Pay-as-you-go pension systems: Automatic balancing mechanism based on nonlinear programming to restore the sustainability

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Abstract

The aim of this paper is to design an automatic balancing mechanism to restore the sustainability and liquidity of a Pay-As-You-Go pension (PAYGO) system, based on minimising changes in the main variables, such as the contribution rate, normal retirement age and indexation of pensions. This mechanism, that uses nonlinear optimisation, identifies and applies an optimal path of these variables into a PAYGO system in the long run and absorbs fluctuations in longevity, fertility rates, life expectancy, salary growth or any other random events in a pension system.

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1 Introduction

The public pension systems are usually financed on a Pay-As-You-Go (PAYGO) basis where pensions for retirees are paid by the contributions of the working-age population. A successfully PAYGO system require a balance between the expenditure on pensions and the income from contributions made by the active workers over time, that is known as intergenerational solidarity[1], and long-term sustainability.

Birth rates have dramatically decreased and with continuous improvements in life expectancy due to improved health care and medical innovations, pensions are paid over a longer time horizon, which causes great difficulties when the system does not expect such improvements and raises serious concerns about the sustainability of the PAYGO pension systems. *The Congressional Budget Office* of the United States of America [14] projected that, in a few decades, their Social Security system will be begin paying out more in benefits than it collects in contributions. This imbalance will be growing and is expected to remain. In Europe, *The European Commission* (White Paper (2012) [18], Green Paper 2010 [17]) shows a clear increase in life expectancy at birth for males and females of 7.9 and 6.5 years respectively from 2010 to 2060. Additionally, the European population aged 60+ is estimated to become almost twice as high in 2060 as it was in the early 2000s. Moreover, in 2012, pension expenditure represented more than 10% of GDP and this is forecast to rise to 12.5% in 2060 in the EU. Furthermore, the recent global financial crisis and a severe worldwide public debt crisis have exerted additional stress on pension systems.

As a result of these events that negatively affect the financial health of pension systems, several countries are implementing changes in their systems. In Europe, the common trend is a wave of parametric pension adjustments including countries, such as France, Greece, Hungary, Romania and Spain, amongst others[2]. Well-defined rules and a transparent accounting about how to share risks, raise the costs connected with breaking the intergenerational contract and decrease the political uncertainty associated with public PAYGO pension systems (See Bovenberg (2003) [13] and OECD (2012, 2013) [28], [29]).

Boado-Penas et al (2008), some politicians, researchers and public opinion mistakenly consider the annual cash-flow deficit or surplus to be an indicator of the pay-as-you-go system's solvency/sustainability[3]; i.e. they confuse a liquidity indicator with a

[1]Haberman [22], define intergenerational solidarity as the willingness of different groups of people to participate in a common pool, sharing actual experience, including any losses emerging. In the context of public pension systems, the concept refers to both young and old generations.

[2]See Whitehouse (2009a) [47], Whitehouse (2009b) [48], OECD (2012) [28].

[3]According to Knell et al. (2006), the term sustainability has many definitions, though it almost always refers to the fiscal policies of a government, the public sector or the pension system. One of the most widely accepted definitions in the area of pensions is that of a position where there is no need to increase the contribution rate in the future. On the other hand, the concept of solvency refers to the

solvency indicator. In order to assess whether or not a system is solvent/sustainable, an actuarial balance must be compiled. The actuarial balance, according to Barr and Diamond (2009) [5], is necessary to have the 'big picture" of the whole system looking for explicit or implicit assets and not only focus on the future liabilities of the PAYGO system. The actuarial balance also supplies a positive incentive to improve financial management by eliminating or at least minimising the traditional mismatch between the planning horizons of electors and politicians -often only four years- and those of the system itself (Boado-Penas and Vidal-Meliá (2013) [11]).

Countries like Canada, Finland, Japan, Sweden, USA and UK publish in a regular basis official actuarial balances to contribute to the transparency of the pension system. In Sweden, the actuarial balance sheet in the accounting sense of the term is the basic source of information for the Swedish pension system. In the U.S., since 1941 an annual actuarial report, based on minimising the difference in present value between the spending on pensions and income from contributions, is produced every year [4].

Even though the information provided by the actuarial balance can help to take better and more informed decisions, some countries decide not to use it, i.e., USA, or take some parametric reforms via emergency modifications in legislation without compiling any balance and knowing the real situation of the financial health of the pension system. In this line, Vidal-Meliá et al. (2009) and (2010) [42] [43] propose *Automatic Balance Mechanisms* (ABM) defined as a set of pre-determined measures established by law to be applied immediately as required according to a solvency/sustainability indicator or any other indicator that reflects the financial health of the system. As examined by Turner (2009) [39] at least 12 countries link the indexation of pensions to life expectancy or any other type of indicator (For more information on ABM's see, for instance, Turner (2007, 2009) [38], [39], Vidal-Meliá et. al. (2009) [42], OECD (2012) [28], OECD (2013) [29]).

In this paper, we propose an automatic balancing mechanism based on nonlinear optimisation to restore not only the liquidity but also the sustainability of the system. The reserve fund associated with the optimisation problem acts as a buffer, fluctuating deliberately in the short run and absorbing partially or completely the uncertainty in mortality, fertility rates or any other random events. Our method also aims to keep the changes in the main variables of the pension system, such as the contribution rate, normal retirement age and indexation on pensions at a minimum level.

Following this introduction, the paper is structured as follows. The following section describes the calculation of the actuarial balance model. The third section describes the optimisation techniques in a PAYGO pension system and the mathematical preliminaries introducing the main notation and definitions. Section 4 describes the ABM

ability of a pension scheme's assets to meet the scheme's liabilities indicator.

[4]For interested readers see Settergren (2008) for Sweden, BOT(2014) for the USA, [36] [12].

proposed to restore the liquidity and sustainability into a PAYGO pension system. Section 5 shows a representative application given a population structure and suggests how an ABM model should be designed for both symmetric and asymmetric cases. Section 6 provides the concluding remarks.

2 The Actuarial Balance

The book, Actuarial Practice in Social Security, by Plamondon et al. (2002), was a first attempt at conceptualising the actuarial balance of the PAYGO system. After that, Boado-Penas and Vidal-Meliá (2013) and Billig and Ménard (2013) [9] describe the different types of actuarial balance for the PAYGO pension systems, including the Swedish NDC model.

According to Boado-Penas and Vidal-Meliá (2013), the main methodology used to compile the actuarial balance in nonfinancial DB systems would be described as an aggregate accounting projection model that compares the spending on pension with the income from contributions.

In general, the actuarial balance (AB) can be expressed as follows:

$$AB = W_0 \sum_{n=0}^{74} c_n N_n \prod_{h=1}^{n} \frac{1+g_h}{1+r_h} - P_0 \sum_{n=0}^{74} R_n \prod_{h=1}^{n} \frac{1+\lambda_h}{1+r_h} \tag{2.1}$$

where W_0 is the contribution base at year 0; c_n is the contribution rate at year n; N_n is the number of contributors at year n; g is the annual real wage growth; r is the discount rate; P_0 is the average pension at year 0; R_n is the number of pensioners at year n; and λ is the indexation of pensions.

Specifically, the actuarial balance in the US, compiled annually since 1941, follows the aggregate accounting methodology and is described as being the benchmark for DB systems [5]. This balance basically involves using the forecast demographic scenario to determine the future evolution of the number of contributors and pensioners according to the rules of the pension system.

In this line, aggregate accounting models are also calculated by Japan (AAD 2009 [2]), the United Kingdom (GAD 2015 [16]), Canada (Office of the Chief Actuary 2012 [30]); and Finland every two years (Risku et al. 2013 [33]). But these balances are not balance sheets in the traditional accounting sense of the term, with a list of assets and liabilities, like in the Swedish NDC pension system.

