Interest rate model comparisons for participating products under Solvency II

Kjersti Aas[a,*], Linda R. Neef[a], Lloyd Williams[b], Dag Raabe[b]

[a] *Norwegian Computing Center*
[b] *SpareBank 1 Forsikring*

Abstract

A key aspect of the Solvency II regulatory framework is to compute the best estimate of the liabilities. This best estimate should be the probability-weighted average of future cash-flows, discounted to its present value. Movements in economic variables are often the driving force of changes in liability present values. Hence, many life insurers need stochastic models for producing future paths for e.g. interest rates, equity and bond returns and currencies. The paths should be risk-neutral, meaning that the expected return of all assets should be equal to the risk-neutral rate used for discounting the cash-flows. Hence, the interest rate model is a key component to consider within the Solvency II framework, particularly for life insurers. In this paper we study three interest rate models; the CIR++-model, the G++-model and the Libor Market model. Even when calibrated to the same historical data, the simulations from these models have very different mean value and volatility characteristics, especially far out into the future. However, when using these simulations when computing the best estimate of the liabilities, the differences between the models are surprisingly small, both for a synthetic and for a real-world insurance portfolio.

Outline

- Desired properties of interest rate models in a Solvency II setting.
- A review of three well-known interest rate models
- A real example from a Norwegian insurance company

Interest rate models and Solvency II

- Desired properties and objectives of an interest rate model used to compute the best estimate liabilities in Solvency II:
  - Adherence to data, judgment and available literature.
  - Intuitive for decision makers.
  - Ability to calibrate to market prices or historical data.
  - Time or cost needed for calibration and simulation.
  - Numerical stability of results.

Interest rate models

Interest rate types

- The **spot rate** R(t,t+i) at time t is the interest rate for the time period from t to t+i.
- The **short rate** r_t at time t is the instantaneous spot rate rate: $r_t = \lim_{i \to 0^+} R(t, t+i)$
- The **forward rate** $F_{t+i}(t)$ at time t is the interest rate for the period from t+i to t+i+1. The forward rates are not observable, but they might be computed from the spot rates as follows:

$$F_{t+i}(t) = \begin{cases} R(t, t+1) & \text{if } i = 0 \\ \frac{[1+R(t,t+i+1)]^{i+1}}{[1+R(t,t+i)]^{i}} - 1 & \text{else} \end{cases}$$

Interest rate models

- Here, we study three interest rate models
  - CIR++: Short rate, one-factor
  - G++: Short rate, two-factor
  - Libor: Forward rate, multi-factor

Brigo and Mercurio (2001):

- One factor explains 68% to 78% of the total variation in the yield curve
- Two factors explain 85% to 90% of the total variation in the yield curve
- Three factors explain 93% to 94% of the total variation in the yield curve

CIR++-model

- The short rate is first simulated, and then simulations of the spot rates with different maturities are derived from the short rate simulations.

- Let P(t,T) be the price at time t of a zero-coupon bond with maturity T. For CIR++-model, P(t,T) is given by

$$P(t,T) = A(t,T)\, e^{B(t,T)\, r_t}$$

- Further, the **spot rate** R(t,t+i) is

$$R(t,t+i) = -\frac{\log(P(t,t+i))}{i}$$

This means that at every time point, instant rates for all maturities in the yield curve are perfectly correlated.

CIR++-model

- The short rate dynamics are given by:

$$r_t = \phi_t + (1-\alpha)\,\mu + \alpha\, r_{t-1} + \sqrt{r_t}\,\epsilon_t$$

- where
  - ϕ_t is a function chosen to fit the initial term structure
  - α is a mean-reversion parameter
  - $\epsilon_t \sim N(0, \sigma^2)$
- ϕ_t is computed as the difference between the model and market based instantaneous forward rates.
- The market based rates are computed using the Svensson model.

G++-model

- The short rate is first simulated, and then simulations of the spot rates with different maturities are derived from the short rate simulations.
- As for the CIR++-model, the **spot rate** R(t,t+i) is given by

$$R(t,t+i) = -\frac{\log(P(t,t+i))}{i}$$

- However, for the G++-model P(t,T) is given by

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \exp\{A(t,T)\}$$

- where $P^M(0,T)$ is the market discount factor for maturity T.

G++-model

- The short rate dynamics are given by:

$$
\begin{aligned}
dx(t) &= -\alpha\, x(t)\, dt + \gamma\, dW_1(t), \quad x(0) = 0 \\
dy(t) &= -\beta\, y(t)\, dt + \eta\, dW_2(t), \quad y(0) = 0 \\
r(t) &= x(t) + y(t) + \varphi(t),
\end{aligned}
$$

- where α and β are constants reflecting the rate of mean reversion, γ and η are volatilities, ϕ_t is a function chosen to fit the initial term structure and $W_1(t)$ and $W_2(t)$ are standard Brownian motions with correlation κ.

