Guarantee valuation in Notional Defined Contribution pension systems

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Abstract

The notional defined contribution pension scheme combines pay-as-you-go financing and a defined contribution pension formula. The return on contributions is based on a notional rate which is linked to an external index set by law, such as the growth rate of GDP, average wages, or contribution payments. The volatility of this return may introduce a pension adequacy problem in the system and therefore guarantees may be needed. Here we focus on the guarantee of a minimum return on the contributions made to the pension scheme and we calculate its price in a utility indifference framework. We obtain a closed-form solution in a general dependence structure with exponential preferences and in presence of stochastic short rates.

Keywords: public pension, pay-as-you-go, nontraded assets, option pricing, incomplete markets, unhedgeable risks, exponential utility

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1 Introduction

A notional defined contribution (NDC) model is a pay-as-you-go (PAYG) financed system that deliberately mimics a financial defined contribution pension scheme, meaning that the pension depends on both contributions and its returns (Palmer, 2006 [20]). The returns on contributions are based on a notional rate that reflects the financial health of the system, which is linked to an external index set by law, such as the growth rate of GDP, average wages, or contribution payments. The account balance is called notional because contributions are not invested in financial markets as the system is based on pay-as-you-go financing. One of the shortcomings of NDC is the participants are exposed to some risks they were not in a defined benefit pension scheme. In particular, they are exposed to the risk that their contributions will have lower investment returns than expected, which will negatively affect their pension adequacy.

Pennacchi (1999) [21] proposes in this framework to offer a minimum return guarantee to each contribution made to the first pillar. He develops these guarantees in a funded public defined contribution scheme, like the ones developed in Chili, Uruguay or Colombia.[1] His setting has the advantage that the underlying asset can be hedged. Therefore the markets are complete and Black & Scholes classical formulae can be used (Black & Scholes (1973) [3]). However, in notional defined contribution schemes the underlying rate of return is based on an index which is not traded. We therefore have to find the price of an option written on a nontraded asset in an incomplete market settings.

There are different ways to price in incomplete markets due to the non-uniqueness of the martingale pricing measure. The academic literature offers different approaches to solve this problem. A first approach is to suppose that all derivatives must have the same market price of risk in order to ensure an internal consistency in the model (Björk (2004) [2]). The price is then the present value of the expected value under a 'risk-neutral' measure that depends on the specific market price of risk, which is usually chosen by the market.

Another way is by using Cochrane and Saa-Requejo's (2000) [8] approach. They relax the no-arbitrage assumption in order to derive tight bounds on asset prices based on the assumption that investors prefer assets with high Sharpe ratios. They use this setting to calculate bounds on options written on nontraded assets and find that common prices fit within these bounds. A slightly modified version of this setting is used in Floriou and Pelsser (2013) in order to obtain closed-form solutions for options in incomplete markets.

A third approach is to price via utility-indifference pricing. It consists on finding the price of risks which can't be hedged by incorporating the investor's or issuer's attitude towards the unhedgeable risk. In our case, the idea is to price the option on the untraded asset by using as a proxy a traded asset which is correlated to it. This approach has been extensively used in the literature since its introduction by Hodges and Neuberger (1989) [15]. They used this approach in order to value European calls in the presence of transaction costs.

Musiela and Zariphopoulou (2004) [18] and Zariphopoulou (2001) [27] developed an intuitive framework where they obtain closed-form formulas for prices written on nontraded

[1] See Queisser (1995) [23] for a discussion on pension reforms in Latin America.

assets in a market environment with lognormal dynamics. They do their study when risk preferences are exponential, which ensures that the pricing measure is independent of risk preferences and that it has the minimal entropy with respect to the historical measure. In the same line, Henderson (2002) [14] uses utility maximization and duality methods in order to obtain the option price in the case of the power and exponential utility. She finds that the prices are not very dependent of the utility chosen when the risk aversions match locally but that they show different behavior when the risk aversion is close to zero.

Rouge and el Karoui (2000) [24] obtain the price for a claim in the exponential utility case and related the price, modeled as a quadratic backward SDE, with minimal entropy. However, these papers assume that the short rate is deterministic. This hypotheses can be a drawback when studying claims of a long-term nature. Young (2004) [26] studies these prices when the short rate is stochastic and has an affine term structure. She uses this theoretical framework to price catastrophe risk bonds and equity-indexed term life insurance.

Other approaches include finding a super-replicating portfolio (see Cvitanic, Pham and Touzi (1998) [9]) with a payoff which is equal or higher than the payoff of the derivative in any state of the world or pricing via coherent risk measures (see Artzner et al. (1999) [1]). Here they use risk measures to study market and nonmarket risks without the complete markets assumption.

In this paper we apply the theory of utility indifference pricing in the particular case of a public pension system where the nontraded asset is driven by two distinct noises. One is associated to the mean wage increase in the economy and the other is linked to the working population increase. These two risks are allowed to be correlated to each other, as well as to all risks in the market. In this sense we generalise the setting of Musiela and Zariphopoulou (2001) [18] and Henderson (2002) [14] and apply it to public pension systems. Furthermore, we present the obtained price as an intermediary price between an insurance premium and a complete-market option price. When the nontraded asset is totally uncorrelated we find that our price is a zero-utility exponential insurance premium (Denuit, (1999) [11]). When the nontraded asset is completely correlated with the market we obtain the same expression as in Black & Scholes (1973) [3] for a deterministic short rate setting and Brigo and Mercurio (2007) [6] for the stochastic rate setting.

The paper is organized as follows. Section 2 presents the pension system as well as the financial market and the unhedgeable risks which will be priced. Details are given about the derivation of the pricing measure. In Section 3 the closed-form price for the option sought is developed and different particular cases are presented. In Section 4 we show some numerical illustrations comparing our prices to the Black & Scholes setting and the totally independent case. Section 6 and two appendices conclude the paper.

2 The Model

2.1 The Pension System

We develop the pension setting for a representative individual 'i' participating at the pension system aged x at time t. She starts her career at age x_0^i and retires at age x_r^i.

She participates in the compulsory public state pension system since $s_x^i = t - x + x_0^i$ and pays a fixed proportion $\pi \in (0,1)$ on her age and time dependent income $L(x,t;i)$ every year. The total individual contribution at time t is then $C(x,t;i)$ and earns a return up to retirement $r_x^i = t - x + x_r^i$. The return is based on a nontraded but observable index such as the growth rate of GDP, average wages, or contribution payments. This observable but unhedgeable rate is commonly known as notional rate in defined contribution pay-as-you-go financed pension systems Palmer (2006) [20]. This index depends on risks such as population evolution, inflation, or productivity.

The main critic concerning pure defined contribution pension schemes is that they transfer too much risk towards the participants. Pension benefits at retirement depend highly on the return associated to their contributions in absence of guarantees. Therefore, in our model the pension provider chooses to provide a minimum return guarantee to the participants in order to increase the pension adequacy. At time of retirement the return on the individual contribution from time t will be worth the maximum of the following expressions:

- $K(x;i) = (1+i_G)^{x_r^i - x}$
- $\frac{Y(r_x^i)}{Y(t)}$

where i_G is the guaranteed yearly rate of return and $\frac{Y(r_x^i)}{Y(t)}$ is the return between time t and time of retirement. The value at retirement for a contribution made at time t is then:

$$\begin{aligned} V(r_x^i, t) &= C(x,t;i)\max(K(x;i), \frac{Y(r_x^i)}{Y(t)}) \\ &= C(x,t;i)\left[\frac{Y(r_x^i)}{Y(t)} + \underbrace{\left(K(x;i) - \frac{Y(r_x^i)}{Y(t)}\right)^+}_{\text{Put Option}}\right] \end{aligned} \tag{2.1}$$

In order to give this guarantee the state should protect itself from the potential losses when the guarantee is triggered. He will then have to buy a put option and hold the nontraded asset. Mathematically, the state will have to price the following put option:

$$\left(K(x;i) - \frac{Y(r_x^i)}{Y(t)}\right)^+ \tag{2.2}$$

There is a guarantee associated to each contribution made during the participant's career. These guarantees share the characteristic that they trigger (or not) at the same moment of time, namely at time of retirement. If we were in a funded pension scheme we would be able to price the put option by means of the Black & Scholes formula. When financing

is pay-as-you-go we don't have this possibility. However, we can calculate the value of this put option by using the theory of indifference utility pricing.