In Canada, actuarial valuation reports on the CPP are prepared every three years. These reports determine a minimum contribution rate and show projections of the

[5]For more details see the Board of Trustees of the Federal Old-Age and Survivors Insurance and Disability Insurance Trust Funds 2014 (OASDI 2014) [12]

Plans contributions, expenditures and assets for the next 75 years. Each actuarial valuation report is based on a number of best-estimate assumptions that reflect the best judgement of the Chief Actuary of the CPP as to future demographic and economic conditions (Billing and Ménard 2013 [9]).
The Japanese model is compiled at least every five years with a 95-year time horizon and includes an ABM that makes the system sustainable. Japan applies what is known as the "limited balance" or "closed period balancing" method, that is, the period of financial equilibrium is finite, whereas the whole future balancing method considers that the period of financial equilibrium is for a perpetual time horizon [6].
Furthermore, many countries have introduced automatic mechanisms linked to an indicator of the financial health of the system, its purpose, through successive application, is to re-establish the financial equilibrium of PAYGO pension systems with the aim of making those systems viable without the repeated intervention of the legislators. Vidal-Meli'a et al (2009) [42] argue that there are three reasons for introducing an ABM method, first, to adapt the system to changes in socioeconomic and demographic conditions; secondly, to create a credible institutional framework in the sense that promises of pension payments are kept; and finally, to minimise the use of the pension system as an electoral weapon.
This paper, that uses a nonlinear optimisation model, identifies and applies an optimal path of the contribution rate, normal retirement age and indexation of pension to make the system sustainable in the long run, based on the sustainability indicator that emerges actuarial balance.

3 Optimisation techniques in a PAYGO pension system

In the case of pension reforms, policymakers are interested in a smooth transition of the key variables of the pension system. With this in mind, this section presents the optimisation[7] techniques used to set the optimal and smooth paths of the key variables involved in the PAYGO system such as the contribution rate, the age of normal retirement and the indexation of pensions. We are particularly interested in getting an optimal path that guarantees not only the short run equilibrium (liquidity in the system) but also the long-run sustainability of the pension finance.
In a general nonlinear optimisation problem (NLO) the functions and parameters involved are:

[6] See Boado-Penas and Vidal-Meliá (2013) [11].
[7] Optimisation techniques are extensive used in Economics (See for example, Obstfeld and Rogoff (1996) [27], Wilcoxen (1989) [46], Walde (2010) [45]). However, there are no many application on pensions systems (See for example, Feldstein and Liebman (2002) [20], Roberts (2002) [34] and Sayan and Kiraci (2001) [35]).

(a) A function $f_n(d_n^j, n)$, that is called, the objective function of the NLO;

(b) The decision variables that are represented by $\{d\}_n = \{d_n^0, d_n^1, ..., d_n^k\} \in D$; where $D \in \Re^n$ is the decision space;

(c) The set of feasible solutions $\mathcal{F} = \{d_n^j \in D | h_k(d_n^j) = 0, k = 1, ...l$ and $g_j(D) \leq 0, j = 1, 2, ...m\}$; and $h_1, ..., h_l, g_1, ..., g_m$ are functions defined on D (or any open set containing D), and;

(d) If d_n^{j*} minimise (or maximise) f, then d_n^{j*} it is call an optimal solution of the NLO. If $\mathcal{F} = \emptyset$ then the NLO is infeasible, i.e., there are no d_n^j that minimise (or maximise) the objective function.

Specifically in this model $d_n^j = (c_n^j, x_n^{(r)j}, \lambda_n^j)$ where $\{c\}_{n\in\mathbb{N}}$ is the contribution rate, $\{x^{(r)}\}_{n\in\mathbb{N}}$ is the age of retirement and $\{\lambda\}_{n\in\mathbb{N}}$ is the indexation of pensions.
The objective function proposed in the ABM minimises the net present value of the difference between expenditure on pensions and income from contributions within a time horizon N. As stated by Nemirovski (1999) [26], numerical optimisation methods. The method is usually unable to find an exact solution in a finite number of computations. Therefore, there is an infinite sequence $\{d_n\}$ of approximate solutions to the NLO and the methods for numerical solving nonlinear optimisation problems are, in their essence, iterative routines . The next iterate d_{n+1} is formed, according to certain rules, on the basis of local information of the problem collected along the previous iterates. What is usually expected is that the sequence of iterates $\{d_n\}$ generated by the method in question converges, as $t \to \infty$, to the solution set of the problem, that is d^* with the measure chose equal to 0. Mathematically, the parts that conform to such a method are: i) an initial iterate d_0; ii) a search direction γ and a step size α.
The iterative method called Generalized Reduced Gradient (GRG) allows us to find the optimal solution of the NLO. The GRG is the extension of the gradient method to constrained and bounded optimisation problem. It was first developed by Abadie and Carpenter (1969) [1] and still today is one of the most powerful nonlinear optimisation algorithms [8]. In the GRG the basic idea is to replace the nonlinear equations (or inequalities) by their linear Taylor approximation at the current value of d, and then apply the reduced gradient algorithm to the resulting problem (Klerk, Roos and Terlaky(2006) [25]). These inequalities are modified to be equalities using a linear slack variable (see Venkataraman (2009) [41]). The new equality constraints are expanded into Taylor series. Then with these linear equations, the constraint equations are used to reduce the number of independent variables. The modified problem minimises $f(D)$ subject to $h_k(D) = 0; k = 1, 2, ..., l$ and $g_j(D) + q_{n+j} = 0; j = 1, 2, ...m$, $d_n^{min} \leq q_i \leq d_n^{max}, i = 1, 2, ..., v$ and $d_{p+j} \geq 0; j = 1, 2, ...m$ with $D = [d_1, d_2, ..., d_N]$.

[8]See, for example, Kallrath (1999) [24], Bazaraa, Sheraldi and Shetly (1993) [6] and Wright (1996) [49].

In general, the aim is to find an optimal $d^* \in D$ such that $f(d^*) \leq f(d), \forall d \in D$. The *cost* or *objective* function $f(d)$ is defined as $f_n(d_n^j, n)$ by eq.(4.1) and (4.4) where d_n^j is the permissible decisions that is chosen from the set D and n is the step of the process with $n \in N$. In our model, D consists of the values that the contribution rate, the age of normal retirement and the indexation of pensions can take every year n. The system constraints U could be, $g_j(D) \leq 0; j = 1, 2, ...m$; nonlinear constraint and $h_k(D) = 0; k = 1, 2, ..., l$; linear constraint equations. In the model, the constraints are defined by the upper and lower bounds, the rates of change of the key variables in each year, the liquidity restriction without including the dynamics of the fund for eq. (4.1) and with the dynamics of the fund for eq. (4.4).
In the next section, the formulation of the ABM is described. In real world problems, the modelling phase is as important and sometimes more difficult than the solving phase (Nemirovski (1999) [26]). This modelling phase is discussed in the following section.

4 Automatic Balancing Mechanism to restore the sustainability

This section proposes an Automatic Balancing Mechanism to restore the sustainability of a Pay-As-You-Go pension system in the long run without the government intervention. Optimisation techniques are applied in a nonlinear framework to find the optimal path for the decision variables -contribution rate, age of normal retirement and indexation of pensions- in a PAYGO pension scheme.
The system's sustainability is measured via the actuarial balance as the difference between the net present value of the income from contributions and the expenditure on pensions. The next subsections describe how different ABMs might be built.

4.1 Sustainability ABM (SA)

Following the actuarial balance methodology, this paper designs a Sustainability ABM (SA) to restore the sustainability of the PAYGO system according to the difference in present value between spending on pensions and income from contributions. This mechanism only takes into account the initial level of financial reserves (buffer fund), F_0, and no other deficit or surplus that may arise during the period of analysis, i.e., the SA only focuses on the actuarial projections of the income from contributions and expenditure on pension and not on the financial assets of the system.
The objective function used to determine the ABM, based on optimisation techniques, that makes the system sustainable calculating the optimal path for the contribution

rate, the indexation of pensions and the age of normal retirement is defined as follows:

$$min_{c_n x_n, \lambda_n} \sum_{n=0}^{N} \frac{c_n W_n(n, g, x_n^{(r)})}{(1+\delta)^n} + F_0 - \sum_{n=0}^{N} \frac{B_n(n, g, x_n^{(r)}, \lambda_n)}{(1+\delta)^n} \tag{4.1}$$

where δ is the discount rate and N the last year of analysis; c_n as the contribution rate during year n; W_n the total contribution base paid at n; B_n as the total expenditure on pensions at n that depends on the growth of salaries g, the retirement age at n, $x_n^{(r)}$, and the indexation of pensions at n, λ_n.

