2015 IAA Colloquium in OSLO

The 2014 West African Ebola outbreak as a Branching Process [Working Paper]

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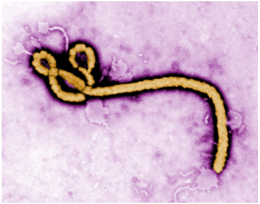
Motivation

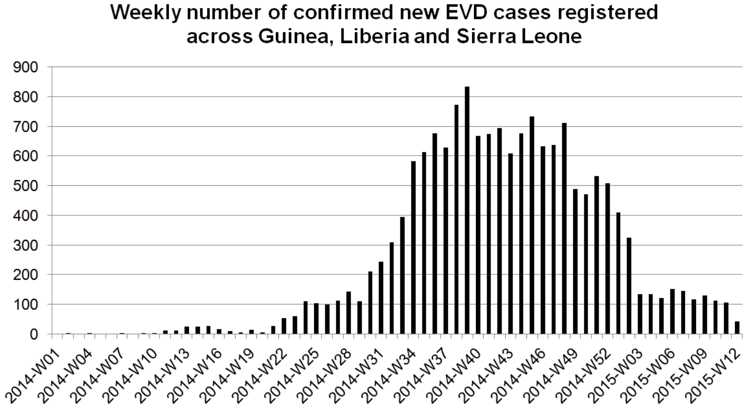
- Being able to assess the propagation risk of a virus disease in a straightforward statistical manner
- Give a mathematical sense to the so-called propagation
- Bayesian Paradigm
- Subcriticality vs. Supercriticality statistical decision in a Branching Process context
- How to cope with the problem of predicting Extinction or Explosion?

Background on Ebola Virus Disease

- EVD: very severe hemorrhagic fever illness
- Fruit bats: maybe natural Ebola virus hosts
- Transmission mechanism: from wild animals, via direct contact with bodily fluids of infected humans
- Average EVD case fatality rate: around 50% (varied from 25% to 90% in past outbreaks)
- Incubation period: 2 to 21 days.
- Symptoms:
  - Onset: fever fatigue, muscle pain, headache, sore throat
  - Then: vomiting, diarrhoea, rash, impaired kidney and liver function, internal and external bleeding (in some cases).
- As yet no proven treatment available for EVD



The current Ebola outbreak in W.A. is the largest and most persistent on record since the virus was discovered in 1976



- Peak of 833 weekly confirmed new cases in week 39 of 2014
- At the end of May, the WHO is reporting 27,049 cases and 11,149 deaths, while admitting these figures are very likely underestimated

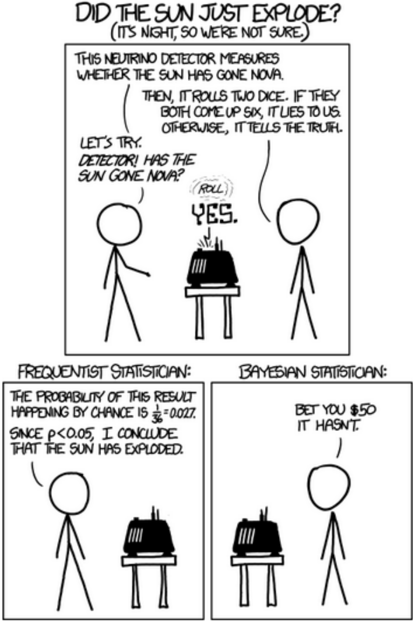
Bayes Philosophy: a (subjective) probabilistic method for updating beliefs

- Hume's Problem of Induction: will the sun rise tomorrow? Mixing experience (Past Observations) and reason (Probability Theory) principles, Laplace assumes an unitial uniform probability for the unknown probability p that the sun rises. The probability for rising tomorrow is given by his Rule of Succession

$$\Pr\left(r_{\text{tomorrow}\sim n+1} \mid r_{\text{today}\sim n}, r_{\text{yesterday}\sim n-1}, \ldots, r_0\right) = \frac{n+1}{n+2}$$

- Given that it has risen every day for the past 5000 years, Laplace gave odds of $1826251 : 1$ in favour of the sun rising tomorrow.
- ⚠The basic assumption for using the rule of succession would be that we have no prior knowledge about the question whether the sun will or will not rise tomorrow, except that it can do either: $0 < p < 1$. This is not the case for sunrises ...