G++-model

- The quantity A(t,T) may be computed from the short rate parameters as follows

$$A(t,T) = \frac{1}{2}\left[V(t,T) - V(0,T) + V(0,t)\right] - \frac{1-e^{-\alpha(T-t)}}{\alpha}\,x(t) - \frac{1-e^{-\beta(T-t)}}{\beta}\,y(t).$$

- where

$$\begin{aligned}
V(t,T) \quad &= \quad \frac{\gamma^2}{\alpha^2}\left[T - t + \frac{2}{\alpha}\,e^{-\alpha(T-t)} - \frac{1}{2\,\alpha}\,e^{-2\,\alpha\,(T-t)} - \frac{3}{2\,\alpha}\right] \\
&+ \quad \frac{\eta^2}{\beta^2}\left[T - t + \frac{2}{\beta}\,e^{-\beta(T-t)} - \frac{1}{2\,\beta}\,e^{-2\beta(T-t)} - \frac{3}{2\beta}\right] \\
&+ \quad 2\,\kappa\,\frac{\gamma\,\eta}{\alpha\,\beta}\left[T - t + \frac{e^{-\alpha(T-t)}-1}{\alpha} + \frac{e^{-\beta(T-t)}-1}{\beta}\right. \\
&- \quad \left.\frac{e^{-(\alpha+\beta)\,(T-t)}-1}{\alpha+\beta}\right]
\end{aligned}$$

The Libor Market Model

- In the Libor Market model $F_i(t)$ is given by:

$$F_i(t) = F_i(t-1)\ \text{exp}\left(\sigma_i(t)\,\mu_i(t) - \frac{1}{2}\sigma_i(t)^2 + \sigma_i(t)\,\epsilon_i(t)\right),$$

$$\mu_i(t) = \sum_{k=t}^{i} \frac{F_k(t-1)\rho_{i,k}(t)\sigma_k(t)}{1+F_k(t-1)}$$

- $\sigma_i(t)$ is the volatility of $\epsilon_i(t)$. We assume that it is given by:

$$\sigma_{i-t} = a\ \text{exp}(-b|i-t|).$$

- $\rho_{i,k}(t)$ is the correlation between $\epsilon_i(t)$ and $\epsilon_k(t)$. We assume that it is given by:

$$\rho_{i,j}(t) = \text{exp}(-c|j-i|).$$

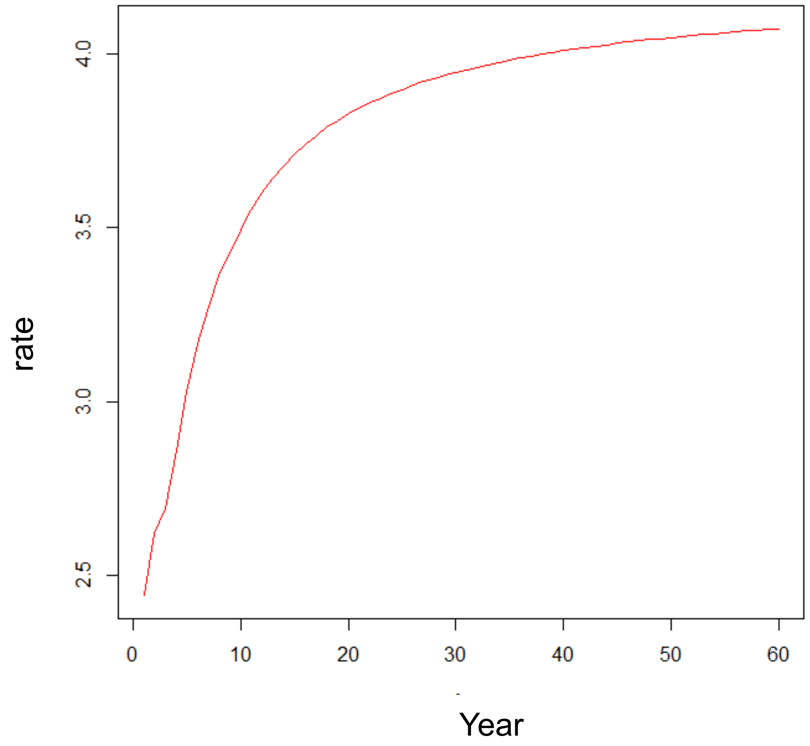
NR SpareBank 1

Calibration

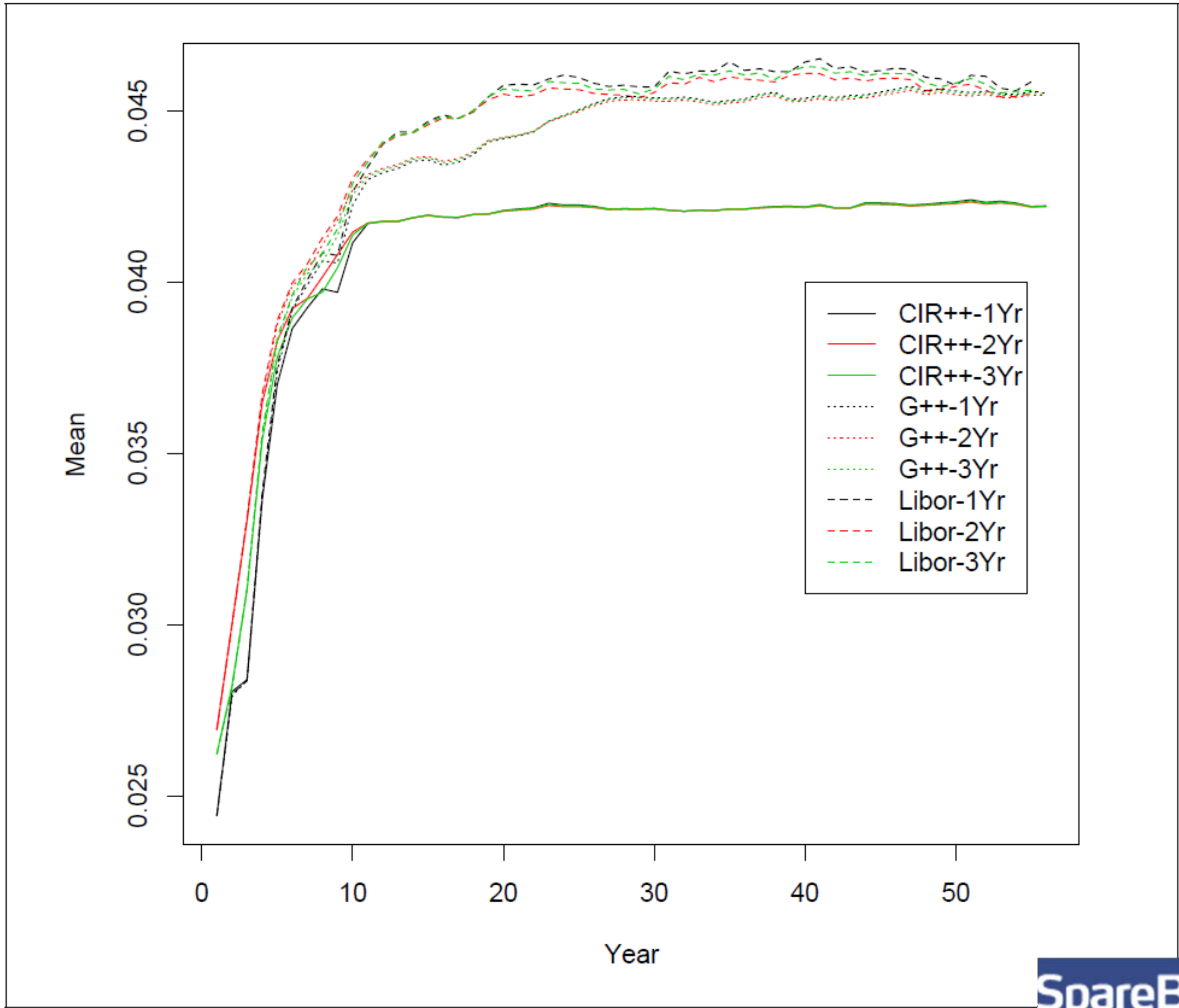
- Have estimated all models based on monthly data for the 3-month rate and swap rates with maturities from 1 to 10 years from the period March, 2001 to March, 2011.
- CIR++-model: Use the maximum likelihood method.
- G++-model: Minimize the sum of squared differences between theoretical and empirical volatilities of monthly absolute spot rate changes.
- Libor Market model: Minimize the sum of squared differences between observed and model-based volatilities/correlations.

Simulation

- Use the yield curve shown in the figure, which was specified by EIOPA in December 2011.
- 10,000 simulations.
- Yearly resolution
- Time horizon 60 years.

