Pricing and Hedging in Incomplete markets with Model Ambiguity

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Outline

1. Introduction
2. Literature Overview
3. Pricing and Hedging with Model Ambiguity
4. Examples
5. Conclusion
Figure: Overview of Pricing in Incomplete Markets
Complete Market (1)

Explain concepts on one-step binomial “fork”:

\[
\begin{array}{c}
\uparrow \quad \downarrow \\
\quad \quad \\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad {\text{t} = 0} \quad \text{buy replicating portfolio } \Delta S + B \text{ to replicate uncertain payoff } (f_u, f_d) \text{ at } t = 1.
\end{array}
\]
Complete Market (2)

Solve system of equations to match payoff:

\[
\begin{align*}
  u\Delta S + RB &= f_u \\
  d\Delta S + RB &= f_d
\end{align*}
\]

Solution given by:

\[
\Delta S = \frac{f_u - f_d}{u - d}, \quad B = \frac{uf_d - df_u}{R(u - d)}
\]

Market price of replio at \( t = 0 \):

\[
\Delta S + B = \frac{(R - d)f_u + (u - R)f_d}{R(u - d)} = \frac{1}{R} (qf_u + (1 - q)f_d)
\]

\( q \) well-defined iff \( u > R > d \) (FTAP: [Delbaen and Schachermayer, 1994])

Hence: “risk-neutral” pricing is convenient representation of the market price of the replio
Binomial Incomplete Market

Note: in real world we never have complete market!

Model for incomplete market:
Combine two binomial forks: financial \((u, d)\) and insurance \((g, b)\)

\[
\begin{align*}
&\uparrow f_{ug} \text{ with prob } p\pi \\
&S \quad \downarrow f_{ub} \text{ with prob } p(1 - \pi) \\
&\uparrow f_{dg} \text{ with prob } (1 - p)\pi \\
&\downarrow f_{db} \text{ with prob } (1 - p)(1 - \pi)
\end{align*}
\]

For simplicity we use independent prob’s
Can only trade \((S, B)\). Insurance event \((g, b)\) makes market incomplete.
How do we price the uncertain payoff \(f\)?
Market-Consistent & Time-Consistent (1)

For pricing insurance and pension contracts we want to consider realistic pricing operators $v[t, X]$ that are:

- market consistent
- time consistent

**Market-consistent (MC):** $v[t, X + H] = v[t, X] + \mathbb{E}_Q[H | \mathcal{F}_t]$ for all hedgeable claims $H$

This is an extension of “cash invariance” to all $H$

**Time-consistent (TC):** if $X_1 \geq X_2$ at time $T$, then $v[t, X_1] \geq v[t, X_2]$ for all $t < T$

This is an extension of no-arbitrage to incomplete markets.

Note: the conditional expectation operator $\mathbb{E}_Q[H | \mathcal{F}_t]$ is TC (and MC)
Result by [Pelsser and Stadje, 2014]: every MC and TC pricing operator can be represented as a “two-step” operator:

\[
\frac{1}{R} (q f_u + (1 - q) f_d)
\]

\[
f_u := \nu[f | uS]
\]

\[
f_d := \nu[f | dS]
\]

“financial” valuation  “actuarial” valuation

Clean separation of financial and actuarial pricing in each “half-step”.

Tech. cond: financial info arrives more frequently than insurance info
Binomial Insurance Risk

We will now concentrate on the “actuarial” pricing step, in a “pure” incomplete market.

One-step binomial “fork” for insurance process $y$:

$$y + a\Delta t + b\sqrt{\Delta t} \quad \text{with prob } \frac{1}{2}$$

$$y + a\Delta t - b\sqrt{\Delta t} \quad \text{with prob } \frac{1}{2}$$

The term $\pm \sqrt{\Delta t}$ is approximation of Brownian Motion increment $\Delta W$.

