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Optimal consumption and investment decisions under time-varying risk attitudes

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Introduction

- Changing risk aversion over the life cycle of an individual is analyzed in several literature:
  - Increasing risk aversion over time (e.g. Morin and Suarez, 1983; Riley and Chow, 1992)
  - Decreasing risk aversion over time (e.g. Bellante and Saba, 1986; Wang and Hanna, 1997)
- Within the optimal consumption and asset allocation problem in Merton (1969), the risk aversion remains constant over time.
- Different approaches in the literature to account for time dependence of the individuals risk attitude:
  - Habit level for consumption (e.g. Constantinides, 1990; Munk, 2008)
  - Time-varying relative risk aversion in power utility (e.g. Aase, 2009; Steffensen, 2011)

Introduction

- Combine both approaches: An additive habit level within a power utility function with a time-varying coefficient of risk aversion
- Optimal consumption and asset allocation in a complete market
- Numerically compare different magnitudes of habit formation and several shapes for the time-varying risk aversion
- Results:
  - Hump-shaped consumption for time-increasing risk aversion and large enough habit formation
  - Decreasing investment in risky asset over time for time-increasing risk aversion
  - For time-constant risk aversion (i.e. CRRA utility), changes in initial wealth affect absolute values of consumption
    For a time-varying risk aversion, changes in initial wealth affect the shape of the consumption

Financial market

- Filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$
- Finite time horizon $T < \infty$
- Individual can invest into a complete market
- Riskless asset B

$$dB_t = rB_t dt$$

- Risky asset S

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

- Market price of risk $\theta = \frac{\mu - r}{\sigma}$
- Since market is complete, there exists a unique state price density ξ:

$$\xi_t = \exp\left\{\left(-r - \frac{1}{2}\theta^2\right)t - \theta W_t\right\}$$

Model for the individual

- Individual has initial wealth of $x_0 > 0$, at time t invests proportion π_t into risky asset and consumes c_t.
- Controls (π, c) are adapted to the filtration and called admissible if
  - $c_t \geq 0$ $\mathbb{P} - a.s.$ $\forall t \in [0, T]$ and $E\left[\left(\int_0^T c_t dt\right)^2\right] < \infty$ $\mathbb{P} - a.s$
  - the portfolio process $(\pi_t)_{t\in[0,T]}$ is self-financing
- Wealth process X of the individual with initial wealth $x_0 > 0$ satisfies

$$dX_t = (r + \pi_t(\mu - r))\, X_t dt - c_t dt + \pi_t \sigma X_t dW_t,\ X_0 = x_0.$$

- Individual aims to maximize expected utility of consumption

$$\mathbb{E}\left[\int_0^T e^{-\rho t} U(t, c_t) dt\right] \to \max$$

with utility $U(t, c)$ and individual discount ρ.

Time varying risk aversion without habit

- Utility with time-varying relative risk aversion γ_t: $U(t,c) = \frac{1}{1-\gamma_t} c^{1-\gamma_t}$
- Problem solved by Steffensen (2011) and Aase (2009).
- Here: use unique state-price density and results from Cox and Huang (1989) and Karatzas et al. (1987) (*martingale method*)

Optimization problem

$$\mathbb{E}\left[\int_0^T e^{-\rho t} \frac{1}{1-\gamma_t} c_t^{1-\gamma_t} dt\right] \to \max$$

$$\text{subject to: } \mathbb{E}\left[\int_0^T \frac{\xi_t}{\xi_0} c_t dt\right] = x_0$$

Optimal solution

- Reciprocal of risk aversion $\Phi_t := \frac{1}{\gamma_t}$
- Optimal consumption $c_t^* = e^{-\rho t \Phi_t}(y\xi_t)^{-\Phi_t}$

  with Lagrangian multiplier y defined by: $\int_0^T y^{-\Phi_t} g(t,0)dt = x_0$,

  where $g(s,t) = e^{-\rho s \Phi_s} e^{-(1-\Phi_s)\left(r+\frac{1}{2}\theta^2\Phi_s\right)(s-t)}$.
- Shape of consumption depends on initial wealth (implicitly through Lagrangian multiplier y).