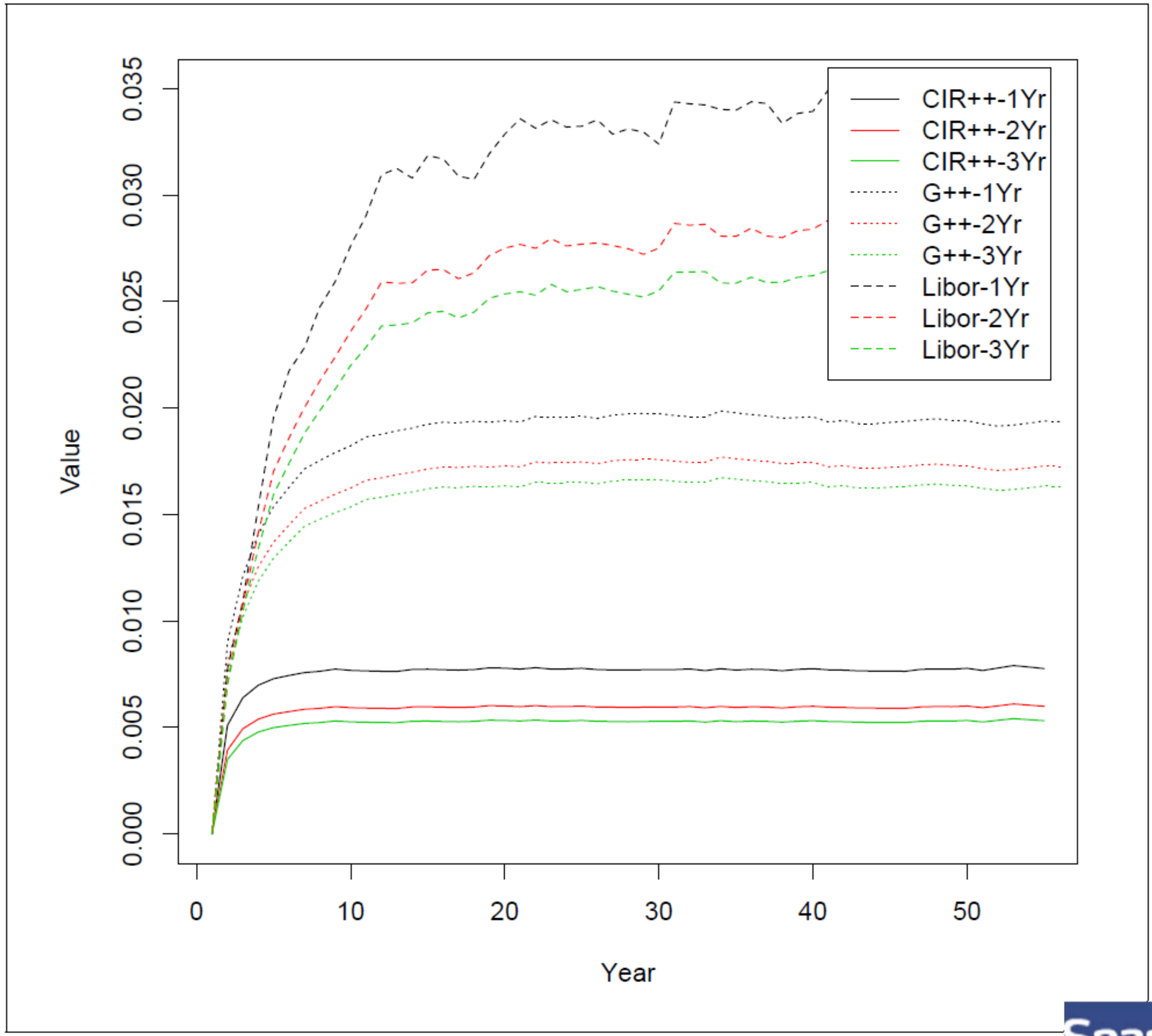


Mean values

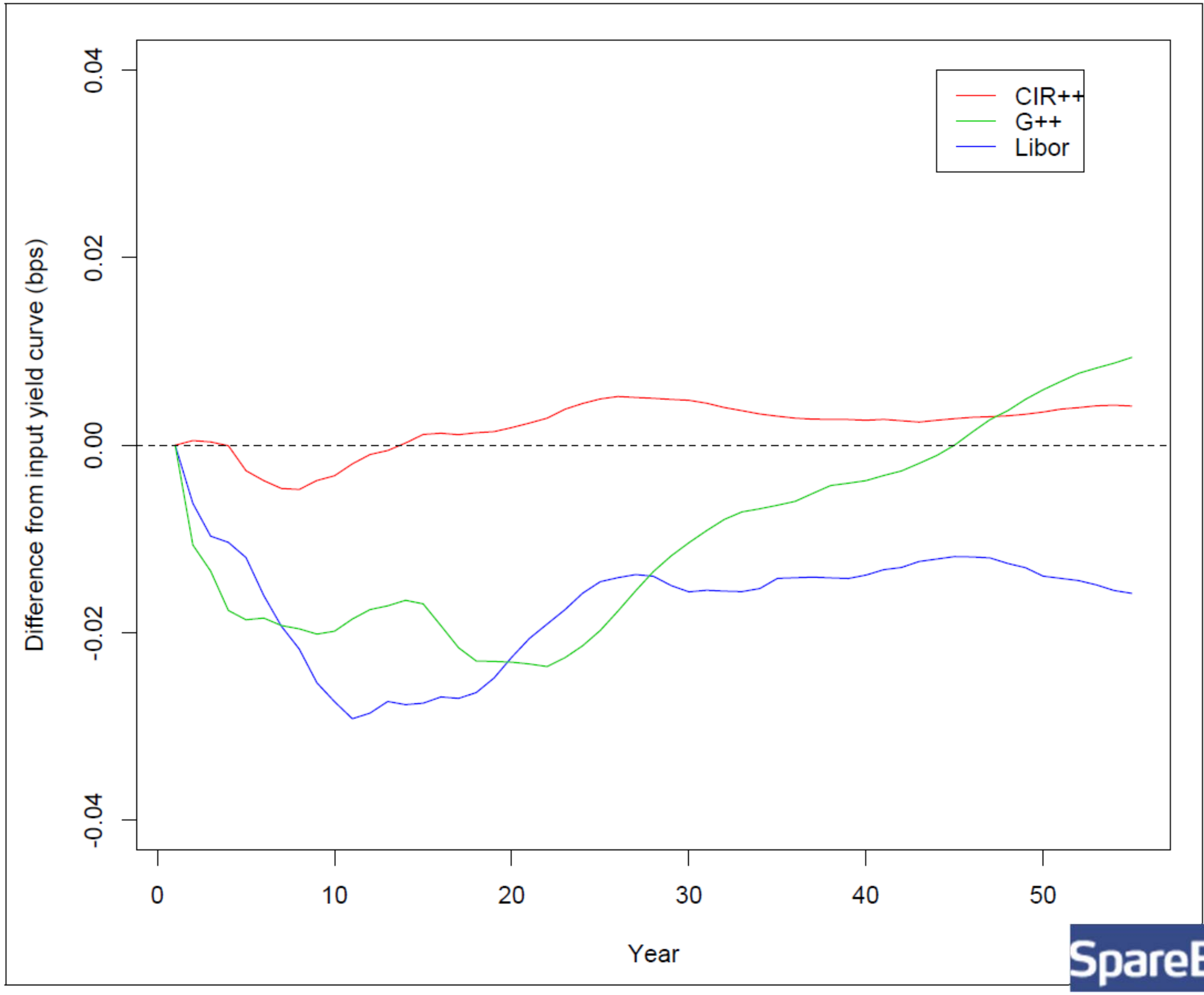


Mean
0.025
0.030
0.035
0.040
0.045
0
10
20
30
40
50
Year
CIR++-1Yr
