Optimal consumption and investment decisions under time-varying risk attitudes

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Introduction

- Changing risk aversion over the life cycle of an individual is analyzed in several literature:
  - Increasing risk aversion over time (e.g. Morin and Suarez, 1983; Riley and Chow, 1992)
  - Decreasing risk aversion over time (e.g. Bellante and Saba, 1986; Wang and Hanna, 1997)
- Within the optimal consumption and asset allocation problem in Merton (1969), the risk aversion remains constant over time.
- Different approaches in the literature to account for time dependence of the individuals risk attitude:
  - Habit level for consumption (e.g. Constantinides, 1990; Munk, 2008)
  - Time-varying relative risk aversion in power utility (e.g. Aase, 2009; Steffensen, 2011)
Introduction

- Combine both approaches: An *additive habit level* within a power utility function with a *time-varying coefficient of risk aversion*
- Optimal consumption and asset allocation in a complete market
- Numerically compare different magnitudes of habit formation and several shapes for the time-varying risk aversion
- Results:
  - *Hump-shaped consumption* for time-increasing risk aversion and large enough habit formation
  - *Decreasing investment* in risky asset over time for time-increasing risk aversion
  - For time-constant risk aversion (i.e. CRRA utility), changes in initial wealth affect absolute values of consumption
    For a time-varying risk aversion, *changes in initial wealth affect the shape of the consumption*
Financial market

- Filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})\)
- Finite time horizon \(T < \infty\)
- Individual can invest into a complete market
- Riskless asset \(B\)
  \[ dB_t = rB_t dt \]
- Risky asset \(S\)
  \[ dS_t = \mu S_t dt + \sigma S_t dW_t \]
- Market price of risk \(\theta = \frac{\mu - r}{\sigma}\)
- Since market is complete, there exists a unique state price density \(\xi\):
  \[ \xi_t = \exp \left\{ \left( -r - \frac{1}{2} \theta^2 \right) t - \theta W_t \right\} \]
Model for the individual

- Individual has initial wealth of $x_0 > 0$, at time $t$ invests proportion $\pi_t$ into risky asset and consumes $c_t$.

- Controls $(\pi, c)$ are adapted to the filtration and called admissible if
  - $c_t \geq 0 \mathbb{P} - a.s. \ \forall t \in [0, T]$ and $E \left[ (\int_0^T c_t \, dt)^2 \right] < \infty \mathbb{P} - a.s$
  - the portfolio process $(\pi_t)_{t \in [0, T]}$ is self-financing

- Wealth process $X$ of the individual with initial wealth $x_0 > 0$ satisfies
  $$dX_t = (r + \pi_t(\mu - r)) X_t \, dt - c_t \, dt + \pi_t \sigma X_t \, dW_t, \quad X_0 = x_0.$$

- Individual aims to maximize expected utility of consumption
  $$\mathbb{E} \left[ \int_0^T e^{-\rho t} U(t, c_t) \, dt \right] \to \max$$
  with utility $U(t, c)$ and individual discount $\rho$. 
Time varying risk aversion without habit

- **Utility** with time-varying relative risk aversion $\gamma_t$: $U(t, c) = \frac{1}{1 - \gamma_t} c^{1 - \gamma_t}$
- Problem solved by Steffensen (2011) and Aase (2009).
- Here: use unique state-price density and results from Cox and Huang (1989) and Karatzas et al. (1987) (*martingale method*)

**Optimization problem**

$$\mathbb{E} \left[ \int_0^T e^{-\rho t} \frac{1}{1 - \gamma_t} c_t^{1 - \gamma_t} dt \right] \rightarrow \text{max}$$

subject to: $\mathbb{E} \left[ \int_0^T \frac{\xi_t}{\xi_0} c_t dt \right] = x_0$
Optimal solution

- **Reciprocal** of risk aversion $\Phi_t := \frac{1}{\gamma_t}$
- **Optimal consumption** $c_t^* = e^{-\rho t\Phi_t}(y\xi_t)^{-\Phi_t}$

with Lagrangian multiplier $y$ defined by: $\int_0^T y^{-\Phi_t} g(t, 0) dt = x_0$,

where $g(s, t) = e^{-\rho s\Phi_s} e^{-(1-\Phi_s)(r+\frac{1}{2}\theta^2\Phi_s)(s-t)}$.

- **Shape of consumption depends on initial wealth** (implicitly through Lagrangian multiplier $y$).

For fixed $t$ as function of wealth:

$$\frac{\partial}{\partial x} c_t^* = (\Phi_t e^{-\rho t\Phi_t}(y\xi_t)^{-\Phi_t}) \left(\int_t^T \Phi_s \cdot g(s, t) \cdot (y\xi_t)^{-\Phi_s} ds\right)^{-1}$$
Time varying risk aversion with habit

- Habit level for consumption → achieve a smoother consumption
- Habit level $h_t = h_0 e^{-\beta t} + \alpha \int_0^t e^{-\beta(t-s)} c_s ds, \quad \alpha < \beta$
  
  dynamics: $dh_t = -(\beta h_t - \alpha c_t) dt$
- Utility with time-varying risk aversion $\gamma_t$ and habit level $h$:
  
  $U(t, c, h) = \frac{1}{1-\gamma_t} (c - h)^{1-\gamma_t}$

  Relative risk aversion $= -\frac{c \cdot U''}{U'} = \gamma_t + \gamma_t \cdot \frac{1}{\frac{c}{h} - 1}$

Optimization problem

$$
\mathbb{E} \left[ \int_0^T e^{-\rho t} \frac{1}{1-\gamma_t} (c_t - h_t)^{1-\gamma_t} dt \right] \rightarrow \text{max}
$$

subject to: $\mathbb{E} \left[ \int_0^T \frac{\xi_t}{\xi_0} c_t dt \right] = x_0$
Solve by using a dual problem (Schroder and Skiadas, 2002).