Equation 4.1, obviously, has a minimum equal to minus infinity so, in order to find a real optimal path of the key variables of the system (the contribution rate, indexation of pensions and age of normal retirement) and make the system sustainable in a N time horizon some restrictions are imposed. Firstly, in order to make the system sustainable, the present value of the income for contributions needs to be greater or equal than the expenditure on pensions, that is: $\sum_{n=0}^{N} \frac{c_n W_n(n, g, x_n^{(r)})}{(1+\delta)^n} \geq \sum_{n=0}^{N} \frac{B_n(n, g, x_n^{(r)})}{(1+\delta)^n}$.

Secondly, upper and lower bounds are set to the key variables $c_{min} \leq c_n \leq c_{max}; x_{min}^{(r)} \leq x_n^{(r)} \leq x_{max}^{(r)}; \lambda_{min} \leq \lambda_n \leq \lambda_{max}$. Thirdly, it is necessary a smooth penalty (increase in contribution and age of retirement or decrease in indexation) in these variables, thus a smooth constraint is imposed:

$$c_{1\Delta} \leq \frac{c_{n+1}}{c_n} \leq c_{2\Delta}; x_{1\Delta}^{(r)} \leq \frac{x_{n+1}^{(r)}}{x_n^{(r)}} \leq x_{2\Delta}^{(r)}; \lambda_{1\Delta} \leq \frac{\lambda_{n+1}}{\lambda_n} \leq \lambda_{2\Delta};\ c_{1\Delta}, c_{2\Delta}, x_{1\Delta}^{(r)}, x_{2\Delta}^{(r)}, \lambda_{1\Delta}, \lambda_{2\Delta} \in \Re$$

However, as we have discussed, a sustainable pension system does not imply a liquid one and vice versa. In order to achieve that the income from contributions covers the expenditure in pensions of the same year, it is necessary to incorporate the following liquidity constraint: $c_n W_n(n, g, x_n^{(r)}) \geq B_n(n, g, x_n^{(r)}, \lambda_n)$ for all n.

In summary, the Sustainability ABM is expressed by the following nonlinear minimisation problem that also guarantees the liquidity into the system:

$$min_{c_n x_n, \lambda_n} \sum_{n=0}^{N} \frac{c_n W_n(n, g, x_n^{(r)})}{(1+\delta)^n} + F_0 - \sum_{n=0}^{N} \frac{B_n(n, g, x_n^{(r)}, \lambda_n)}{(1+\delta)^n}$$

$$s.t. = \begin{cases} c_{min} \leq c_n \leq c_{max}; x_{min}^{(r)} \leq x_n^{(r)} \leq x_{max}^{(r)}; \\ \lambda_{min} \leq \lambda_n \leq \lambda_{max}; \\ c_{1\Delta} \leq \dfrac{c_{n+1}}{c_n} \leq c_{2\Delta}; x_{1\Delta}^{(r)} \leq \dfrac{x_{n+1}^{(r)}}{x_n^{(r)}} \leq x_{2\Delta}^{(r)}; \\ \lambda_{1\Delta} \leq \dfrac{\lambda_{n+1}}{\lambda_n} \leq \lambda_{2\Delta}; \\ \dfrac{c_n W_n(n, g, x_n^{(r)})}{B_n(n, g, x_n^{(r)}, \lambda_n)} \geq 1; \\ \sum\limits_{n=0}^{w} \dfrac{c_n W_n(n, g, x_n^{(r)})}{(1+\delta)^n} \geq \sum\limits_{n=0}^{w} \dfrac{B_n(n, g, x_n^{(r)}, \lambda_n)}{(1+\delta)^n} \end{cases} \tag{4.2}$$

The optimal solution for this ABM built by equation 4.2 should be zero, however, it is well-known that in practice it is impossible to have an exact actuarial equilibrium for the valuation problem, which is the reason to minimise this difference.

4.2 Other design of the Sustainability ABM (SAF)

This section proposes an alternative design of ABM when the buffer fund is considered during the whole analysis. The ABM under this design measures the system's financial health including not only the actuarial projections of the income from contributions and the pensions to be paid but also the accumulated value of the buffer fund. The Sustainability ABM with buffer fund (SAF) restores the sustainability of the pension system including also the buffer fund as a key parameter.

The buffer fund, or even returns that it generates, might have a positive effect on a pension system facing unexpected changes in the demographic or economic projections. For example, the interest generated by the buffer fund in Spain covered the shortfall in contributions during 2010 (Vidal-Meliá (2014) [44]). With this in mind, the SAF incorporates the financial assets of the PAYGO scheme. The premises for this ABM are derived from the basic equation in pay-as-you-go defined benefits pension system and, similar to Haberman and Zimbidis (2002) [22] the buffer fund (financial assets) is taken into account. This contingency fund is completely determined by the actuarial profits (or losses). The buffer fund evolves according to the following formula:

$$F_n = (1 + J_n)F_{n-1} + c_n W_n(n, g, x_n^{(r)}) - B_n(n, g, x_n^{(r)}, \lambda_n), \tag{4.3}$$

where J_n is the return [9] of the fund during year n; $c_n, x_n^{(r)}, \lambda_n$ are the key variables

[9] According to Yermo (2008) [51] one of the most remarkable aspects of the regulatory environment of the reserve funds surveyed is that with the exception of Ireland, Japan, Korea, and Sweden there are no major investment limitations. The only quantitative investment limit applied to the Irish NPRF is the prohibition to invest in Irish government securities. The Japanese GPIFs investments are mainly restricted to domestic listed equities and bonds. Sweden, for example, prohibits that more than 40% of

during year n.
In contrast to ABM build in 4.2, the SAF will focus in the minimisation of the present value of the contingency fund. The objective function to minimise is:

$$min_{c_n x_n, \lambda_n} \sum_{n=0}^{N} \frac{F_n(n, g, x_n^{(r)}, \lambda_n, J_n)}{(1+\delta)^n} \tag{4.4}$$

The same constraints for the contribution rate, the age of retirement and the indexation of pensions as in the Sustainability ABM (eq. 4.2) are imposed. However, the liquidity constraint is set now as: $F_n \geq 0$, for all n, with this restriction the system is assuring to cover the expenditure on pensions each year. Imposing this restriction allows the model to find a minimum greater or equal to zero. So, the optimisation problem for the SAF is as follows:

$$min_{c_n x_n, \lambda_n} \sum_{n=0}^{N} \frac{F_n(n, g, x_n^{(r)}, \lambda_n, J_n)}{(1+\delta)^n}$$

$$s.t. = \begin{cases} c_{min} \leq c_n \leq c_{max}; x_{min}^{(r)} \leq x_n^{(r)} \leq x_{max}^{(r)}; \\ \lambda_{min} \leq \lambda_n \leq \lambda_{max}; \\ c_{1\Delta} \leq \dfrac{c_{n+1}}{c_n} \leq c_{2\Delta}; x_{1\Delta}^{(r)} \leq \dfrac{x_{n+1}^{(r)}}{x_n^{(r)}} \leq x_{2\Delta}^{(r)}; \\ \lambda_{1\Delta} \leq \dfrac{\lambda_{n+1}}{\lambda_n} \leq \lambda_{2\Delta}; \\ F_n \geq 0 \end{cases} \tag{4.5}$$

The variable F_n fluctuates deliberately to absorb changes in fertility, mortality projections and any other events that might affect the liquidity and sustainability indicator in the pension system.
Under this particular design of ABM with the constraint that $F_n \geq 0$, two liquidity indicators can be calculated. The first one, which does not include the value of the buffer fund is expressed as $\dfrac{c_n W_n(n, g, x_n^{(r)})}{B_n(n, g, x_n^{(r)}, \lambda_n)}$. While the second one that includes the value of the accumulated fund is defined as $\dfrac{(1+J_n)F_{n-1} + c_n W_n(n, g, x_n^{(r)})}{B_n(n, g, x_n^{(r)}, \lambda_n)}$.

assets may be exposed to currency exchange risk. The reserve funds in these countries face additional restrictions intended to ensure diversification or to avoid direct control of corporations by reserve funds. By 2010, Belgium, United States and Spain invest the fund only in low-risk fixed income. Chile, Poland, Japan, Portugal and Mexico invest between 60 and 80% in fixed-income. France, Sweden, Norway, Canada and New Zealand invest approximately between 25 and 40% in fixed-income. (Severinson and Stewart (2012) [37]). As the scope of the paper is not the investment strategies for the buffer fund, the analysis is done assuming a risk-free investment rate as in the case of United States, Belgium and Spain.