In the limit $\Delta t \to 0$ this converges to the sde $dy = a(t, y)dt + b(t, y)dW$. 
Binomial Insurance Risk – Girsanov

Consider the change of probability measure:

\[ y + a\Delta t + b\sqrt{\Delta t} \] with prob \( \frac{1}{2}(1 + \lambda \sqrt{\Delta t}) \)

\[ y \rightarrow y + a\Delta t - b\sqrt{\Delta t} \] with prob \( \frac{1}{2}(1 - \lambda \sqrt{\Delta t}) \)

Factor \((1 + \lambda \Delta W)\) is the Radon-Nikodym derivative

Tree above is equivalent to process \( y \) with adjusted drift:

\[ y + (a + \lambda b)\Delta t + b\sqrt{\Delta t} \] with prob \( \frac{1}{2} \)

\[ y \rightarrow y + (a + \lambda b)\Delta t - b\sqrt{\Delta t} \] with prob \( \frac{1}{2} \)

Hence:

\[ \mathbb{E}_t^{(\lambda)}[y_{t+\Delta t}] = y_t + (a + \lambda b)\Delta t = \mathbb{E}_t[y_{t+\Delta t}] + \lambda b\Delta t. \]
Actuarial Variance and Utility Pricing

Variance pricing operator: \( \nu_{\text{var}}(t, y) := \mathbb{E}_t[y_{t+\Delta t}] + \alpha \text{Var}_t[y_{t+\Delta t}] \)

In our binomial model: \( \nu_{\text{var}}(t, y) = \mathbb{E}_t[y_{t+\Delta t}] + \alpha b^2 \Delta t \)

But, this is equivalent to: \( \mathbb{E}_t^{(\lambda)}[y_{t+\Delta t}] \) with \( \lambda = \alpha b \)

Reference: [Kaas et al., 2008]

Utility pricing operator: \( \nu_{\text{util}}(t, y) := u^{-1}(\mathbb{E}_t[u(y_{t+\Delta t})]) \)

In our binomial model: \( \nu_{\text{util}}(t, y) \approx \mathbb{E}_t[y_{t+\Delta t}] + \frac{1}{2} \frac{u''}{u'} b^2 \Delta t \)

But, this is equivalent to: \( \mathbb{E}_t^{(\lambda)}[y_{t+\Delta t}] \) with \( \lambda = \frac{1}{2} \frac{u''}{u'} b \)

Note: \( \lambda(t, y) \) can be state dependent, e.g. with \( u(t, y) \)

References: [Musiela and Zariphopoulou, 2004], [Carmona, 2009]
Prescribed in the standard model of Solvency II
\( \nu_{\text{CoC}}(t, y) := \mathbb{E}_t[y_{t+\Delta t}] + \gamma \sqrt{\Delta t} \text{VaR}_t[y_{t+\Delta t}] \)

Note: standard model uses time-step of \( \Delta t = 1 \) year
We include factor \( \sqrt{\Delta t} \) to allow scaling to different time horizons
For small time-step \( \Delta t \to 0 \) the \( \text{VaR}[\cdot] \to k \sqrt{\text{Var}[\cdot]} \)
Standard model uses 99.5% prob, hence \( k = \Phi^{-1}(0.995) = 2.58 \)

In our binomial model: \( \nu_{\text{CoC}}(t, y) = \mathbb{E}_t[y_{t+\Delta t}] + \gamma k |b| \Delta t \)
But, this is equivalent to: \( \mathbb{E}_t^{(\lambda)}[y_{t+\Delta t}] \) with \( \lambda = \gamma k \text{sign}(b) \)
Standard model corresponds to: \( \lambda = 0.06 \times 2.58 = 0.15 \).
Coherent and Convex Risk Measures

Axiomatic foundation of risk measure / pricing operator (note ±-switch)
Started with coherent: [Artzner et al., 1999]
Extended to convex: [Föllmer and Schied, 2002]
Extension to time-consistent convex: [Cheridito et al., 2006]

Any TC convex pricing operator can be characterised as:
(In our binomial model:) \( \nu_{cvx}(t, y) = \max_\lambda \mathbb{E}_t^{(\lambda)}[y_{t+\Delta t}] - c(\lambda)\Delta t \)
convex penalty function \( c(\lambda) \geq 0 \) and \( c(0) = 0 \)
Equiv to: \( \nu_{cvx}(t, y) = \mathbb{E}_t[y_{t+\Delta t}] + (\max_\lambda \lambda b - c(\lambda))\Delta t \)
Equiv to: \( \nu_{cvx}(t, y) = \mathbb{E}_t[y_{t+\Delta t}] + \tilde{c}(b)\Delta t \)
where \( \tilde{c}(b) \) is the convex dual of \( c(\lambda) \)