Define a dual market, with dual consumption $\hat{c} := c - h$

$\hat{c}$ consists of current consumption reduced by habit $h_t$ (some fictitious consumption derived from past consumption)

Adjust the market to price this fictitious consumption correctly.

Idea:

Keep the consumption from $t$ on at the habit level, i.e. $c_s = h_s$, $s > t$

$\Rightarrow$ dynamics of consumption: $dc_s = - (\beta - \alpha)c_ds$, with $c_t = h_t$

$\Rightarrow c_s = e^{-(\beta-\alpha)(s-t)}h_t$

**costs** for this consumption: $\mathbb{E}_t \left[ \int_t^T e^{-(\beta-\alpha)(s-t)}h_t \frac{\xi_s}{\xi_t} ds \right] = h_tF_t$

with $F_t = \int_t^T e^{-(\beta-\alpha)(s-t)}\mathbb{E}_t \left[ \frac{\xi_s}{\xi_t} \right] ds = \frac{1}{\alpha - \beta - r} \left( e^{(\alpha-\beta-r)(T-t)} - 1 \right)$

**Note**: require $x_0 > h_0F_0$
- **Dual market** is defined by
  - dual state price density $\hat{\xi}_t = \xi_t(1 + \alpha F_t)$
  - dual initial wealth $\hat{X}_0 = \frac{X_0 - h_0 F_0}{1 + \alpha F_0}$
- **Dual problem** is given by
  
  $\mathbb{E} \left[ \int_0^T e^{-\rho t} \frac{1}{1 - \gamma_t} \hat{c}_t^{1 - \gamma_t} dt \right] \rightarrow \text{max}$

  subject to: $\mathbb{E} \left[ \int_0^T \frac{\hat{\xi}_t}{\hat{\xi}_0} \hat{c}_t dt \right] = \hat{X}_0$

- Apply **previous results** to get the solution to the dual problem then transform dual solution to get solution to the actual problem.
Optimal consumption and investment:

\[ c_t^* = e^{-\rho t} \Phi_t (y \xi_t)^{-\Phi_t} (1 + \alpha F_t)^{-\Phi_t} + h_t^*, \]

\[ \pi_t^* = \mu - r \cdot \frac{1}{\sigma^2} \int_t^T \Phi_s \cdot g(s, t) \cdot (1 + \alpha F_s)^{1-\Phi_s} (y \xi_t)^{-\Phi_s} ds, \]

\[ X_t^* = h_t^* F_t + \int_t^T g(s, t) \cdot (1 + \alpha F_s)^{1-\Phi_s} (y \xi_t)^{-\Phi_s} ds, \]

where \( g(s, t) = e^{-\rho s} \Phi_s e^{-(1-\Phi_s)(r + \frac{1}{2} \theta^2 \Phi_s)} (s - t) \)

with Lagrangian multiplier \( y \) being defined by

\[ \int_0^T g(t, 0) \cdot (1 + \alpha F_t)^{1-\Phi_t} y^{-\Phi_t} dt = x_0 - h_0 F_0 \]

Similar as before, the shape of the consumption depends on the initial wealth.
Numerical results

- Parameters: $T = 30$, $r = 2\%$, $\mu = 8\%$, $\sigma = 20\%$, $x_0 = 1000$, $\rho = r$.
- Habit formations:

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<thead>
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<th></th>
<th>No habit</th>
<th>Low habit</th>
<th>Medium habit</th>
<th>High habit</th>
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<td>$h_0 = 20$</td>
<td>$h_0 = 30$</td>
<td>$h_0 = 40$</td>
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- Time-varying risk aversion $\gamma_t$:

- Comparison:
  1. Expected values of decision rules $\rightarrow$ development over time
  2. Decision rules as function of wealth
Expected value of optimal consumption ($c$), habit level ($h$) and investment ($\pi$) for $\gamma_t \equiv 3$ as function of time $t$. 
Expected value of optimal consumption ($c$), habit level ($h$) and investment ($\pi$) for $\gamma_t$ linear increasing as function of time $t$. 
Consumption \((c)\) as function of wealth \(X\) for fixed time points \((t = 5\) and \(t = 20)\), two initial wealth \((x_0 = 1\) and \(x_0 = 1000)\), for case without habit and with habit and three different \(\gamma_t\).
Summary

- Empirical observations suggest that risk attitude changes over time.
- Optimal consumption and investment in a complete market for an individual with an *additive habit level for consumption* and a *time varying risk aversion* in a power utility function.
- For increasing risk aversion and sufficiently large habit formation, optimal decision rules consistent to observations in literature:
  - hump-shaped consumption pattern
  - asset allocation decreasing over the life cycle
- Shape of decision rules *depends on initial wealth* (in contrast to case of time-constant risk aversion).
Literature