4.3 Symmetric and Asymmetric cases for the Sustainability ABM

It is also possible to design symmetric and asymmetric cases under both designs. Palmer [31] states that under a symmetric ABM, any surplus that might arise would be automatically distributed. In the absence of a symmetric ABM, an undistributed surplus is maintained. Furthermore, Alho, Bravo and Palmer [4] state that the balancing mechanism can be symmetric, adjusting for both positive and negative deviations from the financial health indicator, or asymmetric, as in the Swedish system, which balances only if the solvency ratio is less than 1, i.e. the amount of assets is less than the amount of liabilities.

Under the *symmetric* design, our ABMs determines whether the contribution rate, the age of normal retirement (and indexation of pensions) are reduced (increased) when the system has a surplus or increased (decreased) in periods of deficit. Analytically, for the *asymmetric* case the change in the contribution rate and age of normal retirement are enforced to be strictly greater or equal to zero (strictly lower or equal to zero for the indexation of pensions). Under the *symmetric* case, the change in the variables could be positive or negative.

5 Numerical Example

This section presents a numerical example using the ABMs defined by the equations 4.2 and 4.5 developed in Section 4, and briefly summarises the key assumptions needed for this numerical example.

The total contribution base for year 1 is modelled as a function of the individuals' average wage[10], $wage(x)$ at age x, the number of people alive, $l_{x,1}$, at age x at the first year of study (that is, at time 1[11]), and the entry age into the labour market, x_e (for this particular example the entry age is 20). Thus, at time $n = 1$, the total contribution base, W_1, is modelled as:

$$W_1 = \left(\sum_{x=x_e}^{x_1^{(r)}-1} l_{x,1} * wage(x) \right).$$

For $n > 1$, the total contribution base, W_n, depends additionally on variables such as the growth of salaries, g and the normal retirement age, $x_n^{(r)}$, floor function and modulus operation, where the floor function is $\lfloor x_n^{(r)} \rfloor$; i.e. it maps a real number to the largest previous integer number, and $x_n^{(r)} mod \lfloor x_n^{(r)} \rfloor$ is the modulus operation that

[10]For simplicity, the terms wage, salary and contribution base are considered to be synonyms.

[11]We assume that $l_{x,n}$ is uniformly distributed over the year.

finds the remainder of the division $x_n^{(r)} / \lfloor x_n^{(r)} \rfloor$. Therefore, the expression of the total contribution base for n greater than 1 is:

$$W_n = \left(\sum_{x=x_e}^{\lfloor x_n^{(r)} \rfloor - 1} (l_{x,n}) wage(x)(1+g)^n \right) + (x_n^{(r)} mod \lfloor x_n^{(r)} \rfloor) l_{\lfloor x_n^{(r)} \rfloor, n} wage(\lfloor x_n^{(r)} \rfloor)(1+g)^n \tag{5.1}$$

The dynamics of $B(n)$ for $n > 1$ could be written as:

$$B_n = \left(1 - (x_n^{(r)} mod \lfloor x_n^{(r)} \rfloor) l_{\lfloor x_n^{(r)} \rfloor, n} \right) * P_{\lfloor x_n^{(r)} \rfloor, n} + \sum_{x=\lceil x_n^{(r)} \rceil}^{\omega} P_{x,n} l_{x,n} \tag{5.2}$$

where the expenditure on pensions at year n=1 is equal to:

$$B_1 = P_{x_1^{(r)},1} l_{x_1^{(r)},1} + P_{x_1^{(r)}+1,1} l_{x_1^{(r)}+1,1} + P_{x_1^{(r)}+2,1} l_{x_1^{(r)}+2,1} + ... = \sum_{x=x_1^{(r)}}^{\omega} P_{x,1} l_{x,1}$$

Thus, the expenditure on pensions, $B(n)$[12], depends on: the number of people alive, $l_{x,n}$, at age x in time n; the normal retirement age, $x_n^{(r)}$; the growth of salaries, g; the last age to which a person can survive, ω; the average pension at time n for pensioners aged $x_n^{(r)}$, $P_{x_n^{(r)},n}$; and the indexation of pensions, λ_n, that is dynamic over time.
The demographic factor (population structure) used for the analysis is the European Population from 2013 to 2087. The data come from Eurostat, which provides statistical information about the European Union (EU). During 2013, 3.63 contributors finance each pensioner. This ratio, known as the old-dependency ratio[13], worsens over time, reaching values of 1.93 by 2087. As it is important to extrapolate the conclusions of the analysis to other parts of the world with similar population characteristics, the demographic structure has been normalised.
The same salary structure and pensions over time have been used in order to make a comparison between different countries and results. The initial pension is set at 55% of last salary[14] and $P_{x,n}$ can be written as:

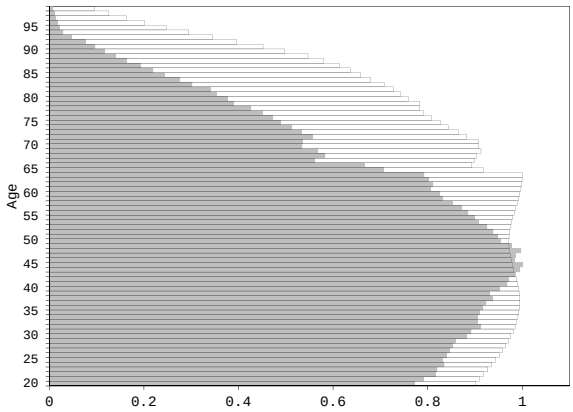
$$P_{x,n} = P_{x-1,n-1} * (1 + \lambda_{n-1}),$$

where $P_{x_n^{(r)},n} = P_{x_n^{(r)},n} \mathbb{1}_{\{x_n^{(r)},n=1\}} + P_{x_n^{(r)},n} * (1+g)^t \mathbb{1}_{\{x_n^{(r)},n>1\}}$.

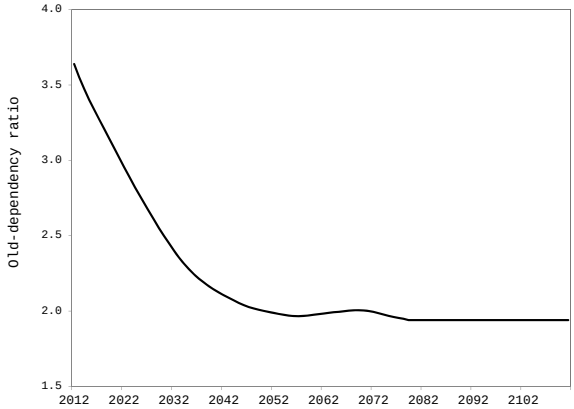
[12]The ceiling function $\lceil x_n^{(r)} \rceil$ maps a real number to the smallest next integer number.
[13]This ratio measures the number of elderly people relative to those of working age. It is calculated as the number of contributors divided by the number of pensioners.
[14]The level is in line with the average replacement rate, which measures the percentage of a worker's pre-retirement income that is maintained by their pension upon retirement in Europe, according to Creighton (2014) [15].

No unemployment is considered in our analysis. The annual salary growth is equal to 2.5% while the contingency fund is assumed to increase at an annual rate of 3%[15].
The lower bounds for the contribution rate, age of normal retirement and indexation of pensions are given respectively by 15%, 65 and -5%; the upper bounds are 20%, 70 and 5% respectively. For smooth changes, it is also assumed that the change in the contribution rate varies between 0.3% and 0.7%, the age of normal retirement between 1.5 and 4 months and the indexation of pensions between -0.5% and 0.5%. These values are in line with the most important reforms in the 34 OECD member countries between January 2009 and September 2013 (interested readers could see OECD (2013) [29])[16].

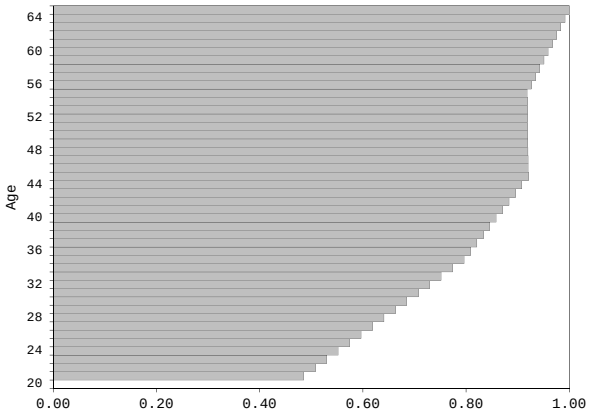
Figure 1: Normalised European population structure in 2013 (grey) and 2087 (transparent); Old-dependency Ratio, and; Salary Structure.