Bayesian vs Frequentist Statistics



Bayesian Testing

- Suppose $\mathbf{X} \mid \theta \sim f(\mathbf{x} \mid \theta)$
- We need to test: $H_0: \theta \in \Theta_0$ vs $H_1: \theta \in \Theta_1$.
- Prior beliefs: $\theta \sim \pi_0(\theta)$ on Θ_0 and $\theta \sim \pi_1(\theta)$ on Θ_1.
- Prior probabilities $\Pr(H_0)$ and $\Pr(H_1)$ of the hypotheses.
- Marginal likelihoods under the hypotheses

$$\mathscr{L}(\mathbf{x} \mid H_i) = \int_{\Theta_i} f(\mathbf{x} \mid \theta) \pi_i(\theta) \, d\theta, \; i = 0, 1.$$

- Posterior distributions of the parameters under the hypotheses:

$$g_i(\theta \mid \mathbf{x}) = \frac{f(\mathbf{x} \mid \theta) \chi_{\Theta_i}(\theta)}{\mathscr{L}(\mathbf{x} \mid H_i)}, \; i = 0, 1.$$

- Posterior probabilities $\Pr(H_0 \mid \mathbf{x})$ and $\Pr(H_1 \mid \mathbf{x})$ of the hyp.
- Bayes Factor of H_0 to H_1

$$B_{01}(\mathbf{x}) = \frac{\dfrac{\Pr(H_0 \mid \mathbf{x})}{\Pr(H_1 \mid \mathbf{x})}}{\dfrac{\Pr(H_0)}{\Pr(H_1)}} = \frac{\mathscr{L}(\mathbf{x} \mid H_0)}{\mathscr{L}(\mathbf{x} \mid H_1)}$$

Jeffreys Scale

B_{01}	Strength of evidence
$1:1$ to $3:1$	Barely worth mentioning
$3:1$ to $10:1$	Substantial
$10:1$ to $30:1$	Strong
$30:1$ to $100:1$	Very strong
$> 100:1$	Decisive

Branching Process

- Let $\mathbf{p}$ be a probability measure on $\mathbb{N}$, defined by $\{p_k;\ k = 0, 1, 2, \ldots\}$.
- Denote $(X_n)_{n \in \mathbb{N}}$ the Markov Chain on $\mathbb{N}$ with transition matrix P

$$P(x, y) = \mathbf{p}^{*x}(y) = p_y^{*x}, \qquad \forall x, y \in \mathbb{N},$$

  where $\mathbf{p}^{*x}$ is the $x-$fold convolution $\mathbf{p}$.
- Generating Function h of $\xi \sim \mathbf{p}$

$$\mathrm{h}(s) = \mathbb{E}\left[s^{\xi}\right] = \sum_{k=0}^{\infty} p_k s^k, \quad s \geq 0,$$

- Mean number of offspring produced by a single parent particle

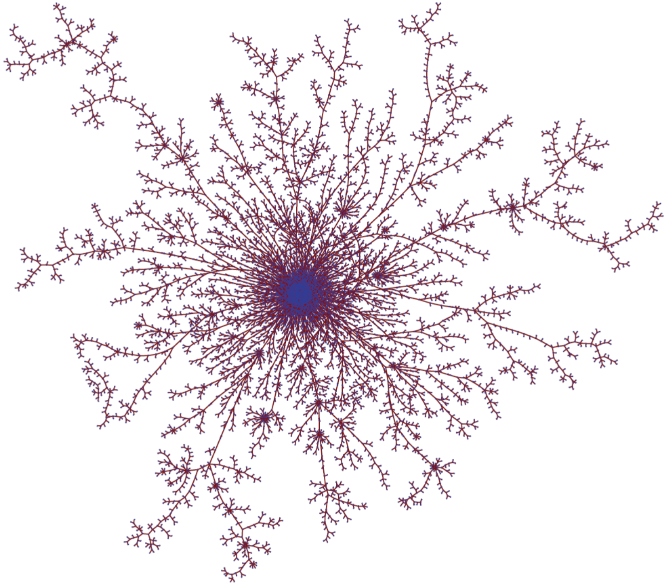
$$\mathbb{E}[\xi] = \mathrm{h}^{'}(1) \equiv m$$

  When $m < 1$, $= 1$, or > 1, we shall refer to the GW process as *subcritical*, *critical* or *supercritical* respectively.
- Extinction Set and Explosion Set provide a partition of Ω:

$$\mathbf{Extinction} = \{\omega \in \Omega,\ X_n(\omega) = 0 \text{ for some } n \geq 1\}$$
$$\mathbf{Explosion} = \{\omega \in \Omega,\ X_n(\omega) \to \infty \text{ as } n \to \infty\}$$

- The extinction probability $q := \mathbb{P}(\mathbf{Extinction})$ is the smallest nonnegative root of the equation $\mathrm{h}(s) = s$.