In the remaining section we present the assets in the market as well as the processes that drive the contribution base of the pension system. The nontraded asset on which an European option is written will be observable at all time and corresponds to the pay-as-you-go contributions $\mathrm{Y(t)}$. Due to the unfunded nature of a pay-as-you-go pension plan we cannot buy or send shares on the underlying asset. Therefore, we work in an incomplete market setting as we cannot hedge the risk related to the nontraded asset.

2.2 The real assets

a) The cash asset level is given by

$$\mathrm{dS_0(t) = r(t)S_0(t)dt} \tag{2.3}$$

where $\mathrm{r(t)}$ is the short rate of return which can be deterministic or stochastic. The discounting process associated to the risk-free asset is denoted by

$$\mathrm{B(t) = B(0)\exp\left(-\int_0^t r(s)ds\right)}$$

b) Short-term stochastic interest rate driven by Vasicek term structure (Vasicek, 1977 [25]) under the real probability measure $\mathcal{P}$:

$$\mathrm{dr(t) = a(b - r(t))dt + \sigma_r dW_r(t)} \tag{2.4}$$

where $\mathrm{a, b}$ and σ_r are positive constants and

$$\mathrm{dr(t) = a(\frac{b^*}{a} - r(t))dt + \sigma_r d\widetilde{W}_r(t)} \tag{2.5}$$

under the risk-neutral measure $\mathcal{Q}$ with $\mathrm{d\widetilde{W}_t^r = dW_t^r + q_t dt}$ and $\mathrm{q_t = \frac{ab - b^*}{\sigma_r} = q}$.

c) The zero-coupon bond from time t to maturity T as given in Brigo and Mercurio (2007) [6]:

$$\mathrm{P(t,T) = E\left[e^{-\int_t^T r(s)ds}\right] = A(t,T)e^{-B(t,T)r(t)}} \tag{2.6}$$

where

$$A(t,T) = \exp\left\{\left(b - \frac{\sigma_r^2}{2a^2}\right)[B(t,T) - T + t] - \frac{\sigma_r^2}{4a}B(t,T)^2\right\} \tag{2.7}$$

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a} \tag{2.8}$$

This zero-coupon bond with maturity T has the following stochastic differential equation (SDE):

$$dP(t,T) = (r(t) - q\sigma_r B(t,T))P(t,T)dt - \sigma_r B(t,T)P(t,T)dW_r(t) \tag{2.9}$$

under the $\mathcal{P}$-dynamics and

$$dP(t,T) = r(t)P(t,T)dt - \sigma_r B(t,T)P(t,T)d\widetilde{W}_r(t) \tag{2.10}$$

under the $\mathcal{Q}$-dynamics. We will denote $\sigma(t,T) = \sigma_r B(t,T)$ for simplicity.

d) The risky asset at time t is solution of the SDE:

$$dS(t) = (r(t) + \lambda_S\sigma_S)S(t)dt + \sigma_S S(t)dW_S(t) \tag{2.11}$$

where $\lambda_S\sigma_S$ is the market price of risk, which is supposed to be constant.

The vector (W_S, W_r) is a two-dimensional brownian motion. The first one is associated to the risky asset, while the second one corresponds to the short term interest risk. These two brownian motions are defined on a probability space $(\Omega, \mathcal{F}, P)$ and the filtration $\mathcal{F}_s$ is the one generated by $W_r(s)$ and $W_S(s)$ for $0 \leq s \leq t$. We suppose furthermore that these two brownian motions are correlated, i.e., $E[W_r(t)W_S(t)] = \rho_{r,S}t$.

2.3 The liabilities

The contribution $C(x,t;i)$ made to the system creates a liability at the pension provider. Furthermore, in our framework we offer the guarantee of a minimum return which can be expressed in terms of a put option. We have stated that the guarantee at retirement depends on a index $Y(t)$ that can represent the growth of the GDP, contribution base, etc. We suppose that the index is based on the contribution base of the pension system (Börsch-Supan (2003) [5]).

a) The total contribution base received by the public pension provider is denoted by:

$$Y(t) = \pi P(t)L(t) \tag{2.12}$$

where π is the constant contribution rate, which is paid by all the workers to the public pension system, $P(t)$ is the total working population and $L(t)$ is the mean salary of the working population.

The working population at time t is the solution of the following stochastic differential equation:

$$dP(t) = \alpha_P P(t)dt + \sigma_P P(t)dW_P(t) \tag{2.13}$$

While the mean salary of the working population is the solution of:

$$dL(t) = gL(t)dt + \sigma_L L(t)dW_L(t) \tag{2.14}$$

The parameters α_P and g (resp. σ_P and σ_L) are the percentage drift (resp. the volatility) of the working population and mean salary, which are supposed to be constant. The vector (W_L, W_P) is a two-dimensional brownian motion. The first one is associated to the salary risk, while the second one corresponds to the population increase risk. We suppose furthermore that mean salary and population are correlated, i.e., $E[W_L(t)W_P(t)] = \rho_{L,P}t$. The stochastic differential equation associated to the contribution (2.12) is then given by the Ito's lemma:

$$dY(t) = \underbrace{\left(g + \alpha_P + \rho_{L,P}\sigma_L\sigma_P\right) Y(t)}_{\mu(Y(t),t)} dt + \underbrace{\sigma_L Y(t)}_{\sigma_1(Y(t),t)} dW_L(t) + \underbrace{\sigma_r Y(t)}_{\sigma_2(Y(t),t)} dW_P(t) \tag{2.15}$$

with $Y(s) = y \in \mathcal{R}$ and $t \in [s, T]$. The risks associated to this two-dimensional brownian motion can't be hedged in the market, which causes the market to be incomplete. However, these brownian motions are correlated to those presented earlier which are traded. In particular, we have:

$$E[W_r(t)W_L(t)] = \rho_{r,L}t; E[W_r(t)W_P(t)] = \rho_{r,P}t$$
$$E[W_S(t)W_L(t)] = \rho_{S,L}t; E[W_S(t)W_P(t)] = \rho_{S,P}t$$

with $\rho_{r,L}$,$\rho_{r,P}$,$\rho_{S,L}$ and $\rho_{S,P} \in [-1, 1]$.

b) The European claim on the nontraded asset $Y(t)$ is denoted by $G = g(Y(T))$ and is supposed to be exercised at maturity T. For example, in the case of a put option we have $g(Y(T)) = (K - Y(T))^+$.

2.4 The pricing measure

Proposition 1. *a) The minimal entropy pricing measure $\mathcal{Q}^T$ is given by the arbitrage-free forward measure:*

$$\eta(T) = \exp\left(-\frac{1}{2}\int_t^T (q+\sigma_r B(s,T))^2\,ds - \frac{1}{2}\int_t^T \left(\frac{\lambda_S - \rho_{r,S}q}{\sqrt{1-\rho_{r,S}^2}}\right)^2 ds\right)$$
$$\exp\left(-\int_t^T (q+\sigma_r B(s,T))\,dW_r(s) - \int_t^T \left(\frac{\lambda_S - \rho_{r,S}q}{\sqrt{1-\rho_{r,S}^2}}\right)^2 dZ_2(t)\right) = \frac{d\mathcal{Q}^T}{d\mathcal{P}} \quad (2.16)$$

where $Z_2(t)$ is a brownian motion, independent of $W_r(t)$ resulting from the Cholesky decomposition: $dW_S(t) = \rho_{r,S}dW_r(t) + \sqrt{1-\rho_{r,S}^2}dZ_2(t)$.

b) The expression of the nontraded asset $Y(t)$ (2.15) under the measure $\mathcal{Q}^T$ (2.16) is given by:

$$\begin{aligned} dY(t) = &\left(\mu(Y(t),t) - \sigma_r B(s,t)A_r - \frac{\lambda_S(A_s - \rho_{r,S}A_r) + q(A_r - \rho_{r,S}A_s)}{1-\rho_{r,S}^2}\right)dt \\ &+ A_r d\widetilde{W}_r(t) + \frac{A_S - \rho_{r,S}A_r}{\sqrt{1-\rho_{r,S}^2}}d\widetilde{Z}_2(t) + \sigma_2(Y(t),t)L_{44}dZ_4(t) \\ &+ (\sigma_1(Y(t),t)L_{33} + \sigma_2(Y(t),t)L_{43})\,dZ_3(t) \end{aligned} \quad (2.17)$$

where $Z_3(t)$ and $Z_4(t)$ are two brownian motions independent of $W_r(t)$ and $Z_2(t)$ issued from the Cholesky decomposition and L_{33}, L_{43} and L_{44} are components of the Cholesky decomposition matrix L.