(a) European Population 2013 and 2087



(b) Old-dependency ratio European Population 2013 and 2087



(c) Salary structure European Population 2013

Figure 1 shows the different demographic factors that are the inputs in this numerical

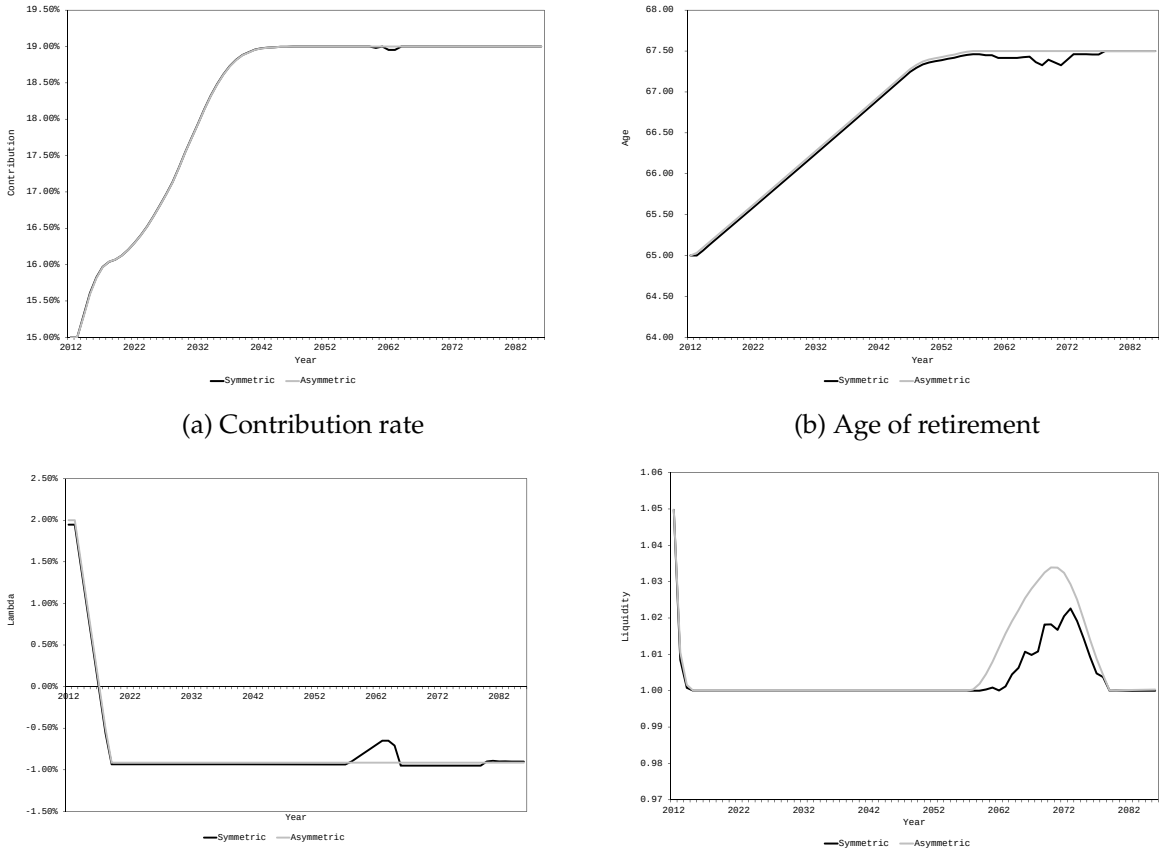
[15]This value, which is only used for the SAF, is in line with the average value for the past 15 years of the Euribor rates.

[16]The indexation of pensions could be less generous; for example, in Austria, Greece, Portugal and Slovenia, indexation of pensions has frozen automatic adjustments for all but the lowest earners. In Spain, the retirement age is being raised gradually from 65 to 67 years in 2027.

example. The pyramid (Figure 1a) represents the normalised mature population of the European Union in 2013 (grey) and 2087 (transparent) to highlight the differences. The old-dependency ratio is shown in Figure 1b, whereas the normalised salary structure is presented in Figure 1c.

The peak of population in 2013 (Figure 1a, grey pyramid), is at ages 40-50 in 2013 corresponding to the demographic boom in the 1960s and early 1970s. By 2087, there are no clear peaks in the population. The median age in 2013 is 55: by 2087, it is 64. From 2055 to 2075, the old-dependency ratio presents a population bump (Figure 1b).

Figure 2: Results of SA when the three variables are projected simultaneously for the symmetric (black line) and asymmetric (grey line) with European population structure.



(a) Contribution rate

(b) Age of retirement

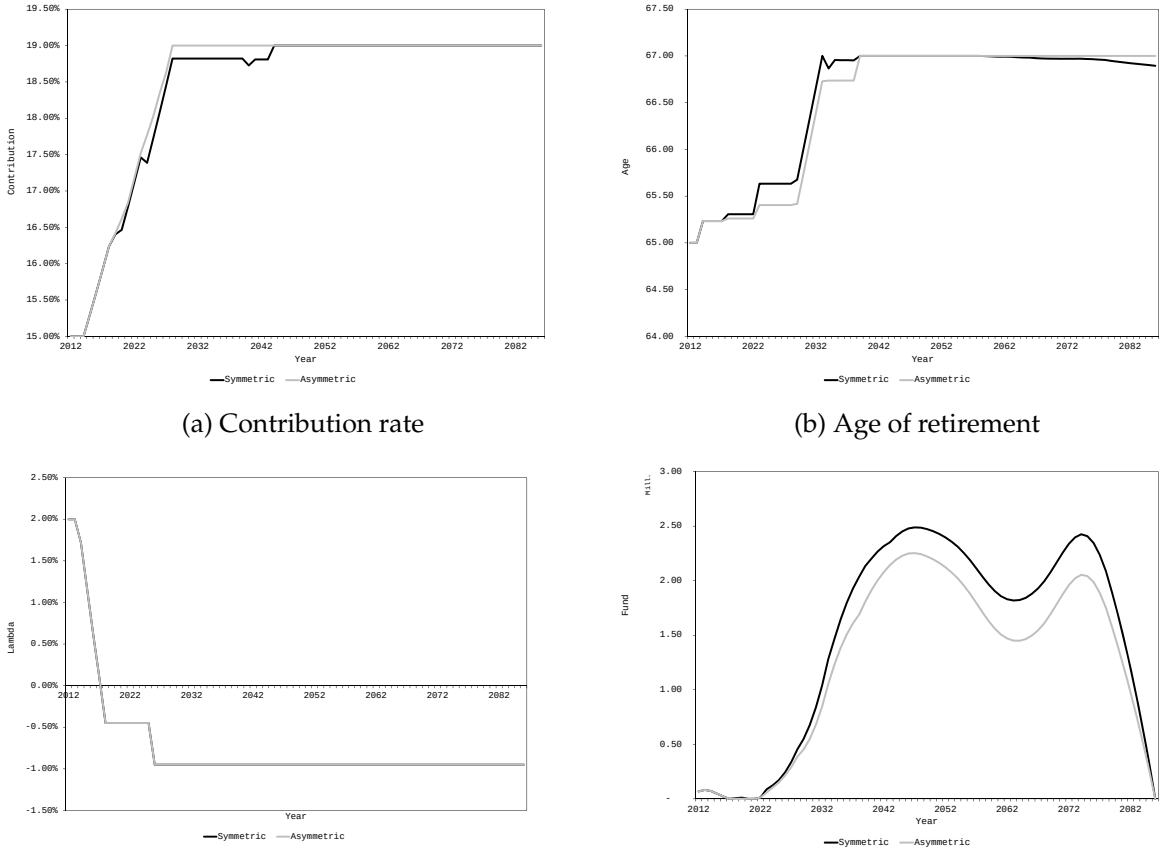
(c) Indexation of pensions

(d) Liquidity indicator without buffer fund

Figure 2 shows the optimal path for the contribution rate, the age of normal retirement and the indexation of pensions under the asymmetric (base case) and the symmetric scenarios for the Sustainability ABM. The results are very similar to each other. Under both scenarios, the contribution rate (Figure 2a) stabilises at 19%, varying only from 2060 to 2064 where the contribution rate is lower under the symmetric design. At the same time, the age of normal retirement (Figure 2b) stabilises at the age of 67.5 under both scenarios, although for the symmetric design the retirement age is lower than under the asymmetric design from 2055 to 2077. The indexation of pensions (Figure 2c) stabilises at -1% at the end of the period of analysis. The liquidity indicator, defined

as the ratio between the income from contribution and expenditure on pensions at every year, stabilises around 1. The difference between the symmetric and asymmetric designs is explained by the demographic factor: the old-dependency ratio increases between 2058 and 2069, so there are more active workers to finance the pensioners.