Subcritical Branching Process with Condensation Phenomenon



Bayesian Approach through an example

- Put $a_k = 1/k!$, so the offspring is Poisson(θ)
- $\exists t^\star < \kappa$ such that

$$H^{\text{sub}} := \{\theta \mid m_\theta < 1\} = (0, t^\star)$$
$$H^{\text{sup}} := \{\theta \mid m_\theta > 1\} = (t^\star, \kappa),$$

- Suppose $\theta \sim \text{Gamma}(\alpha, \beta)$

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}$$

  where $\alpha > 0,\ \beta > 0,$ and $\theta > 0$.
- Denote $\mathbf{x}_n \equiv (x_1, \cdots, x_n)$ weekly epidemic data. Put $y_n := \sum_{i=0}^n x_i$.
- Bayes' Theorem yields the posterior distribution

$$g(\theta \mid \mathbf{x}_n) \propto \frac{\theta^{\alpha + y_n - y_0 - 1}}{e^{(\beta + y_{n-1})\theta}}.$$

- Again we get a Gamma with updated parameters

$$\theta \mid \mathbf{x}_n \sim \text{Gamma}\big(\alpha + x_1 + \cdots + x_n, \beta + x_0 + \cdots + x_{n-1}\big).$$

Numerical Application: Predicting Explosion or Extinction for a Poisson/Gamma GW Process

- Analysis based on the series of weekly number of confirmed new EVD cases registered across Guinea, Liberia and Sierra Leone from the beginning of the outbreak until March 31, 2015.
- As above assume the offspring is Poisson(θ), where $\theta \sim \text{Gamma}(\alpha, \beta)$.
- Set α and β so that the prior odds of both extinction and explosion are identical (equal to 50 %).
- We have favored low values of α and β in order to enable more prior variability. We have eventually retained $\alpha = 1.85$ and $\beta = 1$.
- Bayes Factor

$$B_n^{\text{pred}}(\mathbf{x}_n) = \frac{\left(\dfrac{\mathbb{P}(\textbf{Explosion} \mid \mathbf{x}_n)}{\mathbb{P}(\textbf{Extinction} \mid \mathbf{x}_n)}\right)}{\left(\dfrac{\mathbb{P}(\textbf{Explosion})}{\mathbb{P}(\textbf{Extinction})}\right)}$$

Predicting Explosion or Extinction for a Poisson/Gamma GW Process

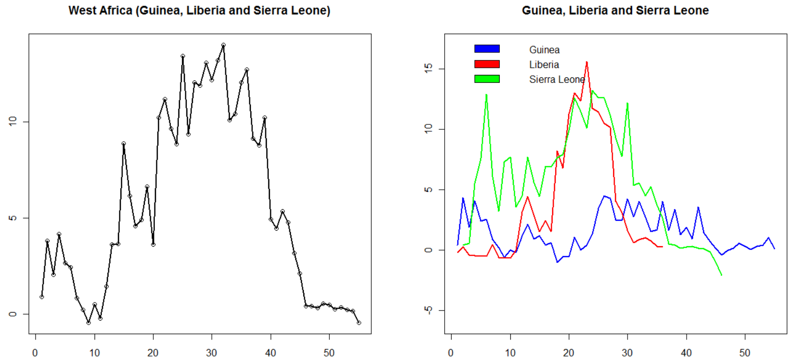


Figure: Weekly evolution of the [Explosion vs. Extinction] Prediction Problem Bayes Factor (base 10 log scale) across Guinea, Liberia and Sierra Leone, from week 9 of 2014 to week 12 of 2015. Data: WHO

Conclusion

- The need to assess the propagation risk may be satisfied within a GW parametric framework and under a Bayesian paradigm
- Probabilistic foundations ensure a fast and reliable implementation of our model and a simple diagnosis for health authorities