CIR++-2Yr
CIR++-3Yr
G++-1Yr
G++-2Yr
G++-3Yr
Libor-1Yr
Libor-2Yr
Libor-3Yr

Standard deviations



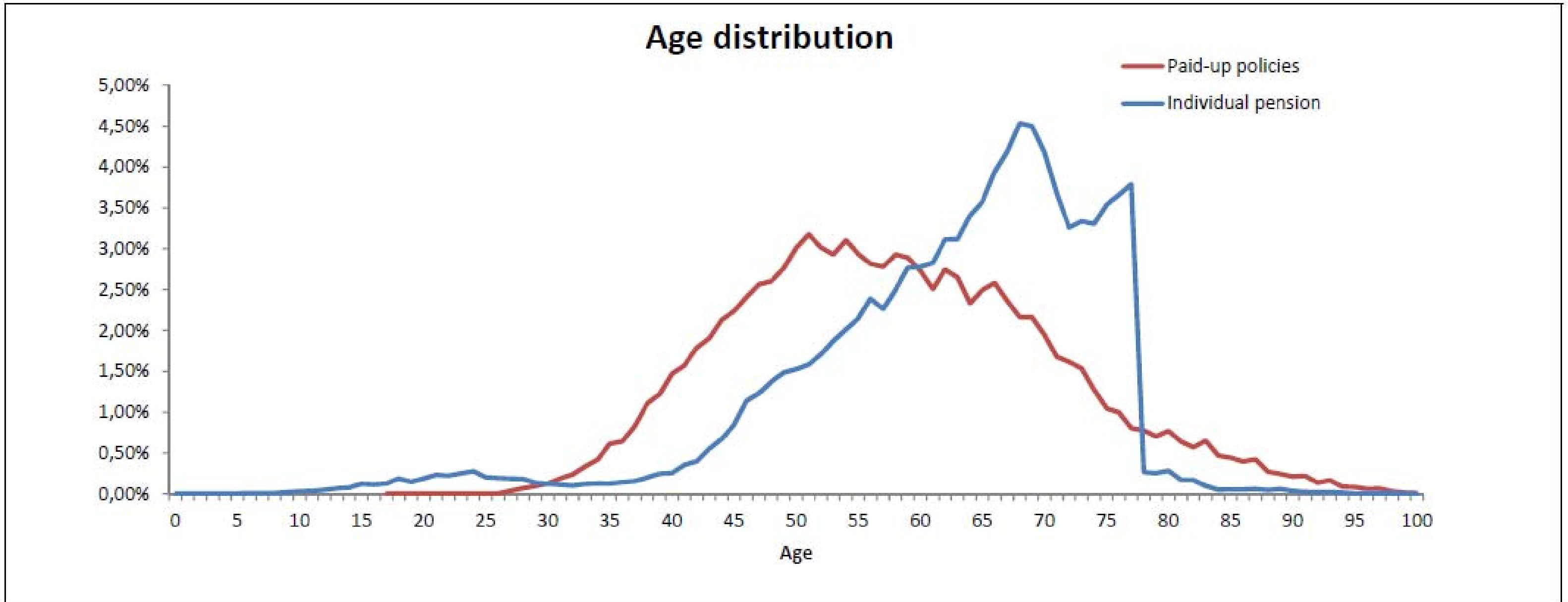
Fitting yield curve?



