far
`FundValue`i	Fund value of the ith investment fund
`FundNum`i	Fund number of the ith investment fund
`FundFee`i	Fund management fee of the ith investment fund

time to maturity is calculated from the current date and the maturity date. The fund fees and the M&E fee are given in Table 3. The rider fees of different guarantee types are presented in Table 1.

4.2 Fair Market Values and Greeks

We used the Monte Carlo simulation engine described in Section 3 to calculate the fair market values, partial dollar deltas, and partial dollar rhos of the guarantees for the synthetic portfolio. Table 8 shows the total fair market value and the total greeks for the synthetic portfolio. From the table, we see that the total fair market value is positive, indicating that the guarantee benefit payoff is more than the risk charge. The total partial dollar deltas are negative because the guarantees are like put options, which have negative deltas. The signs of the total partial dollar rhos for different swap rates are different. Since the variable annuity contracts are usually long-term contracts, the guarantees are more sensitive to long-term interest rates than to short-term interest rates.

The partial greeks shown in Table 8 are calculated by the bump approach as mentioned in Section 3. The fair market values under various shocks are given in Table 9. The partial account values correspond to the amount of

Table 6: Distribution of gender by product type.

Gender	ABRP	ABRU	ABSU	DBAB	DBIB	DBMB	DBRP
F	4068	3974	4054	3974	3948	4013	4002
M	5932	6026	5946	6026	6052	5987	5998
Gender	**DBRU**	**DBSU**	**DBWB**	**IBRP**	**IBRU**	**IBSU**	**MBRP**
F	3952	4038	4022	4007	4027	4007	3909
M	6048	5962	5978	5993	5973	5993	6091
Gender	**MBRU**	**MBSU**	**WBRP**	**WBRU**	**WBSU**		
F	3992	3980	3970	4076	3994		
M	6008	6020	6030	5924	6006		

Table 7: Summary statistics of some fields.

	Min	1st Q	Mean	3rd Q	Max
`gbAmt`	50001.72	179180.9	312370.41	425946.3	989204.5
`gmwbBalance`	0	0	36140.74	0	499708.7
`withdrawal`	0	0	21927.8	0	499585.7
`FundValue1`	0	0	28183.12	41825.59	940769
`FundValue2`	0	0	27745.3	41002.11	904760.1
`FundValue3`	0	0	18765.45	26141.91	825405.7
`FundValue4`	0	0	15864.94	22333.24	939322
`FundValue5`	0	0	22813.6	33814.69	988808
`FundValue6`	0	0	28167.01	41802.33	872706.6
`FundValue7`	0	0	22952.38	33361.79	795151.2
`FundValue8`	0	0	21483.85	31447.77	877957.3
`FundValue9`	0	0	21090.49	30195.66	846460.2
`FundValue10`	0	0	22593.93	33276.01	868970.4
`age`	34.52	42.03	49.49	56.96	64.46
`ttm`	0.59	10.34	14.54	18.76	28.52

Table 8: The total fair market value, the total partial dollar deltas, and the total partial rhos of the synthetic portfolio. Numbers in parenthesis are negative numbers.

Quantity Name	Value	Quantity Name	Value
FMV	18,572,095,089	Rho2y	167,704
Delta1	(4,230,781,199)	Rho3y	85,967
Delta2	(2,602,768,996)	Rho4y	2,856
Delta3	(2,854,233,170)	Rho5y	(96,438)
Delta4	(2,203,726,514)	Rho7y	(546,045)
Delta5	(2,341,793,581)	Rho10y	(1,407,669)
Rho1y	40,479	Rho30y	(62,136,376)

money invested in the five indices that are calculated from the fund mapping.

Table 10 shows some summary statistics of the fair market values and greeks at the individual contract level. From Table 10, we see that some contracts have negative fair market values. For these contracts, the guarantee benefit payoff is less than the risk charge. From the table, we also see that some contracts have positive deltas. The contracts that have positive deltas are contracts with the annual ratchet guarantee benefit. For such contracts, the value of the guarantee may increase when the market goes up because the guarantee benefit is reset to the maximum of the current guarantee benefit and the account value if the later is higher.

Figure 2 and Figure 3 show the histograms of the fair market values and the partial greeks of individual policies. From Figure 2, we see that the distribution of the fair market values is skewed to the right and has a fat tail. The distributions of the partial deltas are skewed to the left. Figure 3 shows that the distributions of short-term rhos are more symmetric than those of long-term rhos. In particular, the distribution of the 30-year rho is skewed to the left and has a long tail in the left. All the histograms in Figure 2 and Figure 3 show that the distributions have extreme values.