Thank you for your attention

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APPENDIX

2014 Ebola Outbreak: Guinea Evolution

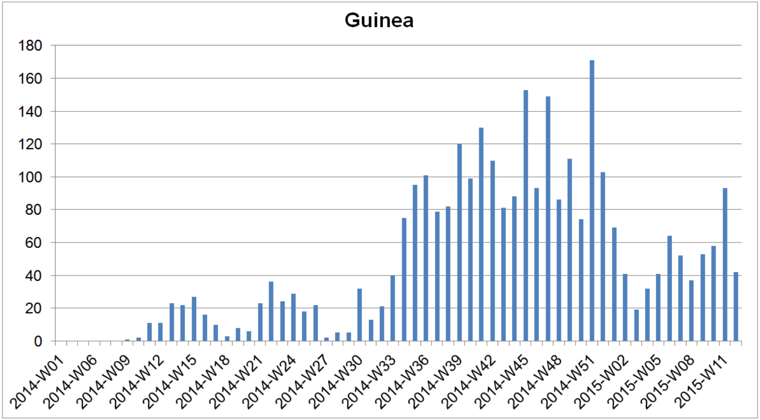


Figure: Weekly number of confirmed new EVD cases across Guinea. Data: WHO

2014 Ebola Outbreak: Liberia Evolution

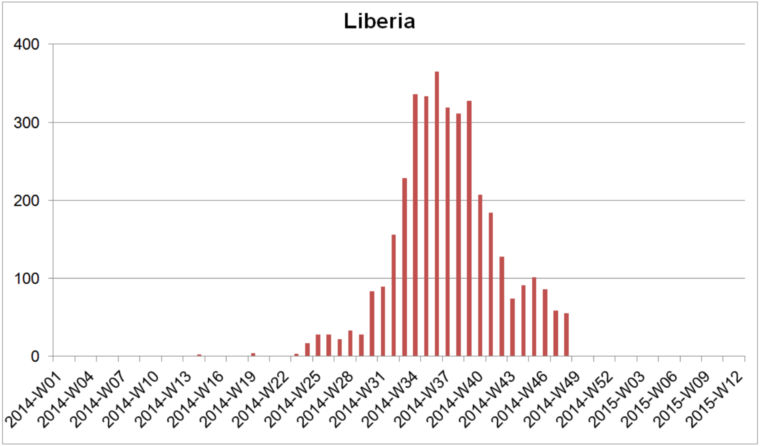


Figure: Weekly number of confirmed new EVD cases across Liberia.
Data: WHO

2014 Ebola Outbreak: Sierra Leone Evolution

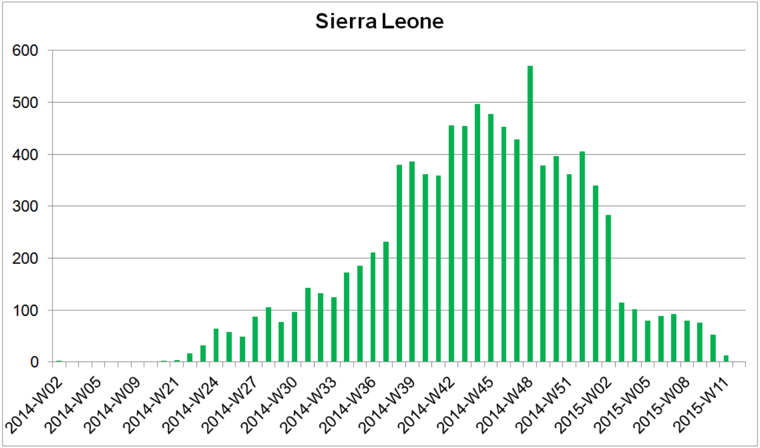


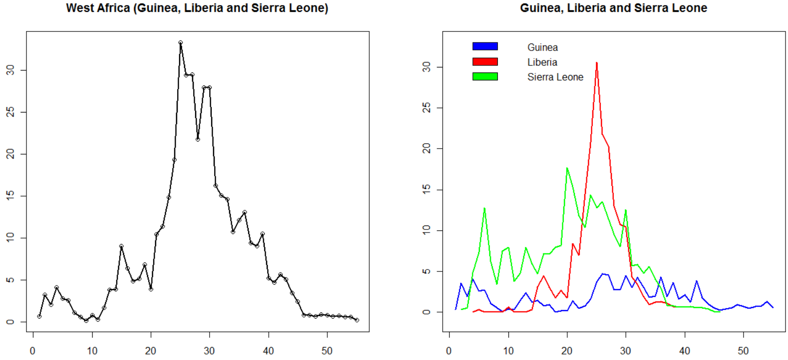
Figure: Weekly number of confirmed new EVD cases across Sierra Leone.
Data: WHO

Numerical Application: Testing Supercriticality vs. Subcriticality a posteriori for a Poisson/Gamma GW Process

- Focusing on low values of α and β in order to allow for more prior variability in the unknown parameter, we have eventually retained $\alpha = 3$ and $\beta = 2.674$.
- Bayes Factor

$$B_n^{\text{crit}}(\mathbf{x}_n) = \frac{\left(\dfrac{\mathbb{P}\left(\theta \in H^{\text{sup}} \mid \mathbf{x}_n\right)}{\mathbb{P}\left(\theta \in H^{\text{sub}} \mid \mathbf{x}_n\right)}\right)}{\left(\dfrac{\mathbb{P}\left(\theta \in H^{\text{sup}}\right)}{\mathbb{P}\left(\theta \in H^{\text{sub}}\right)}\right)}$$

Testing Supercriticality vs. Subcriticality a posteriori for a Poisson/Gamma GW Process



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