  For fixed t as function of wealth:

$$\frac{\partial}{\partial x} c_t^* = \left(\Phi_t e^{-\rho t \Phi_t}(y\xi_t)^{-\Phi_t}\right)\left(\int_t^T \Phi_s \cdot g(s,t) \cdot (y\xi_t)^{-\Phi_s} ds\right)^{-1}$$

Time varying risk aversion with habit

- Habit level for consumption $\rightarrow$ achieve a smoother consumption
- Habit level $h_t = h_0 e^{-\beta t} + \alpha \int_0^t e^{-\beta(t-s)} c_s ds, \quad \alpha < \beta$

  dynamics: $dh_t = -(\beta h_t - \alpha c_t)dt$
- Utility with time-varying risk aversion γ_t and habit level h:

  $U(t, c, h) = \frac{1}{1-\gamma_t}(c-h)^{1-\gamma_t}$

  Relative risk aversion $= -\dfrac{c \cdot U''}{U'} = \gamma_t + \gamma_t \cdot \dfrac{1}{\frac{c}{h} - 1}$

Optimization problem

$$\mathbb{E}\left[\int_0^T e^{-\rho t} \frac{1}{1-\gamma_t}(c_t - h_t)^{1-\gamma_t} dt\right] \rightarrow \max$$

$$\text{subject to: } \mathbb{E}\left[\int_0^T \frac{\xi_t}{\xi_0} c_t dt\right] = x_0$$

- Solve by using a dual problem (Schroder and Skiadas, 2002).
- Define a dual market, with dual consumption $\hat{c} := c - h$
  - $\rightarrow$ consists of current consumption reduced by habit h_t (some fictitious consumption derived from past consumption)
- Adjust the market to price this fictitious consumption correctly.
- Idea:
  - Keep the consumption from t on at the habit level, i.e. $c_s = h_s,\ s > t$
    $\Rightarrow$ dynamics of consumption: $dc_s = -(\beta - \alpha)c_s ds,$ with $c_t = h_t$
    $\Rightarrow c_s = e^{-(\beta-\alpha)(s-t)} h_t$
  - costs for this consumption: $\mathbb{E}_t \left[\int_t^T e^{-(\beta-\alpha)(s-t)} h_t \frac{\xi_s}{\xi_t} ds \right] = h_t F_t$

    with $F_t = \int_t^T e^{-(\beta-\alpha)(s-t)} \mathbb{E}_t \left[\frac{\xi_s}{\xi_t} \right] ds = \frac{1}{\alpha - \beta - r} \left(e^{(\alpha-\beta-r)(T-t)} - 1 \right)$
  - *Note*: require $x_0 > h_0 F_0$

- Dual market is defined by
  - dual state price density $\hat{\xi}_t = \xi_t(1 + \alpha F_t)$
  - dual initial wealth $\hat{X}_0 = \frac{X_0 - h_0 F_0}{1+\alpha F_0}$
- Dual problem is given by

$$\mathbb{E}\left[\int_0^T e^{-\rho t} \frac{1}{1-\gamma_t} \hat{c}_t^{1-\gamma_t} dt\right] \to \max$$

$$\text{subject to: } \mathbb{E}\left[\int_0^T \frac{\hat{\xi}_t}{\hat{\xi}_0} \hat{c}_t dt\right] = \hat{X}_0$$

- Apply previous results to get the solution to the dual problem then transform dual solution to get solution to the actual problem.