Since the Monte Carlo simulation method is time-consuming, we used the HPC (High Performance Computing) cluster at the University of Connecticut with 80 CPUs together to calculate the fair market values and the greeks of the synthetic portfolio. It took these 80 CPUs about 2 hours to finish the calculations. If we add the runtime of all these CPUs, the total runtime was 389925.98 seconds or 108.31 hours.

5 Concluding Remarks

In this paper, we created a large synthetic portfolio of variable annuity contracts and a Monte Carlo simulation engine to facilitate the development and

Table 9: Fair market values and partial account values for different combinations of interest rate shocks and equity shocks. The numbers are in millions and "base" means no shocks are applied.

irShock	eqShock	FMV	AV1	AV2	AV3	AV4	AV5
base	base	18,572	12,825	8,886	8,759	5,454	4,796
1y_D	base	18,572	12,825	8,886	8,759	5,454	4,796
1y_U	base	18,572	12,825	8,886	8,759	5,454	4,796
2y_D	base	18,570	12,825	8,886	8,759	5,454	4,796
2y_U	base	18,574	12,825	8,886	8,759	5,454	4,796
3y_D	base	18,571	12,825	8,886	8,759	5,454	4,796
3y_U	base	18,573	12,825	8,886	8,759	5,454	4,796
4y_D	base	18,572	12,825	8,886	8,759	5,454	4,796
4y_U	base	18,572	12,825	8,886	8,759	5,454	4,796
5y_D	base	18,573	12,825	8,886	8,759	5,454	4,796
5y_U	base	18,571	12,825	8,886	8,759	5,454	4,796
7y_D	base	18,578	12,825	8,886	8,759	5,454	4,796
7y_U	base	18,567	12,825	8,886	8,759	5,454	4,796
10y_D	base	18,587	12,825	8,886	8,759	5,454	4,796
10y_U	base	18,559	12,825	8,886	8,759	5,454	4,796
30y_D	base	19,201	12,825	8,886	8,759	5,454	4,796
30y_U	base	17,959	12,825	8,886	8,759	5,454	4,796
base	1_D	18,615	12,735	8,872	8,747	5,442	4,794
base	1_U	18,530	12,915	8,899	8,770	5,465	4,798
base	2_D	18,598	12,812	8,823	8,749	5,452	4,794
base	2_U	18,546	12,839	8,948	8,768	5,455	4,798
base	3_D	18,601	12,814	8,876	8,695	5,452	4,794
base	3_U	18,544	12,837	8,895	8,822	5,455	4,798
base	4_D	18,594	12,814	8,884	8,757	5,415	4,794
base	4_U	18,550	12,836	8,887	8,760	5,492	4,798
base	5_D	18,596	12,824	8,884	8,757	5,452	4,754
base	5_U	18,549	12,827	8,887	8,760	5,455	4,837