Figure 3: Results of the SAF when the three variables are projected simultaneously - for the symmetric (black line) and asymmetric scenario (grey line) with European population structure.



(a) Contribution rate

(b) Age of retirement

(c) Indexation of pensions

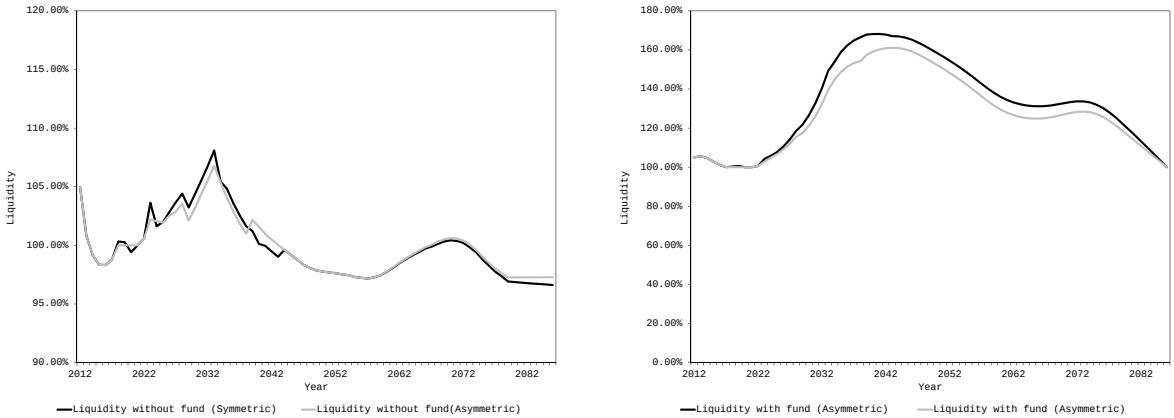
(d) Fund

Figure 3 shows the results of the SAF, which is the modified ABM that takes into account the buffer fund. Under this case, the system accumulates reserves; therefore, it is expected that the optimal path of the contribution rate, age of normal retirement (and indexation of pensions) take lower (higher) values than for the SA. Figure 3 shows that, for example, under the symmetric design (black line) the contribution rate (Figure 3a) needs to increase from 15% to 19%. However, the age of normal retirement, instead of increasing to 67.5, increases to 66.8 at the end of the study. The indexation of pension takes the same path as for the SA. The accumulated value of the fund, Figure 3d, under the symmetric design is slightly higher; this accumulation allows the system to decrease the age of normal retirement at the end of the analysis. Finally, the fund returns to a value equal to zero in the last year of our analysis.

Figure 4 shows the liquidity indicator without buffer fund (Figure 3a) and the liquidity

indicator with buffer fund (Figure 3b) for the SAF. The liquidity indicator with buffer fund remains always greater than 1: that is, the system always has enough money to meet the expenditure on pensions at every year. However, if we do not take into account the buffer fund (Figure 3a), the system presents a permanent deficit from year 2040 until the end of the study.

Figure 4: Liquidity indicator with and without buffer fund for the SAF under the symmetric (black line) and asymmetric (grey line) with European population structure.



(a) Liquidity indicator without buffer fund

(b) Liquidity indicator with buffer fund

The ABMs proposed have the ability to modify only one variable, instead of modifying the three of them simultaneously. For the SA, if the contribution rate is the only decision variable, it would need to increase to 31.43%, whereas if the age of retirement is the only variable it would increase to 78.04 years. If the indexation of pensions is set as the only decision variable, the pensions must decrease by -5.15% for more than 60 years consecutively.

5.1 Sensitivity analysis

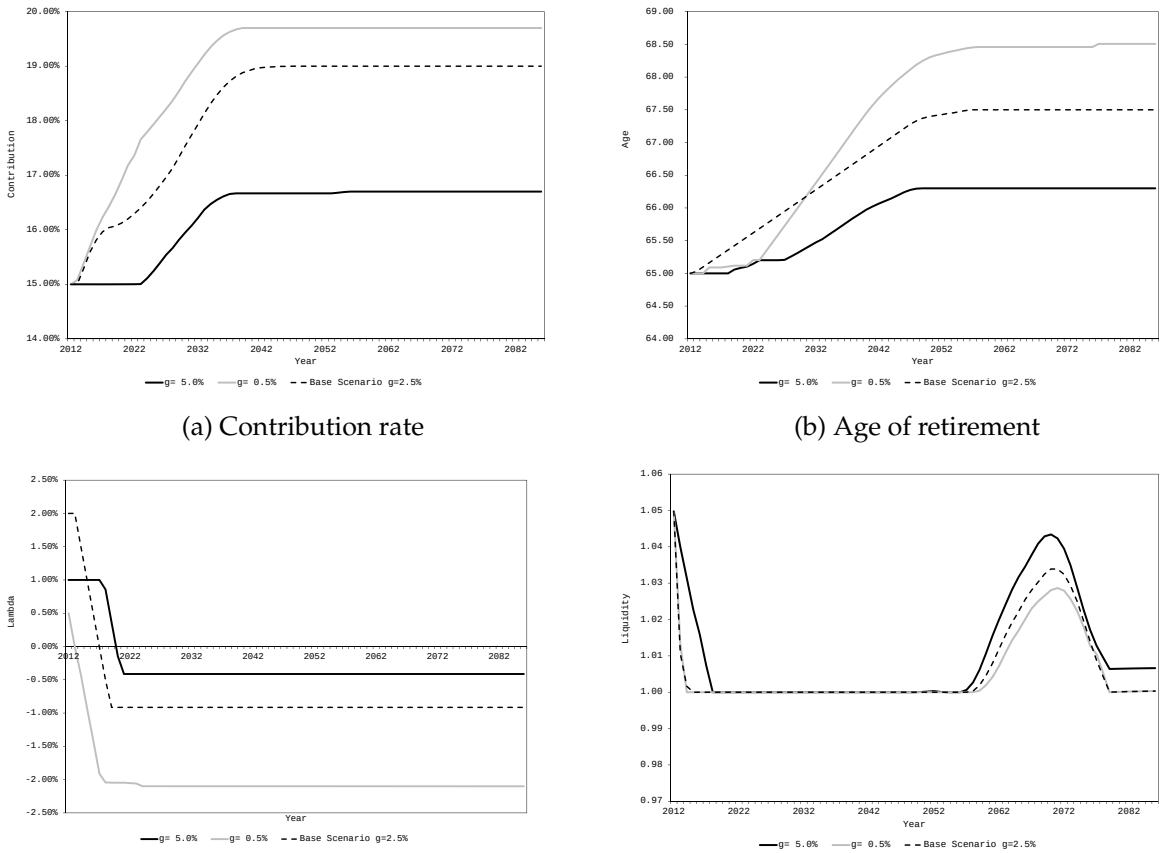
This section performs a sensitivity analysis of different levels of growth rate for salaries (g), percentages of the contribution base to calculate the initial pension and old-dependency ratio into our analysis in order to see how SA reacts to these effects. From now on, only the SA is analysed.

Different levels of growth of salaries (g)

This subsection considers two levels of growth of salaries (g) (Figure 5) equal to 0.5% (grey line) and 5% (black line) respectively. The results show that, in contrast to the base case scenario (g=2.5%), when the growth of salaries is 0.5%, the contribution rate stabilises at 0.7% higher (Figure 5a), the age of normal retirement 1 year above (Figure

5b) and the indexation of pensions decreases to -2.1% (Figure 5c) instead of -1%.
If we set the growth of salaries equal to 5%, the contribution rate stabilises 2.3% lower (Figure 5a), the age of normal retirement 1.2 years below (Figure 5b) and the indexation of pensions decreases to -0.4% (Figure 5c) instead of -1%.
The liquidity of the system (Figure 5d) remains almost within the same path, for both growth levels of 0.5% and 5%, so it is possible to conclude that the objective of the ABM is achieved.