Proof. a) The forward measure $\frac{d\mathcal{Q}^T}{d\mathcal{P}}$ is obtained by using the zero-coupon (2.6) as a numeraire:

$$\frac{d\mathcal{Q}^T}{d\mathcal{P}} = \frac{d\mathcal{Q}^T}{d\mathcal{Q}}\frac{d\mathcal{Q}}{d\mathcal{P}} \quad (2.18)$$

with

$$\begin{aligned} \frac{d\mathcal{Q}^T}{d\mathcal{Q}} &= \frac{\exp\left(-\int_t^T r(u)du\right)}{P(0,T)} \\ &= \exp\left(-\int_t^T q\sigma_r B(s,T)ds - \frac{1}{2}\int_t^T \sigma_r^2 B(s,T)^2 ds - \int_t^T \sigma_r B(s,T)dW_r(s)\right) \end{aligned} \quad (2.19)$$

and the risk-neutral measure is:

$$\frac{d\mathcal{Q}}{d\mathcal{P}} = \exp\left(-\int_t^T \vartheta^T(t)dW(t) - \frac{1}{2}\int_t^T ||\vartheta(t)||^2\right) \quad (2.20)$$

The vector $\vartheta(t)$ is given by:

$$\vartheta(t) = \sigma^T(t)\left(\left(\sigma(t)\sigma^T(t)\right)^{-1}(b(t) - r(t)1)\right)$$

where:

$$W(t) = \begin{pmatrix} W_r(t) \\ Z_2(t) \end{pmatrix}; (b(t) - r(t)1) = \begin{pmatrix} \lambda_S \\ -q\sigma(t,T) \end{pmatrix}$$
$$\sigma(t) = \begin{pmatrix} \rho_{r,S}\sigma_S & \sigma_S\sqrt{1-\rho_{r,S}^2} \\ -\sigma(t,T) & 0 \end{pmatrix}$$

where $Z_2(t)$ is a brownian motion independent of $W_r(t)$ which appears in $dS(t)$ (2.11) after a Cholesky decomposition:

$$dS(t) = (r(t) + \lambda_S\sigma_S)S(t)dt + \sigma_S\rho_{r,S}S(t)dW_r(t) + \sigma_S\sqrt{1-\rho_{r,S}^2}dZ_2(t)$$

We refer the reader to Frittelli (2000) [12] and Miyahara (1996) [17] for the proof concerning the fact that $\mathcal{Q}^T$ is a martingale measure and minimizes the entropy relative to the historical measure $\mathcal{P}$.

b) First of all, we will present the detailed Cholesky decomposition of $W_L(t)$ and $W_P(t)$. Let $\Sigma_W(t)$ be the variance-covariance matrix of the original brownian motion vector $W(t) = (W_r(t), W_S(t), W_L(t), W_P(t))$:

$$\Sigma_W = \begin{pmatrix} 1 & \rho_{r,S} & \rho_{r,L} & \rho_{r,P} \\ \rho_{r,S} & 1 & \rho_{S,L} & \rho_{S,P} \\ \rho_{r,L} & \rho_{S,L} & 1 & \rho_{L,P} \\ \rho_{r,P} & \rho_{S,P} & \rho_{L,P} & 1 \end{pmatrix} \tag{2.21}$$

We search lower-triangular matrix L such that $W = L{\cdot}Z$ where $Z(t) = (Z_1(t), Z_2(t), Z_3(t), Z_4(t))$ is a vector of independent brownian motions. The matrix L is obtained by means of Cholesky decomposition of Σ_W:

$$L_{j,j} = \sqrt{A_{j,j} - \sum_{k=1}^{j-1} L_{j,k}^2}$$
$$L_{i,j} = \frac{1}{L_{j,j}}\left(A_{i,j} - \sum_{k=1}^{j-1} L_{i,k}L_{j,k}\right)$$

where $A_{i,j}$ is the component of Σ_W of the ith row and j^{th} column. In particular, we have:

$$
\begin{aligned}
W_r(t) &= Z_1(t) \\
W_S(t) &= \rho_{r,S} Z_1(t) + \sqrt{1-\rho_{r,S}^2} Z_2(t) \\
W_L(t) &= \rho_{r,L} Z_1(t) + \frac{\rho_{S,L} - \rho_{r,L}\rho_{r,S}}{\sqrt{1-\rho_{r,S}^2}} Z_2(t) + L_{3,3} Z_3(t) \\
W_P(t) &= \rho_{r,P} Z_1(t) + \frac{\rho_{S,P} - \rho_{r,P}\rho_{r,S}}{\sqrt{1-\rho_{r,S}^2}} Z_2(t) + L_{4,3} Z_3(t) + L_{4,4} Z_4(t)
\end{aligned}
$$

The expression (2.15), omitting the dependence to $Y(t)$ and t for simplicity, becomes then:

$$
\begin{aligned}
dY(t) &= \mu dt + \sigma_1 \left(\rho_{r,L} dZ_1(t) + \frac{\rho_{S,L} - \rho_{r,L}\rho_{r,S}}{\sqrt{1-\rho_{r,S}^2}} dZ_2(t) + L_{3,3} dZ_3(t) \right) \\
&+ \sigma_2 \left(\rho_{r,P} dZ_1(t) + \frac{\rho_{S,P} - \rho_{r,P}\rho_{r,S}}{\sqrt{1-\rho_{r,S}^2}} dZ_2(t) + L_{4,3} dZ_3(t) + L_{4,4} dZ_4(t) \right) \\
&= \left(\mu(Y(t),t) - \sigma_r B(s,t) A_r - \frac{\lambda_S (A_s - \rho_{r,S} A_r) + q(A_r - \rho_{r,S} A_s)}{1-\rho_{r,S}^2} \right) dt \\
&+ A_r d\widetilde{W}_r(t) + \frac{A_S - \rho_{r,S} A_r}{\sqrt{1-\rho_{r,S}^2}} d\widetilde{Z}_2(t) + \sigma_2(Y(t),t) L_{44} dZ_4(t) \\
&+ \left(\sigma_1(Y(t),t) L_{33} + \sigma_2(Y(t),t) L_{43} \right) dZ_3(t)
\end{aligned}
$$

where $d\widetilde{W}_r(t) = dW_r(t) + (q + \sigma_r B(t,T))dt$ and $d\widetilde{Z}_2(t) = dZ_2(t) + \frac{\lambda_S - \rho_{r,S} q}{\sqrt{1-\rho_{r,S}^2}} dt$ are martingales under $\mathcal{Q}^T$ (2.16). The brownian motions $dZ_3(t)$ and $dZ_4(t)$ which are orthogonal to the space are unchanged by law. See Föllmer & Schweizer [13] for details.