Real example

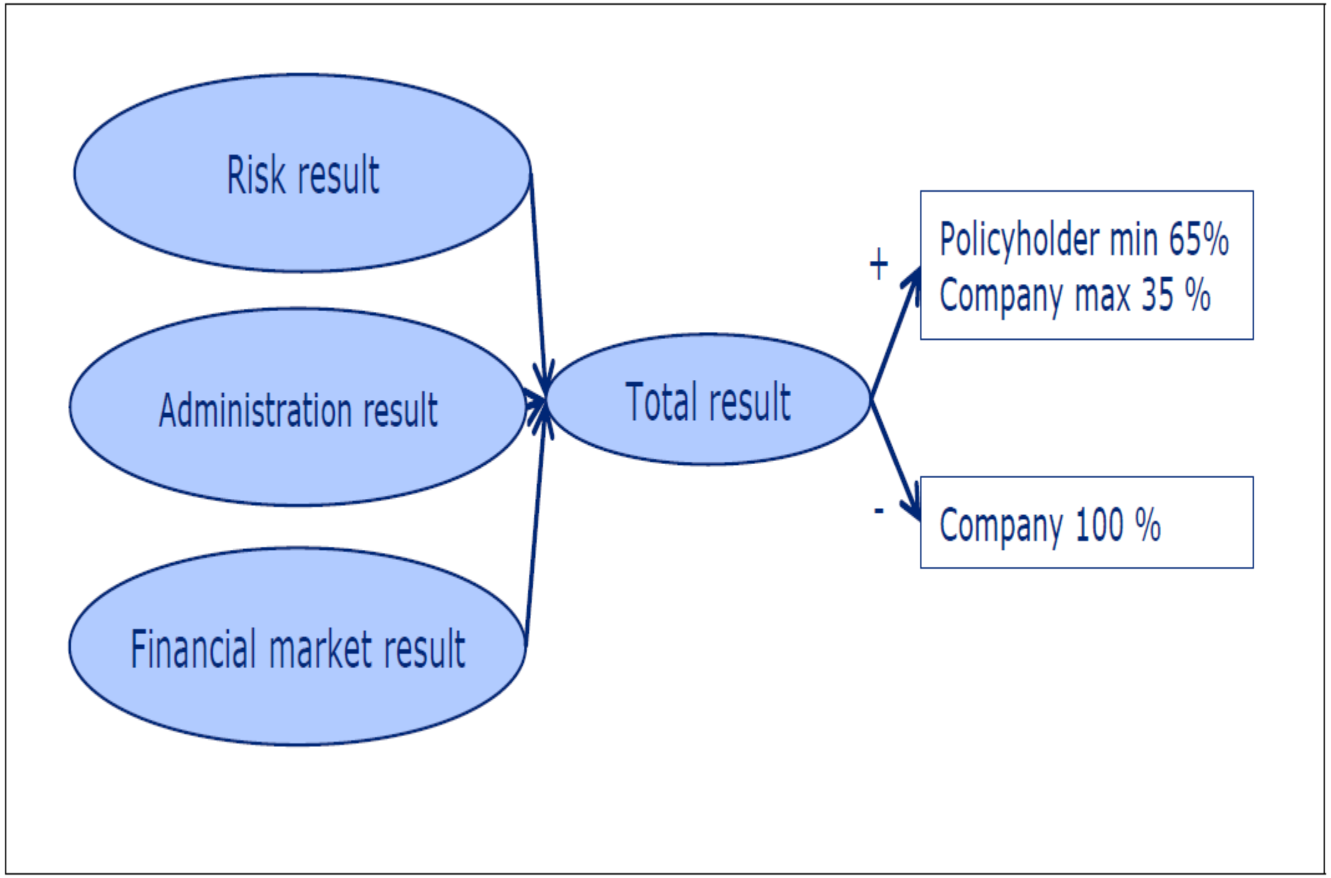
Products

- Two different products:
  - Old-age pension for individuals with profit sharing
  - Paid-up defined benefit pension policies with profit sharing.