Figure 5: Results of the Sustainability ABM under different levels of growth of salaries: 0.5% (grey), 5% (black) and base scenario (dashed).



(a) Contribution rate

(b) Age of retirement

(c) Indexation of pensions

(d) Liquidity

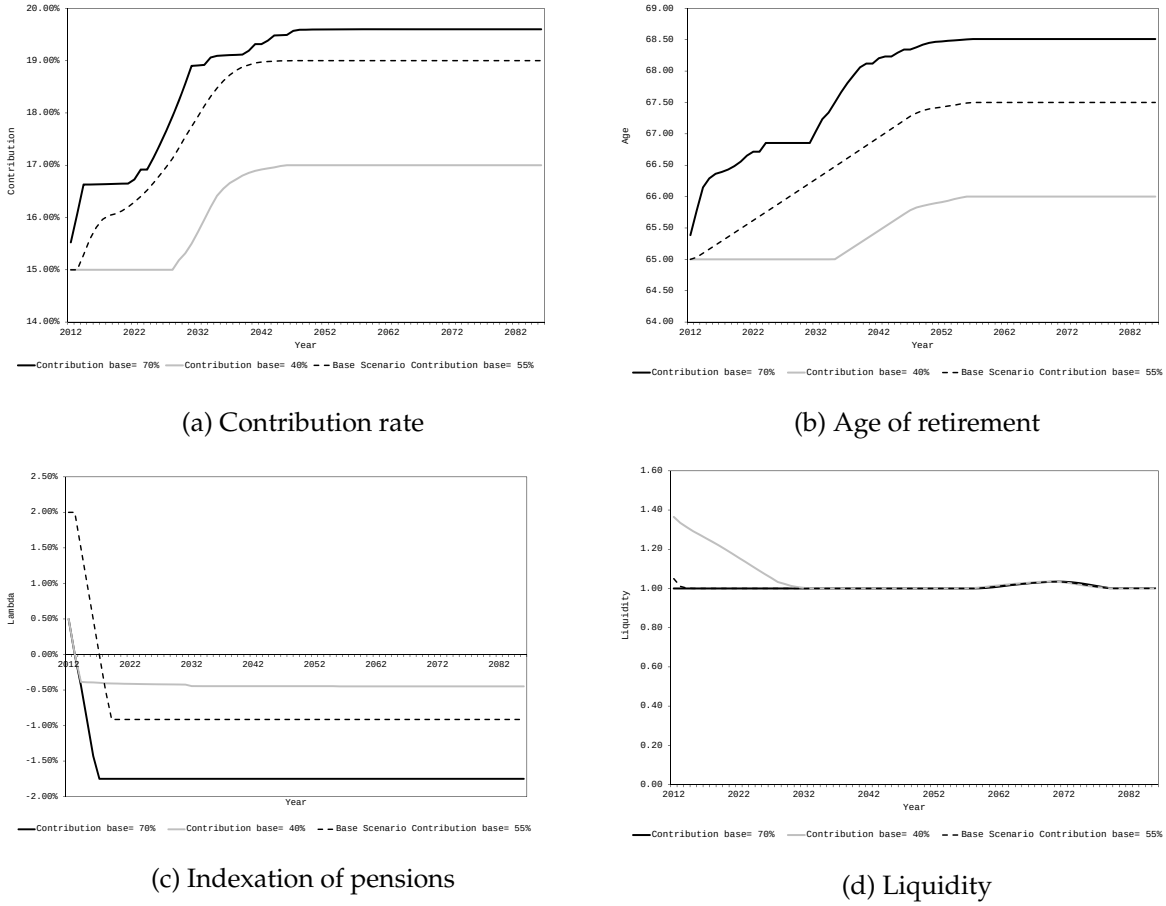
Different level of the contribution base

In this subsection, we analyse different percentage levels of the average contribution base to calculate the initial pension. For sensitivity analysis, the initial pension is set at 40% or 70% of the contribution base. The results are shown in Figure 6. The contribution rate (Figure 6a) stabilises at 17% and 19.6% respectively, instead of 19% in the base case scenario. The age of normal retirement stabilises at age 66 and 68.5 respectively, while under the base scenario it stabilises at 67.

The value of the indexation of pensions stabilises at -0.45% (for 40% of the contribution

bases) and -1.75% (for 70% of the contribution bases). The liquidity of the system is maintained around one for the whole of the period analysed; thus, the target of the ABM is better achieved with these levels of contribution base.

Figure 6: Results of the Sustainability ABM under different levels of the contribution base: 40% (grey), 70% (black) and base scenario (dashed).



(a) Contribution rate

(b) Age of retirement

(c) Indexation of pensions

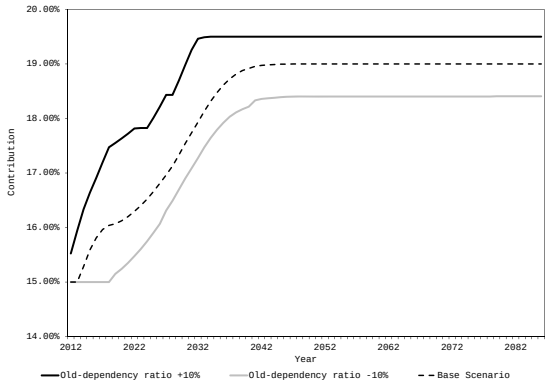
(d) Liquidity

Different levels of old-dependency ratio

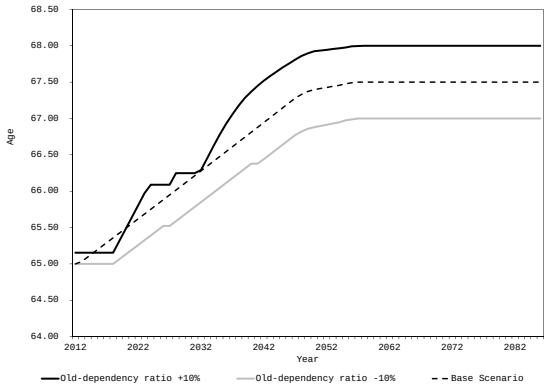
In this subsection, we consider a lower or higher life expectancy in the population, which translates into a smaller or greater number of pensioners in our study period. We suppose a constant of 10% more and 10% fewer pensioners[17]. When the old-dependency ratio (grey line) is lower (Figure 7a) the contribution rate stabilises below the base case scenario by 0.6%, the age of retirement (Figure 7b) by 0.5 years and the indexation of pensions above by 0.41%, whereas when the old-dependency ratio is higher by 10%, the contribution rate and the age of retirement are higher by 0.5% and 0.5 years respectively. The indexation of pensions (Figure 7c) is lower by 0.20%. In this case, liquidity of the system (Figure 7d) is obtained and, again, from 2058 to 2069 the system shows a surplus due to the demographic factor presented during these years.

[17]The constant increase or decrease in life expectancy affects all ages by the same percentage.

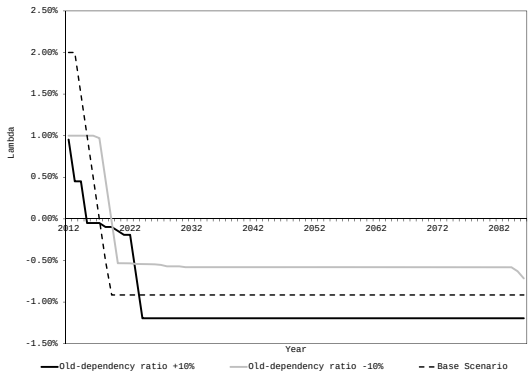
Figure 7: Results of the Sustainability ABM under different levels of old-dependency ratio: -10% (grey), +10% (black) and base scenario (dashed).