Interpretation: pricing based on “worst case” measure \( \lambda \), but with a penalty for choosing “too large” values of \( \lambda \).
Coherent Risk Measures

Special case of convex risk measure, with penalty function:

\[ c(\lambda) = \begin{cases} 
0 & \text{for } |\lambda| \leq k; \\
\infty & \text{for } |\lambda| > k. 
\end{cases} \]

Convex dual is given by: \( \tilde{c}(b) = \max_{|\lambda| \leq k} \lambda b \)

LP with explicit solution: \( \tilde{c}(b) = k|b| \)

Hence: \( v_{\text{coh}}(t,y) = E_t[y_{t+\Delta t}] + k|b|\Delta t \)

Note: Cost-of-Capital is a coherent pricing operator.
Good Deal Bound

Consider the change of probability measure:

\[ y + a \Delta t + b \sqrt{\Delta t} \quad \text{with prob } \frac{1}{2} (1 + \lambda \sqrt{\Delta t}) \]

\[ y \quad \leftrightarrow \quad y + a \Delta t - b \sqrt{\Delta t} \quad \text{with prob } \frac{1}{2} (1 - \lambda \sqrt{\Delta t}) \]

We can also interpret \( \lambda \) as *market price of risk* (MPR)

Arbitrage opportunity is “ridiculous good deal” with \( \lambda \to \pm \infty \)

Good deal bound: only look at measures with MPR not too high

Hence: only consider measures with \( |\lambda| \leq k \)

Hence: \( \nu_{\text{GDB}}(t, y) = \nu_{\text{coh}}(t, y) = E_t[y_{t+\Delta t}] + k|b|\Delta t \)

References: [Cochrane and Saá-Requejo, 2000], [Björk and Slinko, 2006]
Figure: Overview of Pricing in Incomplete Markets
Ambition: “general theory” for pricing and hedging in incomplete markets

Minimal # of assumptions:
1. Agent (or company) wants to maximise surplus
2. Agent (or company) is able to trade financial risk
3. Concerned about model mis-specification
Model Ambiguity

How can we handle model mis-specification mathematically?
Consider models near “central model” \( dy = a(t, y)dt + b(t, y)dW: \)

\[
\begin{align*}
    &y + a\Delta t + b\sqrt{\Delta t} & \text{with prob } & \frac{1}{2}(1 + \lambda \sqrt{\Delta t}) \\
    &y + a\Delta t - b\sqrt{\Delta t} & \text{with prob } & \frac{1}{2}(1 - \lambda \sqrt{\Delta t})
\end{align*}
\]

Models near “central model” correspond to small values of \( \lambda \)

We consider the set of models \( |\lambda| \leq k \), where \( k \) is chosen based on estimation error or statistical test procedure.
Note: stochastic \( \lambda(t, y) \) give a very rich class of alternative models.
Intuition (2)

1. Objective of agent

$$\max_{\tilde{\theta}(t)} \mathbb{E}[A(T, \bar{x}_T, \bar{\theta}_T) - L(T, \bar{x}_T, y_T)]$$

2. Set of alternative models

$$\mathcal{K} = \{ \mu(t, z) + \epsilon(t) | \epsilon(t)' \Sigma^{-1}(t) \epsilon(t) \leq k^2 \}$$

3. Robust optimisation specification via “two-player game”:

$$\max_{\tilde{\theta}(t)} \min_{Q \in \mathcal{K}} \mathbb{E}^Q[A(T, \bar{x}_T, \bar{\theta}_T) - L(T, \bar{x}_T, y_T) | \mathcal{F}_t]$$
Model

- \( n \) tradeable assets \( \mathbf{x}(t) = \{x_i(t)|i = \{1, \ldots, n\}\} \)
- \( k \) untradeable assets \( \mathbf{y}(t) = \{y_i(t)|i = \{1, \ldots, k\}\} \)
- Bank account \( dx_0 = r x_0 dt \)

\[
\begin{align*}
    d \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} \mu_x(t, x) \\ \mu_y(t, y) \end{pmatrix} dt + \Sigma^{1/2}(t) \begin{pmatrix} dW_x(t) \\ dW_y(t) \end{pmatrix}
\end{align*}
\]
Model (2)

\[ \bar{\theta}(t) = [\theta_0(t), \theta_1(t), ..., \theta_n(t) 0, ..., 0] \]