□

2.5 The indifference price

In this market we have three real assets, the riskless cash asset $S_0(t)$, the zero-coupon bond $P(t,T)$ as well as the risky asset $S(t)$. The investor has an initial capital x and chooses dynamically his portfolio allocations $\vartheta_0(s)$, $\vartheta_P(s)$ and $\vartheta_S(s)$, respectively for the riskless asset, the zero-coupon bond with fixed maturity T and the risky asset for $t \leq s \leq T$. We suppose that there are no intermediate cash-flows corresponding to consumption or

contributions and that the portfolio on the traded assets is self-financing. The SDE of the wealth is:

$$\begin{aligned}\frac{dX(s)}{X(s)} &= \vartheta_0(s)\frac{dS_0(s)}{S_0(s)} + \vartheta_P(s)\frac{dP(t,T)}{P(t,T)} + \vartheta_S(s)\frac{dS(t)}{S(t)} \\ &= (r(s) + \vartheta_S(s)\lambda_S\sigma_S - \vartheta_P(s)\sigma(s,T))dt + \vartheta_S(s)\lambda_S dW_S(s) - \vartheta_P(s)q\sigma(s,T)dW_r(s)\end{aligned} \tag{2.22}$$

with $X_t = x$, $0 \leq t \leq s \leq T$. We assume that $\vartheta_S(s)$ and $\vartheta_P(s)$ are $\mathcal{F}_t$ progressively measurable and satisfy the integrability condition $E\left[\int_t^T \vartheta_i^2(s)ds\right] < \infty$ for $i = P, S$. The set of admissible policies which satisfy this condition is denoted by A.

The individual risk preferences are modeled via the exponential utility function (2.23):

$$U(x) = -e^{-\gamma x}, \gamma > 0 \tag{2.23}$$

The indifference price of the option is deduced by comparing the expected utility of the wealth at maturity with or without presence of the option as presented in the following definition:

Definition 1. *The indifference price on the claim* $G = g(Y(T))$ *is defined as the function* $h = h(x, y, r, t)$*, such that the investor is indifferent to optimize its expected utility of the wealth without taking the derivative into account or to optimize it taking into account the price of the derivative at writing's time and its claim at maturity , i.e.,*

$$V(x, r, t) = u(x + \lambda h(x, y, t), y, r, t) \tag{2.24}$$

where

$$V(x, r, t) = \sup_A E_P\left[U(X(T))|X(t) = x, r(t) = r\right] \tag{2.25}$$

is the writer's value function when the derivative is not taken into account and

$$\begin{aligned}u(x + \lambda h(x, y, t), y, r, t) &= \sup_A E_P\left[U(X(T) - \lambda G)|X(t) = x, Y(t) = y, r(t) = r\right] \\ &= \sup_A E_P\left[U(X(T))h(Y(T))|X(t) = x, Y(t) = y, r(t) = r\right]\end{aligned} \tag{2.26}$$

is the writer's value function when the derivative is taken into account, and λ *is the amount of claims sold or bought. A parameter* $\lambda > 0$ *entails that we sell the claim, and* $\lambda < 0$ *means that we buy the claim. The claim is supposed to be written at time* t *and no trading of the asset is allowed in the remaining time period up to maturity* T *(European claim).*

Remark 1. Note that the expression of $h(Y(T))$ involves the calculation of the expected value of an exponential function. There have to be extra constraints concerning the function G and λ in order to have a finite expected value. The price as stated in Definition 1 is finite for bounded functions, for instance, for put options (for all λ) and for call options when $\lambda = -1$. If we price a selling call option ($\lambda = 1$) we would have in the calculation the integral of an unbounded process up to infinity and the price would be unbounded.

In the following section we calculate the expression of the price in this market setting.

3 The price of the guarantee

We calculate the expression of (2.25) in Theorem 1, then the expression of (2.26) in Theorem 2 and use condition (2.24) in order to deduce the price h in Proposition 2. The expression of the value function is in both cases calculated by means of Hamilton-Jacobi-Bellman (HJB) equation methodology. We would like to note that in our HJB equation a term $r(t) \cdot x$ appears which makes it impossible to apply the Lipschitz conditions of the usual verification theorems, as they are both unbounded processes. However, Korn & Kratz (2002) [16] give a suitable verification result in their paper which allows to proceed with the HJB equation in presence of stochastic interest rates with the usual three-step procedure. For a detail of the verification result in presence of stochastic interest rate, as well as the verification of the solution we invite the reader to consult Korn & Kratz (2002).

Theorem 1 (Value function in absence of the derivative)**.** *The writer's value function* $V(x, r, t)$ *(2.25) when the derivative is not taken into account is given by:*

$$V(x, r, t) = -\exp\left(-\frac{\gamma x}{P_Q(t, T)} - \int_t^T \frac{1}{2}\left(\frac{\lambda_S^2 + q^2 - 2\rho_{r,S}\lambda_S q}{1 - \rho_{r,S}^2} + \sigma_r^2 B(s, T)^2 + 2q\sigma_r B(s, T)\right) ds\right) \tag{3.1}$$

where $B(s, T)$ *is given by (2.8) and* $P_Q(t, T)$ *is the expression of the zero-coupon bond under the* $\mathcal{Q}$*-dynamics.*

Proof. See Appendix A.1. □

In the following theorem we develop the value function in the presence of the derivative at maturity.

Theorem 2 (Value function in presence of the derivative)**.** *The writer's value function* $u(x, y, r, t)$ *(2.26) when the derivative is taken into account is given by:*

$$\begin{aligned} u(x, y, r, t) = &-e^{\left(-\gamma \frac{x}{P_Q(t,T)} - \int_t^T \frac{1}{2}\left(\frac{\lambda_S^2 + q^2 - 2\rho_{r,S}\lambda_S q}{1-\rho_{r,S}^2} + \sigma_r^2 B(s,T)^2 + 2q\sigma_r B(s,T)\right) ds\right)} \\ &\times \left(E_{\mathcal{Q}^T}\left[e^{\frac{\gamma}{\delta} g(Y(T))} | Y(t) = y\right]\right)^{\delta} \end{aligned} \tag{3.2}$$

for $(x, y, r, t) \in \mathcal{R} \times \mathcal{R} \times \mathcal{R} \times [t, T]$ *with*

$$\mathcal{Q}^T(A) = E_{\mathcal{P}}\left[\eta(T) I_A\right]; A \in \mathcal{F}_T.$$

where $\eta(T)$ *is given by (2.16),* $Y(t)$ *is given by (2.17) and* δ *as (A.15).*

Proof. See Appendix A.2 □

Proposition 2 (Price of the European option)**.** *The price of an European option* $G = g(Y(T))$ *in a market where:*

- *the cash asset earns a stochastic short-term rate given by (2.4)*

- *the zero-coupon asset's dynamics are given by (2.6),*
- *the risky asset's dynamics are given by (2.11)*
- *the nontraded asset is given by the dynamics (2.15),*
- *and the individual preference is given by the exponential utility (2.23),*

is given by

$$h(x,y,r,t) = P_Q(t,T)\frac{\delta}{\gamma}\log\left(E_{\mathcal{Q}^T}\left[e^{\frac{\gamma}{\delta}g(Y(T))}|Y(t)=y\right]\right) \tag{3.3}$$

Proof. The price is issued by searching the h which makes the equivalence (2.25)=(2.26) hold. □

Remark 2. Note that the price of the derivative doesn't depend on the initial capital and only on the expression of the index under the pricing measure, the utility function, its level of risk aversion as well as the correlation. This is a consequence of using the exponential utility. Rouge and El Karoui (2000) [24] note that the initial capital independence may not be desirable in some cases because it is unlikely that wealthier individuals would give the same price to a claim as poorer agents would, as in the case of stock options.

The corollaries that follow present different particular cases of this general setting.