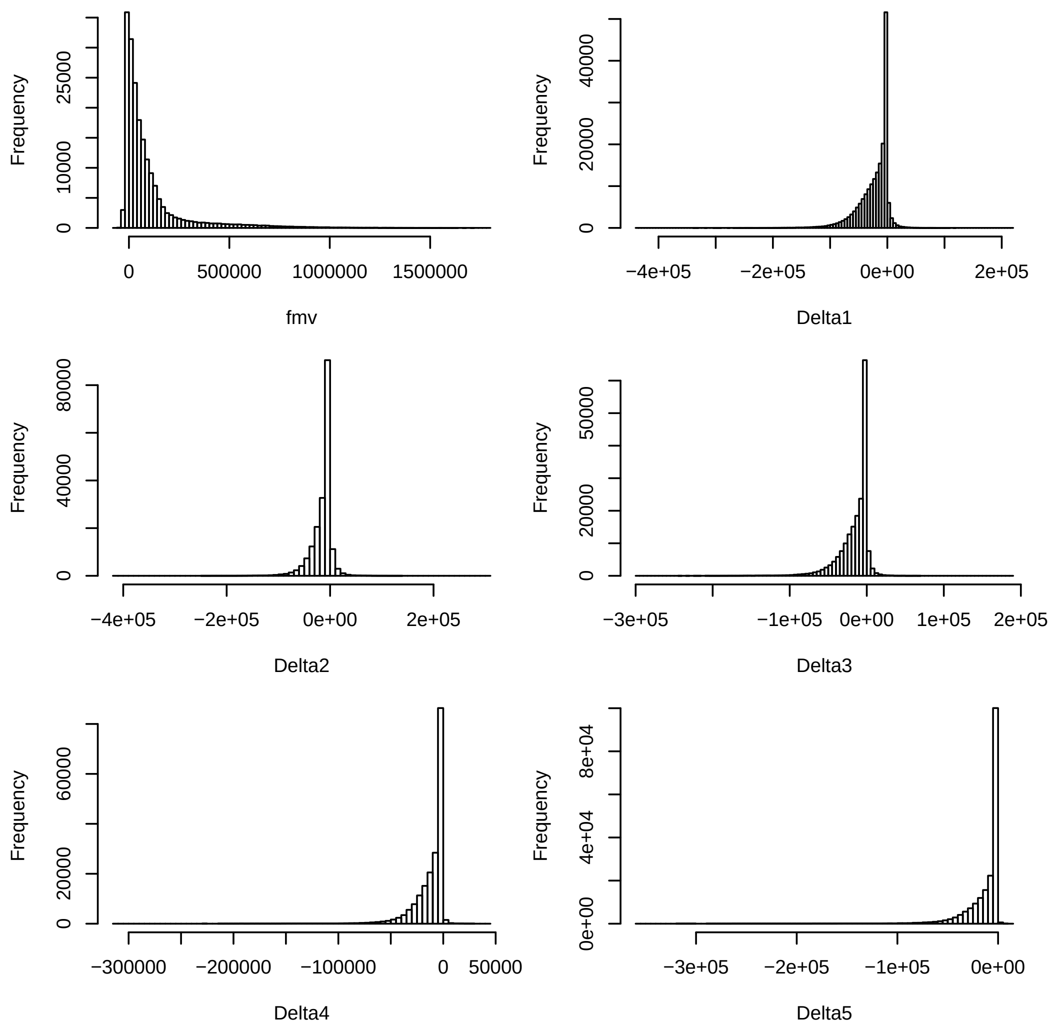

Figure 2: Histogram of the fair market values and the deltas of individual policies.