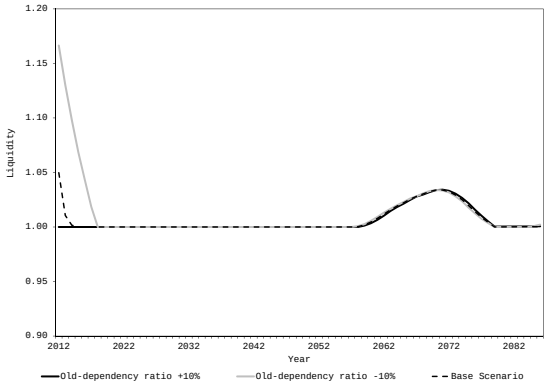
(a) Contribution rate



(b) Age of retirement



(c) Indexation of pensions



(d) Liquidity

6 Conclusions

Restoring the long-term sustainability of a Pay-As-You-Go pension system is on the agenda for most of the governments. The expenditure on pensions is usually increasing in line with the increase in longevity and decline in the fertility rates amongst any other random events while the income from contributions does not increase at the same rate.

Social security decisions to restore the sustainability usually involve political risk in the sense that the horizon of the policymakers is less than the system itself. This paper aims to provide a solution to restore the long-term sustainability of a PAYGO system using automatic balancing mechanisms, isolating the measures to be taken from the political arena. Using optimisation techniques, the ABMs presented in this paper involves the calculation of the optimal path of the contribution rate, age of retirement and indexation of pensions.

Some politicians, researchers and public opinion mistakenly consider the annual cash-flow deficit or surplus, that is, the liquidity indicator to be an indicator of the pay-as-you-go system's solvency/sustainability. Therefore, the two ABMs presented in

this paper focuses not only on restoring the sustainability of the system but also re-establish the liquidity into the system each year to face the expenditure on pensions. The ABMs built are based on minimising the present value of future cash flows, taking and not taking into account the contingency fund of the system. Under the base case ABM, no surpluses are accumulated in the system, whereas it is possible to build other ABM including the buffer fund, and accumulating reserves. Furthermore, the paper discusses two different designs that could be implemented, a symmetric or an asymmetric design.

Finally, based on the model presented in this paper, a number of important directions for future research can be identified. First, the ABMs presented in this paper involve the calculation of the optimal path of the contribution rate, age of retirement and indexation of pensions in a static framework. The model might calculate the optimal path of the key variables in a dynamic scenario, that is, calculating every 1, 3, 5 years the optimal path over the next N years, thus the model will recalibrate the parameters in a dynamic framework.

Second, it would be interesting to apply our model to different countries which have recently carried out parametric reforms and evaluate whether these reforms have any mathematical basis, in terms our optimisation method, given a long-term horizon.

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Appendices

Appendix 1 Actuarial Balance and Automatic Balancing Mechanism.

Table 1 shows some of the main countries that compile an Actuarial Balance Sheet and have an ABM in place[18]. The list is sorted alphabetically.

Table 1

Country	Type	AB	ABM	Comments
Canada	DB	Yes	Yes	Actuarial projections are carried every three years for a time horizon of 75 years, if the plan is not financially sustainable a (semi-automatic) mechanism will be triggered by increasing the contribution rate by the amount necessary to cover 50% of the deficit, and the benefits are frozen for three years.
Finland	DB	Yes	Yes	Every two years, an actuarial balance based on an aggregate accounting model projecting pension spending. The life expectancy coefficient automatically adjusts the amount of pensions in payment as longevity increases (or decreases).
France	DB	No	Yes	France's automatic adjustment mechanism operates maintaining a constant ratio between the duration of activity and the expected duration of retirement. However, the government has the right not to make these adjustments if labour market conditions do not support the extra years of work.
Germany	DB	Yes	Yes	Rather than draw up an actuarial balance at regular intervals in Germany, the European Union's Ageing Working Group makes a projection of pension spending using a time horizon of 53 years. The formula for revaluing pensions includes a sustainability factor that takes into account the system's rate of dependence.
Italy	NDC	No	Yes	Italy uses a transformation coefficient. This coefficient is reviewed every three years in line with changes in mortality rates at different ages up to 2019 and every two years after that date. However, the adjustment is not completely automatic, because it requires legislative approval.

[18]For a deep understanding see The American Academy of Actuaries 2011 [3], Boado-Penas 2009 [42], Boado-Penas and Vidal-Meliá 2013 [11], Billing and Ménard 2013 [9], The British Government Actuary's Department 2015 (GAD 2015) [16], Fall and Bloch 2014 [19], The Old-Age, Survivors, and Disability Insurance (OASDI) [12], OECD 2012 [28], OECD 2013 [29], Turner 2007 [38] and Turner 2009 [39].

Table 1

Country	Type	AB	ABM	Comments
Japan	DB	Yes	Yes	An actuarial report is compiled every five years. Based on this report, a modified indexation applied to both the revaluation of the contribution bases and the revaluation of pensions in payment. It takes into account both improvements in life expectancy and population increases (or decreases).
Latvia	NDC	No	Yes	Latvia uses unisex life expectancy at retirement age to convert the NDC account balance to an annuity. It bases life expectancy on projected cohort life tables, which are adjusted annually.
Norway	NDC	No	Yes	The system has unisex life-expectancy indexing of benefits at retirement. The life expectancy divisors are determined for each cohort, based mainly on remaining life expectancy.
Poland	NDC	No	Yes	Poland uses an annuity divisor which is revised annually. It is based on average life expectancy at retirement age.
Portugal	DB	No	Yes	The pension reform includes an indexation on benefits taking into account improvements in life expectancy. The reduction of benefits is based directly on the percentage change in life expectancy.
Sweden	NDC	Yes	Yes	The ABM is triggered if a solvency ratio is less than one and it consists basically of reducing the growth in pension liability, that is, the pension in payment and the contributors' notional accumulated capital.
US	DB	Yes	No	The actuarial balance of the OASDI program is aimed at measuring the system's financial sustainability with a 75-year time horizon. It measures the difference in present value - discounted by the projected yield on trust fund assets - between spending on pensions and income from contributions, expressed as a percentage of the present value of the contribution bases for that time horizon, taking into account that the level of financial reserves (trust fund) at the end of the time horizon reaches a magnitude of one-year's expenditure.
UK	DB	Yes	No	The British Government Actuary's Department is required to review the operation of the National Insurance Scheme of Great Britain at least every five years with a projection horizon of 62 years. Even though an official actuarial balance is compiled at five-year intervals in the United Kingdom, so far no kind of ABM has been incorporated to correct financial imbalances.

Appendix 2 Relationship between the two objective functions.

The two ABMs proposed are strongly linked since both of them are based on the income from contribution and the expenditure of pensions. Mathematically, from 4.4 the objective function for the SAF is :

$$min_{c_n x_n, \lambda_n} \sum_{n=0}^{N} \frac{F_n(n, g, x_n^{(r)}, \lambda_n, J_n)}{(1+\delta)^n} = \sum_{n=0}^{N} \frac{(1+J_n)F_{n-1} + c_n W_n(n, g, x_n^{(r)}) - B_n(n, g, x_n^{(r)}, \lambda_n)}{(1+\delta)^n}$$

$$= \sum_{n=0}^{N} \frac{(1+J_n)F_{n-1}}{(1+\delta)^n} + \sum_{n=0}^{N} \frac{c_n W_n(n, g, x_n^{(r)})}{(1+\delta)^n} - \sum_{n=0}^{N} \frac{B_n(n, g, x_n^{(r)}, \lambda_n)}{(1+\delta)^n}$$

$$= \sum_{n=0}^{N} \frac{c_n W_n(n, g, x_n^{(r)})}{(1+\delta)^n} + \sum_{n=0}^{N} \frac{(1+J_n)F_{n-1}}{(1+\delta)^n} - \sum_{n=0}^{N} \frac{B_n(n, g, x_n^{(r)}, \lambda_n)}{(1+\delta)^n} \tag{2.1}$$

Now, from equation 4.1, the SA is:

$$min_{c_n x_n, \lambda_n} \sum_{n=0}^{N} \frac{c_n W_n(n, g, x_n^{(r)})}{(1+\delta)^n} + F_0 - \sum_{n=0}^{N} \frac{B_n(n, g, x_n^{(r)}, \lambda_n)}{(1+\delta)^n} \tag{2.2}$$

Equation 2.1 and 2.2 differ only in one term, $\sum_{n=0}^{N} \frac{(1+J_n)F_{n-1}}{(1+\delta)^n}$ and F_0.

So, the relationship between the SA and the SAF lies in the way each one treats the buffer fund. The SA only takes into account the initial level of the buffer fund, F_0, whereas the SAF takes into account the accumulated value of the buffer fund at every year.