Corollary 1 (Price with deterministic short interest rate)**.** *The price of an European option* $G = g(Y(T))$ *in a market where:*

- *the cash asset earns a deterministic short-term rate,*
- *the risky asset's dynamics are given by (2.11)*
- *the nontraded asset is given by the dynamics (2.15),*
- *and the individual preference is given by the exponential utility (2.23),*

is given by the expected value of the utility of the contingent claim under the risk-neutral measure (2.20):

$$h(x,y,r,t) = e^{-\int_t^T r(s)ds}\frac{\delta}{\gamma}\log\left(E_{\mathcal{Q}}\left[e^{\frac{\gamma}{\delta}g(Y(T))}|Y(t)=y\right]\right) \tag{3.4}$$

with $Y(t)$ *given by:*

$$\begin{aligned} dY(t) &= (\mu(Y,t) - \lambda_S A_s)\,Y(t)dt + A_s\,(dW_S(t) + \lambda_S dt) \\ &\quad + \sigma_1(Y,t)\sqrt{1-\rho_{S,L}^2}dW_1(t) + \sigma_2(Y,t)\sqrt{1-\rho_{S,P}^2}dW_2(t) \end{aligned}$$

where $W_S(t)$ *is independent of* $W_1(t)$ *and* $W_2(t)$ *(Cholesky decomposition). In this case* δ *becomes*

$$\delta = \frac{\sigma_1^2 + \sigma_2^2 + 2\rho_{L,P}\sigma_1\sigma_2}{\sigma_1^2\left(1-\rho_{S,L}^2\right) + \sigma_2^2\left(1-\rho_{S,P}^2\right) + 2\sigma_1\sigma_2\left(\rho_{L,P} - \rho_{S,L}\rho_{S,P}\right)} \tag{3.5}$$

Proof. If the short interest rate is deterministic we have $\sigma_r = 0$, $q = 0$ and $\rho_{r,i} = 0$ for $i = S, L, P$. Therefore the zero-coupon bond is equal to the cash asset (2.3). □

Remark 3. Note that if $r = 0$ and $\sigma_2 = 0$ we obtain the same value as in Musiela & Zariphopoulou (2004) [18].

Corollary 2 (Price when population is independent)**.** *The price of an European option* $G = g(Y(T))$ *in the market of Proposition 2 when the population is independent from the market and salary risks is given by:*

$$h(x, y, r, t) = P_Q(t, T)\frac{\delta}{\gamma} \log\left(E_{\mathcal{Q}^T}\left[e^{\frac{\gamma}{\delta}g(Y(T))}|Y(t) = y\right]\right) \tag{3.6}$$

with $Y(t)$ *given by (2.17) when* $\rho_{i,P} = 0$ *with* $i = r, S, L$ *and* δ *becomes*

$$\delta = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2\left(1 - \frac{\rho_{S,L}^2 + \rho_{r,L}^2 - 2\rho_{r,S}\rho_{S,L}\rho_{r,L}}{1-\rho_{r,S}^2}\right) + \sigma_2^2} \tag{3.7}$$

Proof. In this case we have that $\rho_{i,P} = 0$ for $i = r, S, L$ therefore $A_r = \rho_{r,L}\sigma_1$, $A_S = \rho_{S,L}\sigma_1$. □

Corollary 3 (Price when the nontraded asset is uncorrelated)**.** *The price of an European option* $G = g(Y(T))$ *in the market of Proposition 2 is given by the exponential premium under the real measure* $\mathcal{P}$*:*

$$h(x, y, r, t) = P_Q(t, T)\frac{1}{\gamma} \log\left(E_{\mathcal{P}}\left[e^{\gamma g(Y(T))}|Y(t) = y\right]\right) \tag{3.8}$$

with $Y(t)$ *given by its original* $\mathcal{P}$*-form (2.15) when* $\rho_{i,L} = \rho_{i,P} = 0$ *with* $i = r, S$.

Proof. In this case we have that $\rho_{i,L} = \rho_{i,P} = 0$ for $i = r, S$, therefore $A_r = 0$, $A_S = 0$. The coefficient δ becomes

$$\delta = \frac{\sigma_1^2 + \sigma_2^2 + 2\rho_{L,P}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 + 2\rho_{L,P}\sigma_1\sigma_2} = 1 \tag{3.9}$$

□

Remark 4. Note that the price of the derivative in this case collapses to the exponential insurance premium, see Denuit (1999) [11] .

Corollary 4 (Limit to the Black & Scholes setting)**.** *The price of an European option* $G = g(Y(T))$ *(3.3) in the Black & Scholes setting tends to the utility-free expectation of* G *under the forward measure:*

$$h(x, y, r, t) = P_Q(t, T)E_{\mathcal{Q}^T}\left[g(Y(T))|Y(t) = y\right] \tag{3.10}$$

with $Y(t)$ *given by (2.17) when* $\rho_{S,L} = \rho_{S,P} = 1$ *and* $\rho_{r,L} = \rho_{r,P} = 0$.

Proof. Let $\rho_{S,L} = 1$ and $\rho_{S,P} = 1$. This means that the salary and population risks are totally correlated with the market risk $W_S(t)$. Furthermore, let the correlation between the stochastic interest rate and the risky asset $\rho_{r,S}$ be equal to 0. In this case we have $A_S = \sigma_1 + \sigma_2$, $A_r = 0$ and δ becomes:

$$\delta = \underbrace{\frac{\sigma_1^2 + \sigma_2^2 + 2\rho_{L,P}\sigma_1\sigma_2}{2\sigma_1\sigma_2}}_{A(\rho_{L,P})} \underbrace{\frac{1}{(\rho_{L,P} - 1)}}_{1/B(\rho_{L,P})} = \frac{A(\rho_{L,P})}{B(\rho_{L,P})} \tag{3.11}$$

When $\rho_{L,P} = 1$, $\frac{1}{\delta} = 0$ and the SDE has only one source of risk and mimics the traded asset $S(t)$ completely. Let:

$$f(G) = \exp\left(\frac{\gamma}{\delta}G\right) = \exp\left(\frac{\gamma B(\rho_{L,P})}{A(\rho_{L,P})}G\right) \tag{3.12}$$

The Taylor series of $f(G)$ around 0 is:

$$f(G) \approx f(0) + f^{'}(0)G + \sum_{n=2}^{\infty} \frac{f^n(0)}{n!}G^n \tag{3.13}$$

with

$$f^i(0) = \frac{\gamma^i B(\rho_{L,P})^i}{A(\rho_{L,P})^i}$$

Replacing (3.13) in (3.3), and simplifying the notation $A(\rho_{L,P}) = A(\rho)$,$B(\rho_{L,P}) = B(\rho)$, $E_{\mathcal{Q}^T}[G|Y(t) = y] = E_{\mathcal{Q}^T}[G]$ and $h(x, y, r, t)^* = h^*$:

$$h^* = P_Q(t,T)\frac{A(\rho)}{\gamma B(\rho)} \log\left(1 + \frac{\gamma B(\rho)}{A(\rho)}E_{\mathcal{Q}^T}[G] + \sum_{n=2}^{\infty} \frac{\gamma^n}{n!}\frac{B(\rho)^n}{A(\rho)^n}E_{\mathcal{Q}^T}[G^n]\right)$$