- Hedging portfolio \( A(t, x_t) \)
- Liability \( L(t, x_t, y_t) \)

Optimisation by Hamilton-Jacobi-Bellman

\[ \max_{\theta(t)} \min_{\epsilon(t)} \mathbb{E}[A(T) - L(T, z_T)|\mathcal{F}_t] = \nu(t, A, x, y) \]

Indifference pricing:

- No contract \( \nu(t, A, x, y) = A(t)e^{r(T-t)} \)
- Extra cash needed to make agent indifferent

\[ \nu(t, A, x, y) = (A(t) + \pi(t, x, y))e^{r(T-t)} - w(t, x, y) \]

- Solves for \( \pi(t, x, y) = e^{-r(T-t)}w(t, x, y) \)

- Represent price \( \pi() \) as semi-linear PDE
Theorem (Intuitive representation of Theorem)

The indifference price $\pi$ is given by the PDE

$$\pi_t + \mathcal{L}_{\mu,\sigma} \pi - r \pi + c \sqrt{\pi'_y S \pi_y} = 0$$

where $\mathcal{L}_{\mu,\sigma} \pi$ is the Feynman-Kač part, with terminal value $\pi(T, x, y) = L(T, x, y)$ and where $c = \sqrt{k^2 - q_x' \Sigma_{xx}^{-1} q_x}$

The term $c \sqrt{\pi'_y S \pi_y}$ is the $k$-dimensional generalisation of “$k|b|$”. 
Lemma (Optimal hedging portfolio and robustness adjustments)

The agent’s optimal dynamic hedging portfolio is

\[ \theta^* = \left[ \pi_x + \Sigma_{xx}^{-1} \Sigma_{xy} \pi_y + h \Sigma_{xx}^{-1} q_x \right] \]

Mother nature’s optimal drift adjustments are

\[ \epsilon^* = \left[ \Sigma h^{-1} \pi_y - \Sigma_{xy}' \Sigma_{xx}^{-1} q_x \right] \]

where

\[ h = \sqrt{\frac{\pi_y' S \pi_y}{k^2 - q_x' \Sigma_{xx}^{-1} q_x}}. \]
Example: Pure hedgeable risk

- Let $k = 0$
- Complete market

$$\pi_t + \mathcal{L}_{\mu,\sigma} \pi - r\pi + 0 = 0$$

- Risk-neutral pricing
Example: Uncorrelated non-traded asset

- Liability function $L(T, S, N) = \max(S_T, g)N_T$
- Unit-linked contract with $N_T$ survivors
- The stochastic processes are

$$d \begin{bmatrix} S_t \\ N_t \end{bmatrix} = \begin{bmatrix} \mu S_t \\ -\alpha N_t \end{bmatrix} dt + \begin{bmatrix} \sigma S_t & 0 \\ 0 & \beta N_t \end{bmatrix} \begin{bmatrix} dW_S \\ dW_N \end{bmatrix}$$

- Price in our model:

$$\pi(t, S_t, N_t) = \left( S_t \Phi(d_1) - e^{-r(T-t)}g \Phi(d_2) + e^{-r(T-t)}g \right) \times N_t e^{\left( -\alpha + \beta \sqrt{k^2 - ((\mu-r)/\sigma)^2} \right) (T-t)}$$

- This is actuarial prudence with adjusted drift $-\alpha + \beta \sqrt{\cdots}$
Graphical Representation of Model Ambiguity
Conclusion

Based on minimal assumptions
- Agent maximising surplus
- Uncertain about model
- Able to hedge

Complete market $\Rightarrow$ risk-neutral measure
Incomplete $\Rightarrow$ pricing with actuarial prudence
Hedgeable $+$ unhedgeable risk $\Rightarrow$ MC and TC pricing and hedging
Multivariate $\Rightarrow$ unique solution


References II


An example of indifference prices under exponential preferences.
*Finance and Stochastics, 8*(2):229–239.

Time-consistent and market-consistent evaluations.