- Optimal consumption and investment:

$$
\begin{aligned}
c_t^* &= e^{-\rho t \Phi_t}(y\xi_t)^{-\Phi_t}(1+\alpha F_t)^{-\Phi_t} + h_t^*, \\
\pi_t^* &= \frac{\mu - r}{\sigma^2} \cdot \frac{1}{X_t^*} \int_t^T \Phi_s \cdot g(s,t) \cdot (1+\alpha F_s)^{1-\Phi_s}(y\xi_t)^{-\Phi_s} ds, \\
X_t^* &= h_t^* F_t + \int_t^T g(s,t) \cdot (1+\alpha F_s)^{1-\Phi_s}(y\xi_t)^{-\Phi_s} ds,
\end{aligned}
$$

where $g(s,t) = e^{-\rho s \Phi_s} e^{-(1-\Phi_s)\left(r+\frac{1}{2}\theta^2\Phi_s\right)(s-t)}$

- with Lagrangian multiplier y being defined by

$$
\int_0^T g(t,0) \cdot (1+\alpha F_t)^{1-\Phi_t} y^{-\Phi_t} dt = x_0 - h_0 F_0
$$

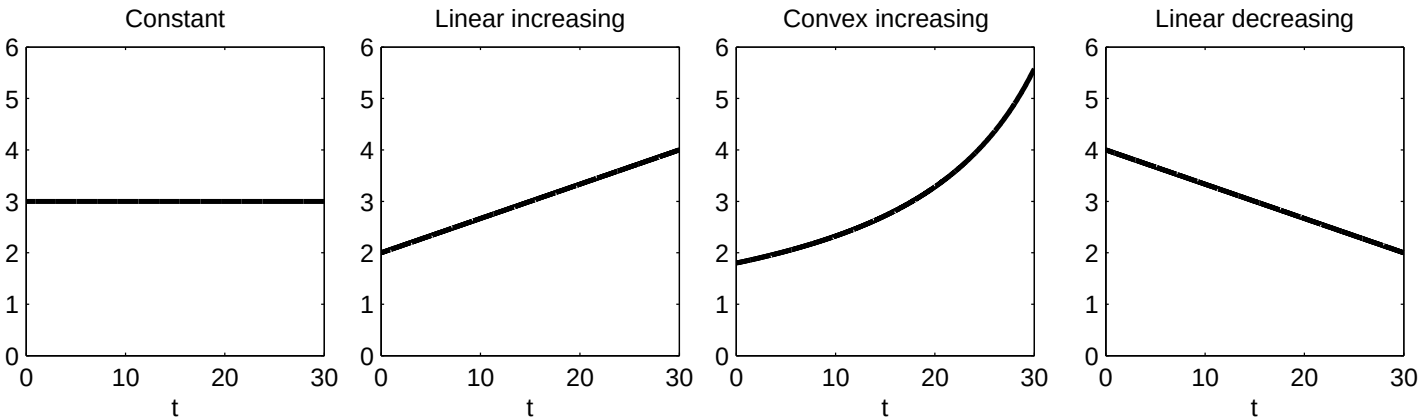
- Similar as before, the shape of the consumption depends on the initial wealth.

Numerical results

- Parameters: $T = 30$, $r = 2\%$, $\mu = 8\%$, $\sigma = 20\%$, $x_0 = 1000$, $\rho = r$.
- Habit formations:

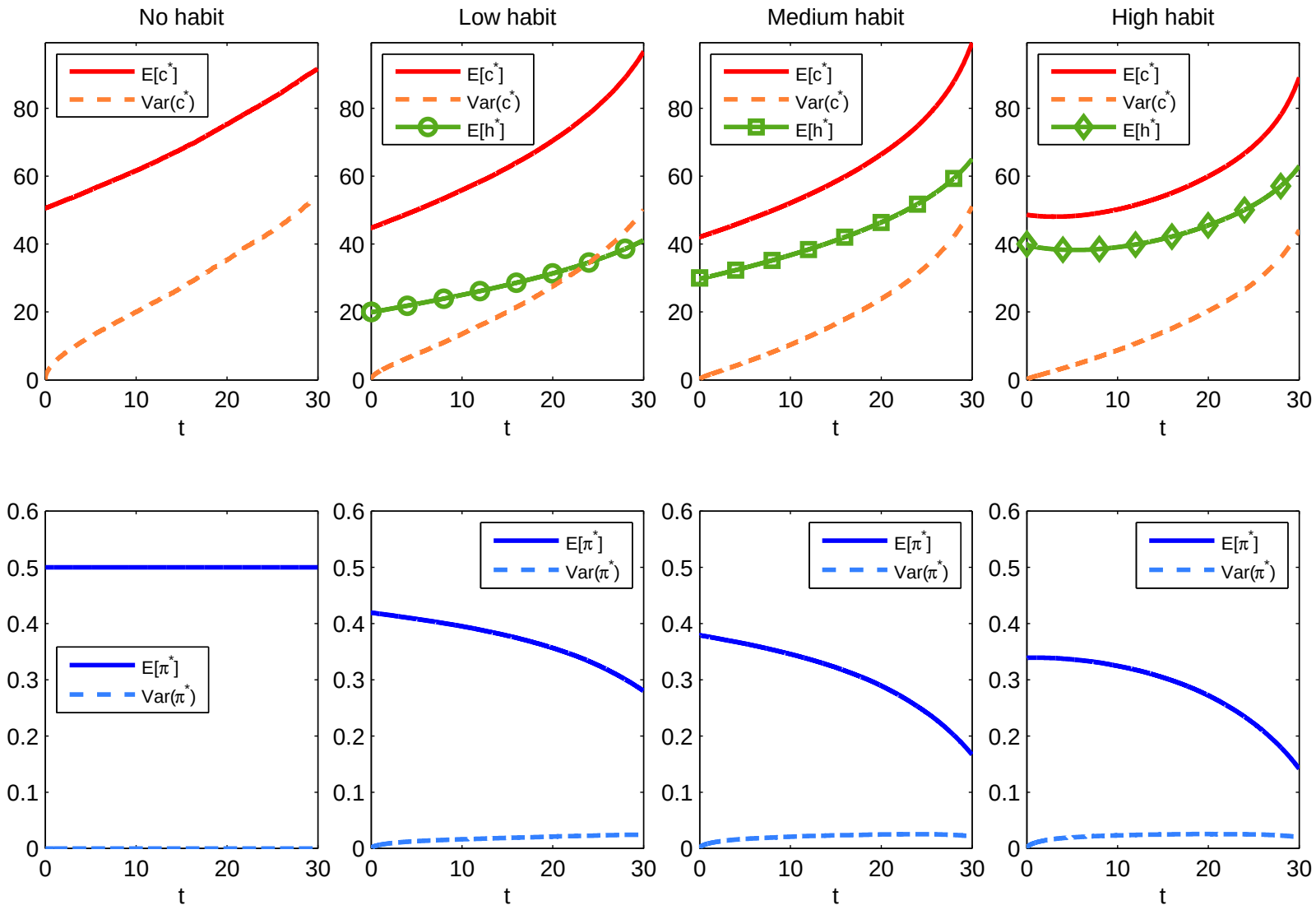
No habit	Low habit	Medium habit	High habit
$\alpha = 0$	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.4$
$\beta = 0$	$\beta = 0.2$	$\beta = 0.4$	$\beta = 0.5$
$h_0 = 0$	$h_0 = 20$	$h_0 = 30$	$h_0 = 40$

- Time-varying risk aversion γ_t:

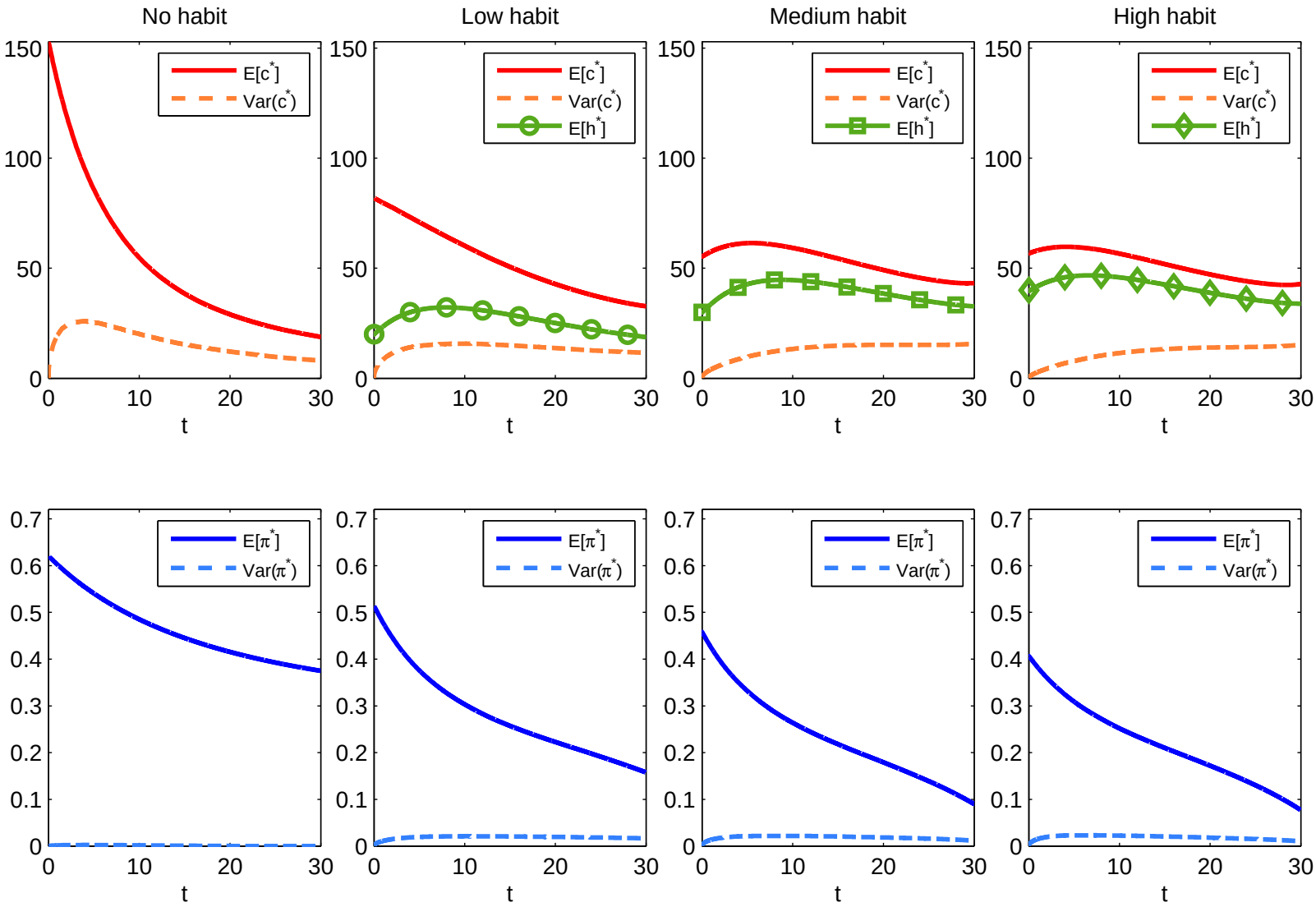


- Comparison:
  1. Expected values of decision rules $\rightarrow$ development over time
  2. Decision rules as function of wealth