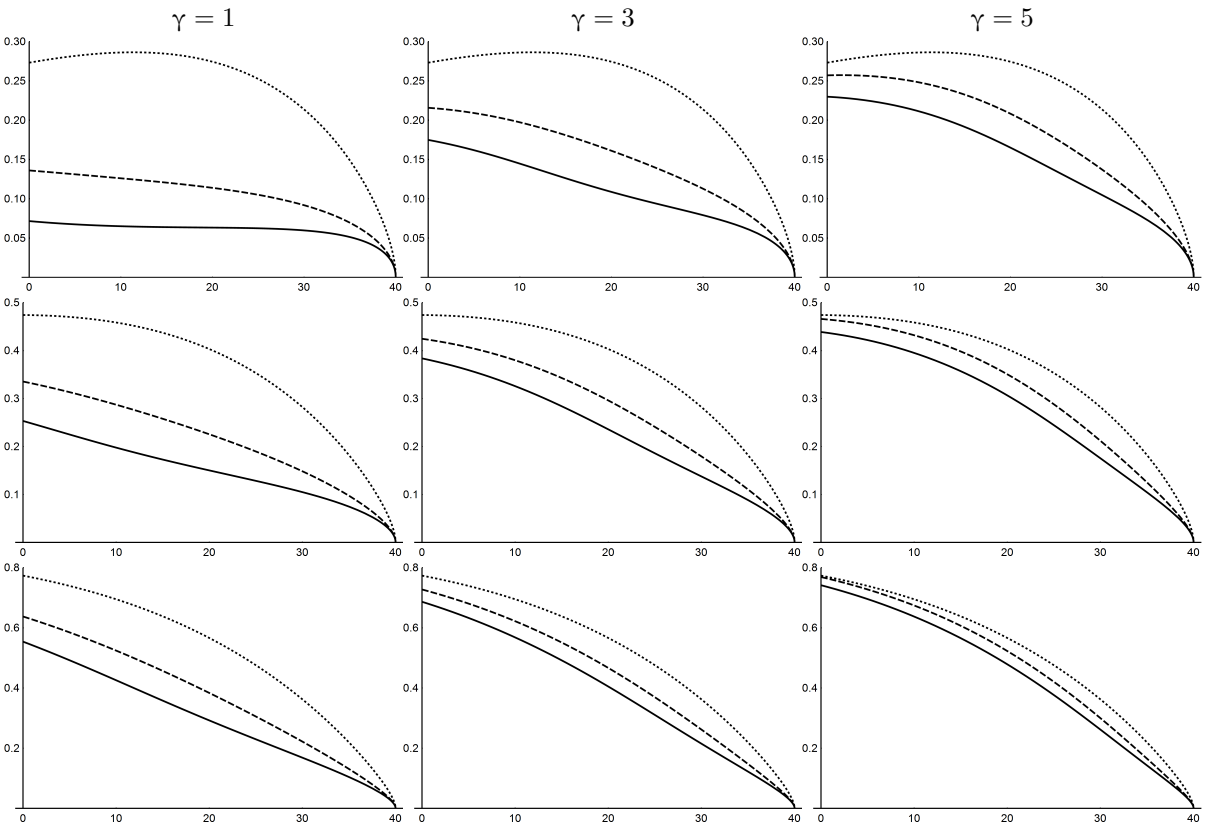
The limit when $\rho_{L,P}$ (denoted ρ for simplicity) tends to 1 is then:

$$\begin{aligned}
&\lim_{\rho\to 1} h^* = \frac{0}{0} = (\text{L'Hôpital}) \\
&= \lim_{x\to\infty} P_Q(t,T)\frac{1}{\gamma}\left(\log\left(1 + \frac{\gamma B(\rho)}{A(\rho)}E_{\mathcal{Q}^T}[G] + \sum_{n=2}^{\infty}\frac{\gamma^n}{n!}\frac{B(\rho)^n}{A(\rho)^n}E_{\mathcal{Q}^T}[G^n]\right)\right. \\
&\left. + A(\rho)\frac{\frac{A(\rho)-B(\rho)}{A(\rho)^2}\gamma E_{\mathcal{Q}^T}[G] + \sum_{n=2}^{\infty}\frac{\gamma^n}{n!}\frac{nB(\rho)^{n-1}A(\rho)^n - nA(\rho)^{n-1}B(\rho)^n}{A(\rho)^{2n}}E_{\mathcal{Q}^T}[G^n]}{\left(1 + \frac{\gamma B(\rho)}{A(\rho)}E_{\mathcal{Q}^T}[G] + \sum_{n=2}^{\infty}\frac{\gamma^n}{n!}\frac{B(\rho)^n}{A(\rho)^n}E_{\mathcal{Q}^T}[G^n]\right)}\right) \\
&= P_Q(t,T)\frac{1}{\gamma}\left(0 + A(\rho)\frac{\gamma E_{\mathcal{Q}^T}[G]}{A(\rho)}\right) = P_Q(t,T)E_{\mathcal{Q}^T}[G]
\end{aligned}$$

This is the price in the Black & Scholes setting with stochastic interest rates (see Brigo and Mercurio (2007) [6]) under the forward measure, which is given by:

$$\begin{aligned}
\frac{d\mathcal{Q}^T}{d\mathcal{P}} = \exp\Bigg(&-\frac{1}{2}\int_t^T (q + \sigma_r B(s,T))^2\, ds - \frac{1}{2}\lambda_S^2(T-t) \\
&- \int_t^T (q + \sigma_r B(s,T))\, dW_r(s) - \int_t^T \lambda_S dW_S(s)\Bigg)
\end{aligned}$$

Figure 1: The value of a put option guaranteeing $i_G = 4\%$ (first row), $i_G = 5\%$ (second row), $i_G = 6\%$ (third row): un-correlated exponential price (continuous line), imperfect correlation exponential price (discontinuous line) and perfect correlation B&S case (pointed line). *Source:* the authors.



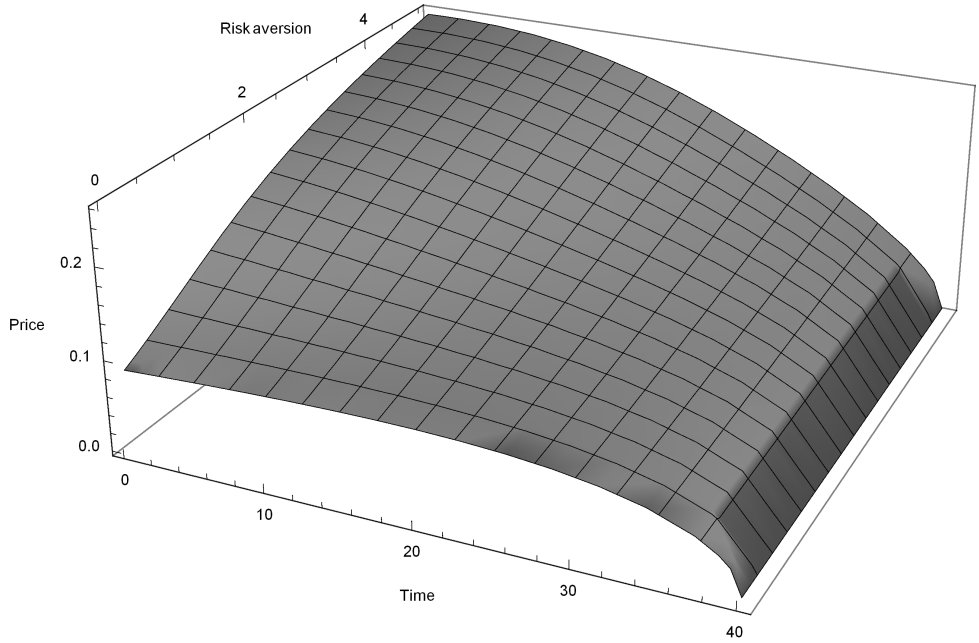
which is the forward measure when the stochastic interest rate has a Vasicek structure independent of the risky asset S(t). □

4 Numerical illustration

In this section we will present different graphics which will show the influence of risk aversion, time, guaranteed return and correlation in the price of a put option as described in (2.2). The parameters used are taken from the literature or are consistent with it.

The parameters related to the short term interest rate and zero-coupon bond are $a = 0.20$, $b = 0.05$, $\sigma_r = 2\%$, $r_0 = 5\%$, $q = 0,1527\%$. The risky asset has $\lambda_S = 30\%$ and $\sigma_S = 20\%$ and $\text{rho}_{r,S} = 0.30$. These parameters are taken from Boulier et al. (2001) [4]. The drift and volatility of the population has been taken from Devolder and Melis (2015) [10] and is $\alpha_P = 2\%$ and $\sigma_P = 5\%$. We suppose that the salaries are less stable than the population and have a higher drift and volatility, i.e., $g = 3\%$ and $\sigma_L = 7\%$. The wiener processes for salaries and population are correlated with $\rho_{L,P} = -0.1$. The negative correlation reflects the fact that cohort size negatively affects earnings (see Brunello, 2010 [7]). The mean one-period notional rate is thus $5,054\%$, so we will study the following interest rate guarantees: $i_G = 4\%$, $i_G = 5\%$ and $i_G = 6\%$.

Figure 2: 3D-Plot showing the evolution of the value of a put option for risk aversions $\gamma \in (0, 5)$ at different purchase time $t \in (0, 40)$ for a guaranteed interest rate of $i_G = 5\%$. *Source:* the authors.