SpareBank 1

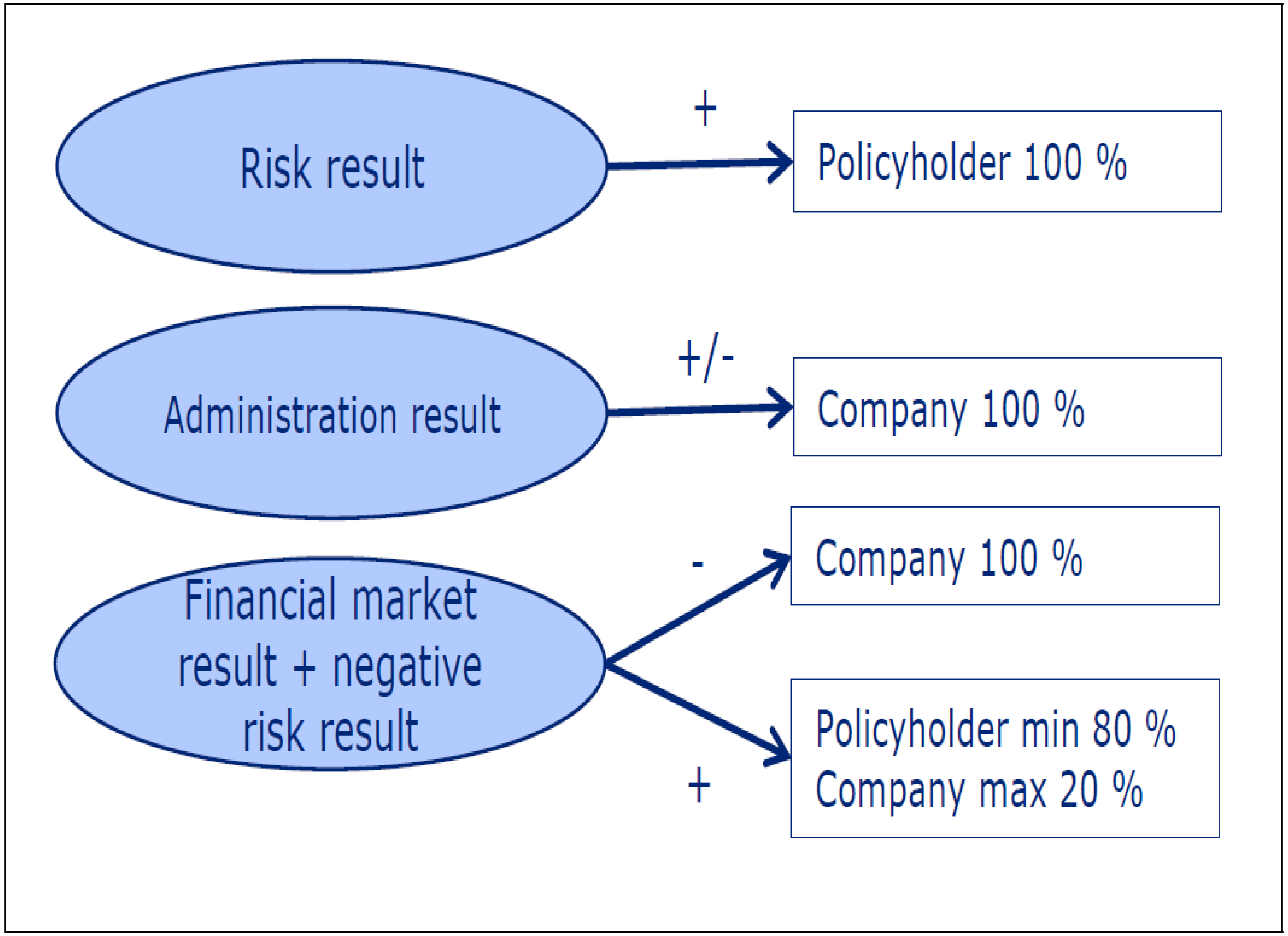
Old-age pension



Interest guarantee between 2.5 and 4%

Pension is either paid out in a defined number of years or as a lifelong benefit, usually starting at the age 67.

Paid-up defined benefit pension



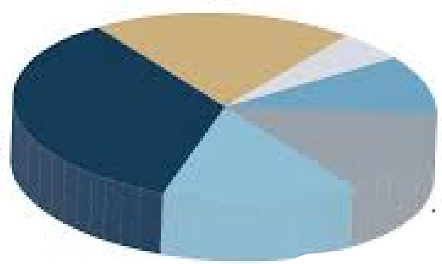
Interest guarantee between 2.5 and 4%

Fully paid contracts from a defined benefit plan. The benefits are old-age pensions, spouse pension and disability pension.

The old-age and spouse pensions are either paid out in a defined number of years, or as a lifelong benefit. The disability pension may be paid until age 67.

Asset model

- The asset portfolio of the life insurance company may be divided into 5 main asset classes:
  - Norwegian stocks(2%)
  - International stocks (10%)
  - Real estate (20%)
  - Credit bonds (33%)
  - Government bonds(35%)



- It is assumed that for all assets the value develops as

$$\log V_t = \log V_{t-1} + \epsilon_t.$$

where the E[ϵ_t] is chosen such that the relative return of the asset equals the risk-free 1-year interest rate.