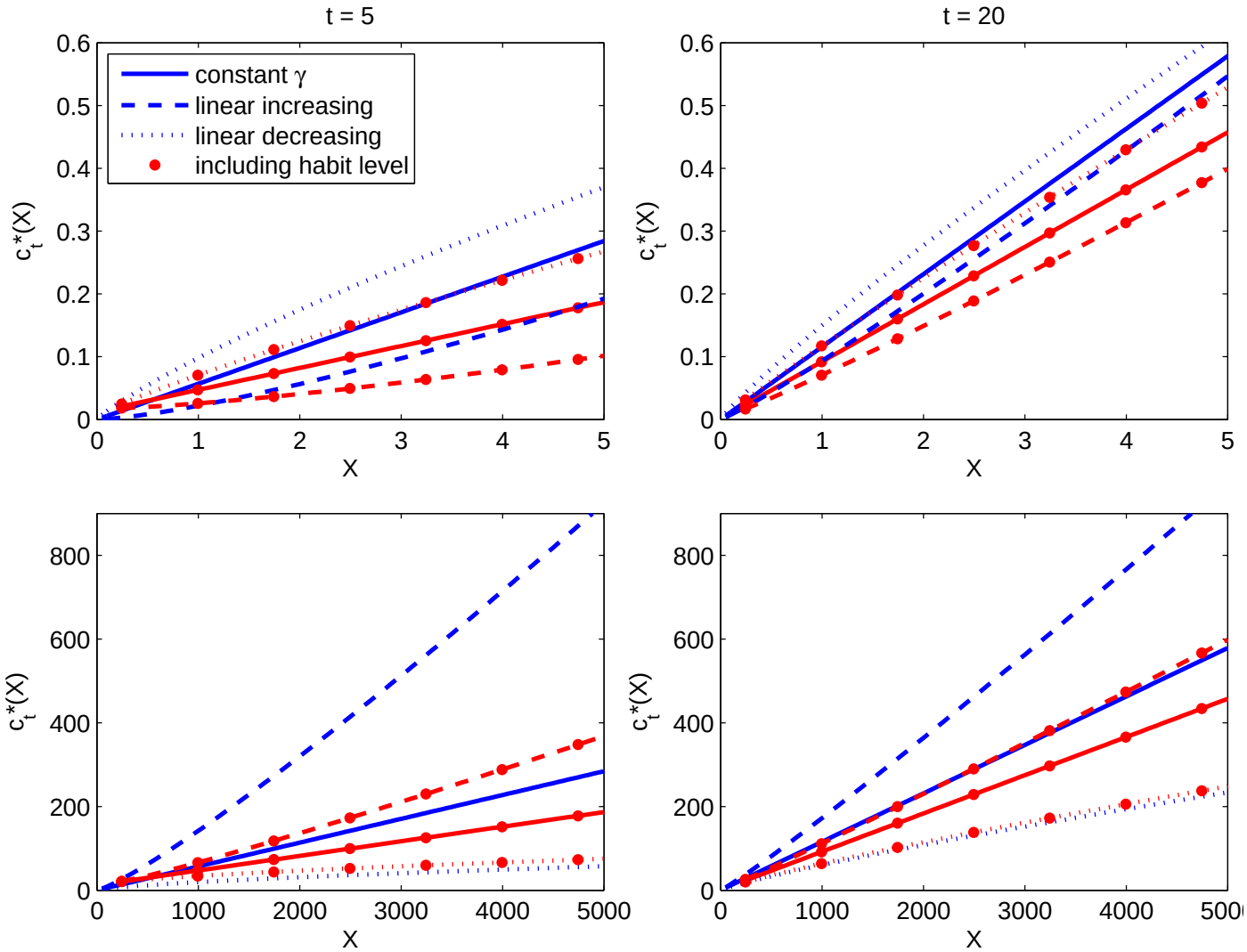
- Expected value of optimal consumption (c), habit level (h) and investment (π) for $\gamma_t \equiv 3$ as function of time t.



- Expected value of optimal consumption (c), habit level (h) and investment (π) for γ_t linear increasing as function of time t.



- Consumption (c) as function of wealth X for fixed time points ($t = 5$ and $t = 20$), two initial wealth ($x_0 = 1$ and $x_0 = 1000$), for case without habit and with habit and three different γ_t.



Summary

- Empirical observations suggest that risk attitude changes over time.
- Optimal consumption and investment in a complete market for an individual with an additive habit level for consumption and a time varying risk aversion in a power utility function
- For increasing risk aversion and sufficiently large habit formation, optimal decision rules consistent to observations in literature:
  - hump-shaped consumption pattern
  - asset allocation decreasing over the life cycle
- Shape of decision rules depends on initial wealth (in contrast to case of time-constant risk aversion).

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