We assume further that the correlation between the population and the markets are $\rho_{r,P} = -0.25$ and $\rho_{S,P} = -0.05$. These correlations are based on the fact that government issued zero-coupon bond are more negatively correlated to demographics than stock returns are (Poterba (2001) [22]). Finally we suppose that salary increase is positively correlated to the short term interest rate risk and risky asset risk: $\rho_{r,L} = 0.6$ and $\rho_{S,L} = 0.4$. The correlation matrix (2.21) is thus:

$$\Sigma_W = \begin{pmatrix} 1 & 0.30 & 0.40 & -0.25 \\ 0.30 & 1 & 0.6 & -0.05 \\ 0.40 & 0.6 & 1 & -0.1 \\ -0.25 & -0.05 & -0.1 & 1 \end{pmatrix}$$

Figure 1 compares the prices for the same market structure in three different cases: totally uncorrelated, which we choose to name 'insurance' case, totally correlated or hedged, denoted by 'Black & Scholes' case, and the case with intermediate correlation structure, denoted by 'intermediate' price. The x-axis represent the time when the option would be written and the y-axis is the price. For instance, the price for an option written at time $t = 20$ when $\gamma = 3$ and $i_G = 5\%$ would be $h_{BS} = 0.27442$, $h_{Ins} = 0.1087$ and $h_{int} = 0.1611$. This means that if an individual would make a contribution of $C = 1000$ € we would have to put aside $C \cdot h_{int} = 161,1$ € in order to ensure 4% return per year. We observe in Figure 1 that the intermediate price for the presented correlation structure and the independence insurance premium are lower than the fully-hedged Black & Scholes case. This relation may be dependent of the parameters. However, we can state than people would be ready to pay more to be better hedged.[2] We observe as well

[2]We have taken into account other parameters and correlation structures for the numerical illustrations

Figure 3: 3D-Plot showing the evolution of the value of a put option for guarantees $i_G \in (3\%, 6\%)$ at different purchase time $t \in (0, 40)$ for a risk aversion $\gamma = 3$. *Source:* the authors.

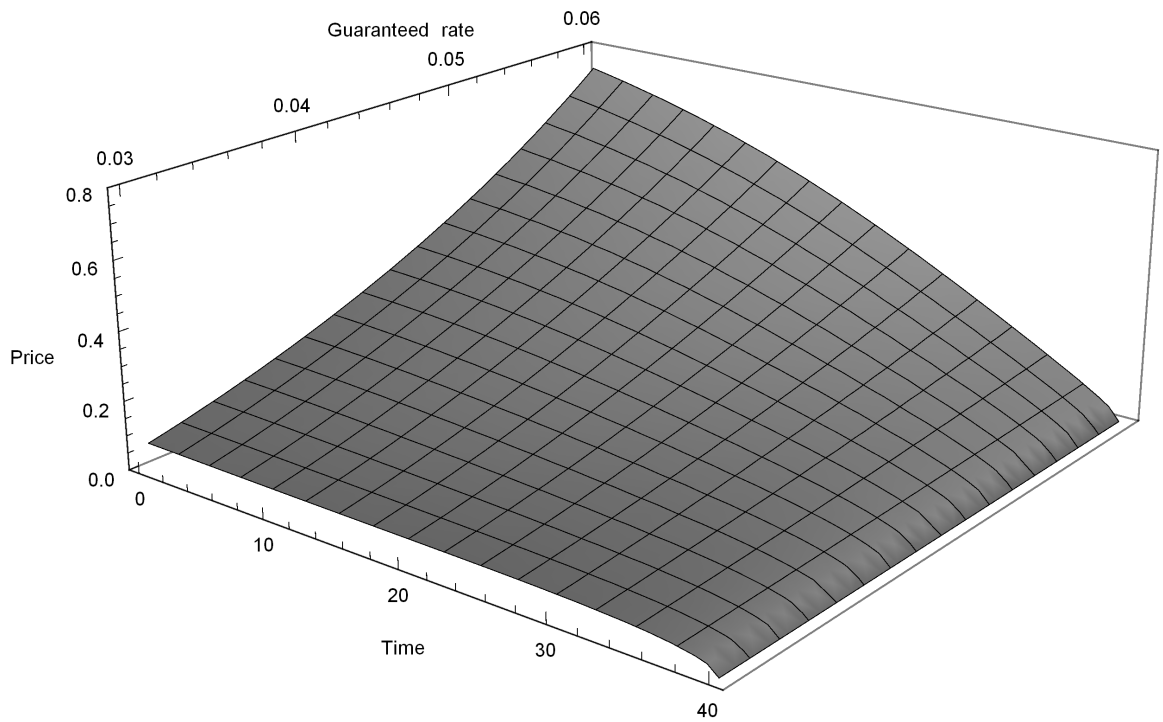
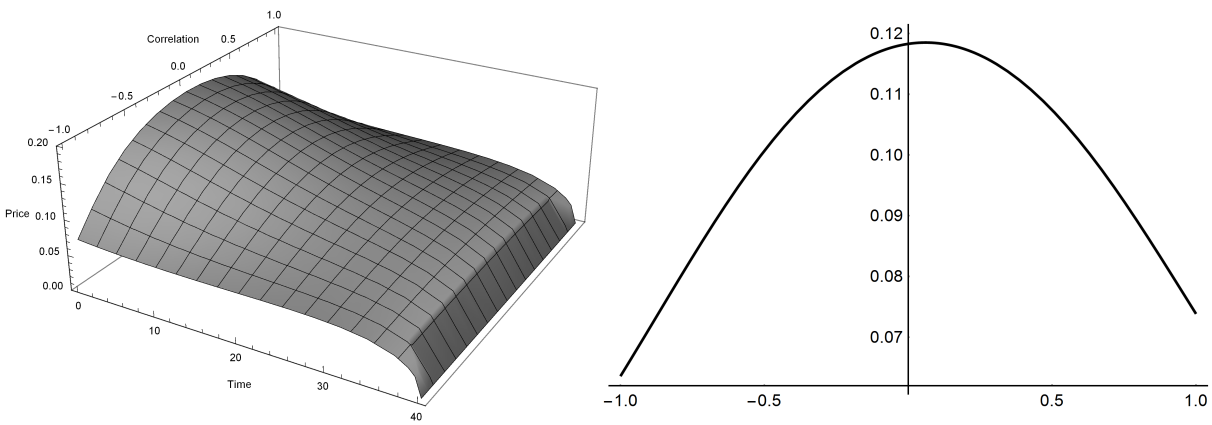


Figure 4: 3D-Plot showing the evolution of the value of a put option for correlation $\rho_{r,L} \in (-1, 1)$, with all other correlations equal to 0, guarantee of $i_G = 4\%$, and risk aversion coefficient $\gamma = 3$. The second graph shows the evolution of the price according to the different correlations for options writte at $t = 20$ *Source:* the authors.



3-D Plot T=20

that prices increase when the return guarantee and/or the risk aversion increases in the three cases.

Figure 2 and 3 show the evolution of the price for different writing times from $t \in (0, 40)$

and obtained similar results. However, in some cases, we have noted that for the same return guarantee, high risk aversion coefficients may lead to higher intermediate and insurance prices than in the Black & Scholes case, while low risk aversion coefficients may lead to lower intermediate and insurance prices than in the complete market case.

and risk aversion, as well as interest rate guarantee. The price increases for higher risk aversions, meaning that issuers are ready to pay a higher price when they are more averse to them. It also increases for higher return guarantees, which seems straightforward. Buyer's are ready to pay more to have higher potential returns.

Figure 4 shows the evolution of the price when all risks are independent except of the salary risk and the short rate. In this case $\rho_{r,L} \neq 0$ and all others $\rho_{i,j} = 0$. We observe that the correlation has a clear influence in the intermediate price. It seems that the price attains its highest point when it's closed to 0, i.e., when the nontraded asset is totally uncorrelated.