Asset model

Volatilities:

Nor. stocks	21%,
Int. stocks	14%,
Real estate	8%,
Gov. bonds	1%

- For stocks, government bonds, and real estate: $\epsilon_t \sim N(\mu_t, \sigma^2)$
- For credit bonds: $\epsilon_t = \log(1 + R(t, t+D) + \gamma_t) + \psi_t,$
- Here γ_t is the change in the market value of the bond portfolio in year t, and ψ_t is the credit spread.
- The change in market value is computed by

$$\gamma_t = \frac{\exp(-(D-1)\,R(t+1, t+D))}{\exp(-(D-1)\,R(t, t+D))} - 1,$$

We assume that every year the credit bond portfolio is rebalanced to maintain a fixed duration D.

- The spread is assumed to be Gaussian: $\psi_t \sim N(0, \sigma^2_{spread})$

Best estimate

- The best estimate of the liabilities is computed as the probability-weighted average of future cash-flows, discounted to its present value:

$$\hat{L} = \frac{1}{S} \sum_{s=1}^{S} \sum_{t=1}^{T} d_{t,s} \, X_{t,s}.$$

We used
S=10,000 simulations
T=55 years

- Here, T is the time to ultimate run-off, S is the number of simulations, $X_{t,s}$ are the liability cash flows in year t and simulation s, and $d_{t,s}$ is the discount factor in the same year/simulation.

- The cash flows are given as the sum of guaranteed benefits, future discretionary benefits (FDB) and operating expenses minus the premiums.

Results: Old age pension

	CIR++	G++	Libor
Premiums	85.74	85.81	85.81
Guaranteed Benefits	7373.75	7383.09	7384.36
FDB	961.26	969.54	946.57
Expenses	326.03	327.15	326.78
Best estimate	**8575.32**	**8593.97**	**8571.92**

The development of the guaranteed benefits and the premiums is assumed to be deterministic, but the resulting cash flows are discounted at a stochastic interest rate.

Results: Paid-up polices

	CIR++	G++	Libor
Premiums	0.00	0.00	0.00
Guaranteed Benefits	9217.96	9237.49	9241.40
FDB	1202.97	1292.65	1290.46
Expenses	650.05	653.41	654.10
Best estimate	**11070.98**	**11183.52**	**11185.96**

Summary

- We have studied three interest rate models; the CIR++-model, the G++-model and the Libor Market model.
- Even when calibrated to the same historical data, the simulations from these models have very different mean value and volatility characteristics, especially far out into the future.
- However, when using these simulations when computing the best estimate of the liabilities, the differences between the models are surprisingly small.

Summary

- There might be several reasons for the small differences:
  - First, the discounted cash-flows far out in the future are less important for the best estimate than those in the first 10 years.
  - Second, we simulate interest rates that have similar maturities (1-3 years), meaning the perfect correlation induced by the one-factor model probably is not very wrong in principle.
- If our findings also are valid for other yield curve shapes and other portfolio weights, we would conclude that model transparency and ease of use should be the deciding factors, rather than which model is ideal in theory alone.

References

Aas, K., Neef, L. R., Raabe, D., and Vårli, I. D. (2014). A Simulation-Based ALM Model in Practical Use by a Norwegian Life Insurance Company. In D. Silvestrov and A. Martin-Löf, editor, *Modern Problems in Insurance Mathematics*. Springer.

Brace, A., Gatarek, D., and Musiela, M. (1997). The Market Model of Interest Rate Dynamics. *Mathematical Finance*, 7(2):127–155.

Brigo, D. and Mercurio, F. (2001a). A Deterministic-Shift Extension of Analytically-Tractable and Time-Homogeneous Short-Rate Models. *Finance and Stochastics*, 5(3):369–387.

Brigo, D. and Mercurio, F. (2001b). *Interest Rate Models - Theory and Practice*. Springer, Berlin.

Jong, F. D., Driessen, J., and Pelsser, A. (2000). Libor and swap market models for the pricing of interest rate derivatives: An empirical analysis. Discussion Paper 2000-35, Tilburg University, Center for Economic Research.

Miltersen, K., Sandmann, K., and Sondermann, D. (1997). Closed Form Solutions for Term Structure Derivatives with Lognormal Interest Rates. *Journal of Finance*, 52(1):409–430.

Packham, N. (2005). Correlation parameterization and calibration for the libor market model. Master Thesis at HfB - Business School of Finance & Management,Frankfurt am Main, Germany.

Park, F. C. (2004). Implementing Interest Rate Models: A practical guide. Capital Markets & Portfolio Research, Inc.

Svensson, L. E. O. (1994). Estimating and Interpreting Forward Interest Rates: Sweden 1992-1994. NBER Working Paper No. 4871.