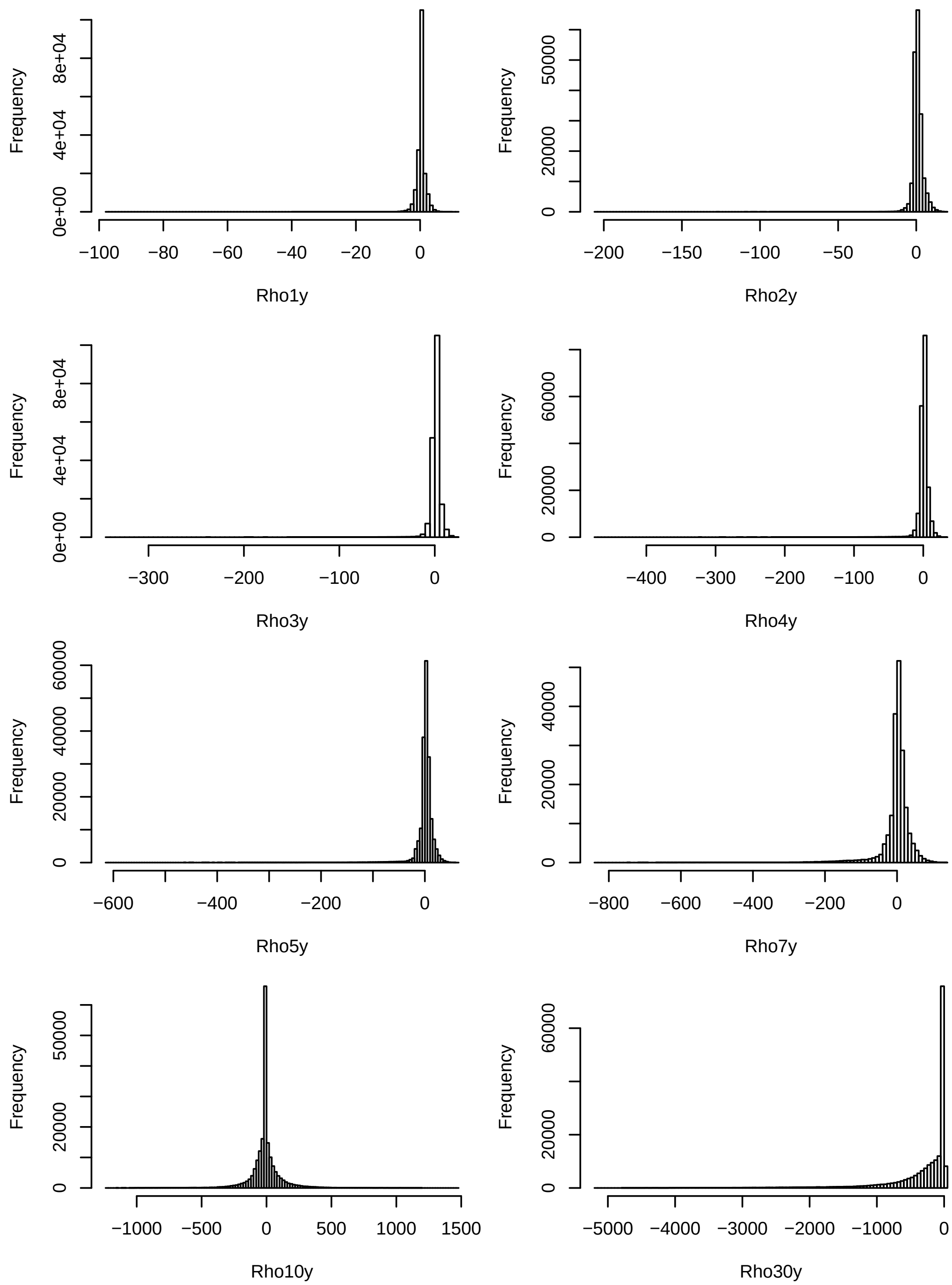

Figure 3: Histogram of the rhos of individual policies.

Table 10: Summary statistics of the fair market values and greeks of individual policies. Numbers in parenthesis are negative numbers.

	Min	1st Q	Mean	3rd Q	Max
FMV	(69,938)	4,542	97,748	108,141	1,784,549
Delta1	(435,070)	(34,584)	(22,267)	(2,219)	216,125
Delta2	(412,496)	(21,234)	(13,699)	(493)	300,019
Delta3	(296,035)	(22,704)	(15,022)	(866)	187,414
Delta4	(312,777)	(16,569)	(11,599)	(656)	40,924
Delta5	(355,741)	(16,517)	(12,325)	0	10,391
Rho1y	(98)	(0)	0	1	11
Rho2y	(205)	0	1	2	20
Rho3y	(343)	(0)	0	3	23
Rho4y	(473)	(0)	0	3	32
Rho5y	(610)	(0)	(1)	7	63
Rho7y	(834)	(5)	(3)	14	137
Rho10y	(1,236)	(41)	(7)	16	1,472
Rho30y	(5,159)	(388)	(327)	0	0

dissemination of research related to the efficient valuation of large variable annuity portfolios. The synthetic portfolio is created to mimic real portfolios in several major aspects. The Monte Carlo simulation method uses a multivariate Balck-Scholes model to simulate asset returns.

The synthetic portfolio, the fair market values, and the greeks can be used to test the performance of metamodeling techniques in terms of speed and accuracy. The full datasets or subsets can be used to test different models. Interested researchers and practitioners can also download the source code of the software from `http://www.math.uconn.edu/~gan/software.html` and extend it to consider other guarantee types or other Monte Carlo simulation methods.

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A Software Implementation

We implemented the yield curve construction program, the synthetic portfolio generator, the risk-neutral scenario generator, and the Monte Carlo simulation engine in Java. The software package can be downloaded from `http://www.math.uconn.edu/~gan/software.html`.

The yield curve construction program is used to bootstrap a yield curve from swap rates. The resulting yield curve is used by the risk-neural scenario generator to generate risk-neutral scenarios. The synthetic portfolio generator is used to create synthetic portfolios of variable annuity contracts. The Monte Carlo simulation engine is used to calculate the fair market values of the synthetic variable annuity contracts. Since Monte Carlo simulation is extremely time-consuming, the simulation engine is implemented in such a way that it can run on multiple machines with multiple threads. The parameters for the four major programs are saved in XML files. The XML parameter files used to create the datasets in this paper can also be found in the software package mentioned above.

In additional to the four major programs, we also developed two utility programs for consolidating the fair market values from different output files and calculating the greeks.