5 Conclusion

This paper studies the development of a pricing framework which could be suitable for the pricing of minimum return guarantees in a pay-as-you-go financed, defined contribution public pension scheme. Defined contributions-based pension systems are usually actuarially fairer, but have the main shortcoming that return risk is borne by the participants. Here we seek to give a solution to this problem in order to correctly measure and finance these guarantees. This framework would help increasing the attractiveness of this pension scheme. The price of the guarantee is the one that makes the writer (resp. buyer) of the option indifferent between only holding a financial portfolio and holding a financial portfolio plus (minus) the price at underwriting time, and the portfolio minus (plus) the payoff at maturity.

We obtain a closed-form formula for a general dependence case which is easy to calculate and has desirable properties: prices increase when risk aversion increases, and/or when the return guaranteed increases, i.e., people are ready to pay more when they are more averse to the potential losses, and/or have the guarantee to have a higher return on their contributions. We have shown that the price is an intermediate price between a zero-utility exponential premium and the complete markets Black & Scholes formula. The zero-utility exponential premium is obtained when the non-traded asset is totally uncorrelated to the asset markets, and could be interpreted as the case when insurance risks are priced. On the other hand, the Black & Scholes formula represents the price of the asset when the risks are totally correlated to the asset markets.

References

[1] Artzner, P., Delbaen, F., Eber, J. M., & Heath, D. (1999): Coherent measures of risk. *Mathematical finance* **9**(3), 203-228.

[2] Björk, T. (2004): *Arbitrage theory in continuous time.* Oxford university press.

[3] Black, F., & Scholes, M. (1973): The pricing of options and corporate liabilities. *The Journal of Political Economy*, 637-654.

[4] Boulier, J. F., Huang, S., & Taillard, G. (2001): Optimal management under stochastic interest rates: the case of a protected defined contribution pension fund. *Insurance: Mathematics and Economics*, **28**(2), 173-189.

[5] Börsch-Supan, A. (2006): What are NDC Systems? What do they bring to Reform Strategies?, in: R. Holzmann and E. Palmer, eds., *Pension Reform: Issues and Prospects for Notional Defined Contribution (NDC) Schemes*, (Washington, D.C.: The World Bank), chapter 3. ISBN 0-8213-6038-8.

[6] Brigo, D., & Mercurio, F. (2007): *Interest rate models-theory and practice: with smile, inflation and credit*. Springer.

[7] Brunello, G. (2010): The Effects of Cohort Size on European Earnings. *Journal of Population Economics*, **23**, 273-290.

[8] Cochrane, J. H., & Saa-Requejo, J. (2000): Beyond Arbitrage: Good-Deal Asset Price Bounds in Incomplete Markets. *Journal of Political Economy* **108**(1), 79-119.

[9] Cvitanić, J., Pham, H., & Touzi, N. (1999): A closed-form solution to the problem of super-replication under transaction costs. *Finance and stochastics* **3**(1), 35-54.

[10] Devolder, P., Melis, R., Miller, A. (2014): Optimal mix between pay as you go and funding for pension liabilities in a stochastic framework. *Submitted*.

[11] Denuit, M. (1999): The exponential premium calculation principle revisited. *Astin Bulletin* **29**(02), 215-226.

[12] Frittelli, M. (2000): The minimal entropy martingale measure and the valuation problem in incomplete markets. *Mathematical finance*: **10**(1), 39-52.

[13] Föllmer, H., & Schweizer, M. (1991): Hedging of contingent claims. *Applied stochastic analysis* **5**, 389.

[14] Henderson, V. (2002): Valuation of claims on nontraded assets using utility maximization. *Mathematical Finance* **12**(4), 351-373.

[15] Hodges, S. D., & Neuberger, A. (1989): Optimal replication of contingent claims under transaction costs. *Review of futures markets* **8**(2), 222-239.

[16] Korn, R., & Kraft, H. (2002): A stochastic control approach to portfolio problems with stochastic interest rates. *SIAM Journal on Control and Optimization* **40**(4), 1250-1269.

[17] Miyahara, Y. (1996): Canonical martingale measures of incomplete assets markets. *Probability theory and mathematical statistics, Proc* 7th *Japan-Russia Symp. Tokyo (1995)*, 343-352.

[18] Musiela, M., & Zariphopoulou, T. (2004): An example of indifference prices under exponential preferences. *Finance and Stochastics* **8**(2), 229-239.

[19] Floroiu, O., & Pelsser, A. (2013): Closed-form solutions for options in incomplete markets. *Netspar Discussion Papers Series* **02/2013-004**.

[20] Palmer, E. (2006): What is NDC?, in: R. Holzmann and E. Palmer, eds., *Pension Reform: Issues and Prospects for Notional Defined Contribution (NDC) Schemes*, (Washington, D.C.: The World Bank), chapter 2. ISBN 0-8213-6038-8.

[21] Pennacchi, G. G. (1999): The value of guarantees on pension fund returns. *Journal of Risk and Insurance*, 219-237.

[22] Poterba, J. M. (2001): Demographic structure and asset returns. *Review of Economics and Statistics,* **83**(4), 565-584.

[23] Queisser, M. (1995): Chile and beyond: The secondgeneration pension reforms in Latin America. *International Social Security Review* **48**(3-4), 23-39.

[24] Rouge, R., & El Karoui, N. (2000): Pricing via utility maximization and entropy. *Mathematical Finance* **10**(2), 259-276.

[25] Vasicek, O. (1977): An equilibrium characterization of the term structure. *Journal of financial economics* **5**(2), 177-188.

[26] Young, V. R. (2004): Pricing in an incomplete market with an affine term structure. *Mathematical Finance* **14**(3), 359-381.

[27] Zariphopoulou, T. (2001): A solution approach to valuation with unhedgeable risks. *Finance and stochastics* **5**(1), 61-82.

A Appendix

A.1 Proof of Theorem 1

The maximum principle applied to the value function (2.25) gives the following Hamilton-Jacobi-Bellman (HJB) relation:

$$\begin{aligned}
0 &= V_t + a(b - r(t))V_r + \frac{1}{2}\sigma_r^2 V_{rr} + r(t)xV_x \\
&+ \max_{\vartheta(t)} \Big\{ \frac{1}{2}\sigma(t,T)^2 x^2 \vartheta_P(t)^2 V_{xx} - \vartheta_P(t)\sigma(t,T)x(qV_x + \sigma_r V_{xr}) \\
&+ \frac{1}{2}\sigma_S^2 x^2 \vartheta_S(t)^2 V_{xx} + \vartheta_S(t)\sigma_S x\left(\lambda_S V_x + \rho_{r,S}\sigma_r\right) - \vartheta_S(t)\vartheta_P(t)\rho_{r,S}\sigma_S\sigma(t,T)x^2 \Big\}
\end{aligned}$$

where $\vartheta(t) = (\vartheta_S(t), \vartheta_P(t))$. The HJB can be written as the maximum of a functional $\psi(\vartheta(t))$. At the optimal control $\vartheta^*(t) = (\vartheta_P^*(t), \vartheta_S^*(t))$ we must have simultaneously:

$$\psi(\vartheta^*(t)) = 0 \tag{A.1}$$

$$\frac{d\psi}{d\vartheta}(\vartheta^*(t)) = 0 \tag{A.2}$$

Note that the presence of $\rho_{r,S}$ produces that the optimal $\vartheta(t)$ has to be calculated as an equation system. The second condition (A.2) entails the following system of equations:

$$\begin{cases} \vartheta_S(t)\sigma_S x V_{xx} - \rho_{r,S}\vartheta_P(t)\sigma(t,T)xV_{xx} = -(\lambda_S V_x + \rho_{r,S}\sigma_r V_{xr}) \\ -\rho_{r,S}\vartheta_S(t)\sigma_S x V_{xx} + \vartheta_P(t)\sigma(t,T)xV_{xx} = qV_x + \sigma_r V_{xr} \end{cases}$$

The optimal investment strategy $\vartheta^*(t)$ is then:

$$\vartheta_S^*(t) = \frac{\left(-\lambda_S + \rho_{r,S}q\right)V_x}{\sigma_S x V_{xx}\left(1-\rho_{r,S}^2\right)}$$

$$\vartheta_P^*(t) = \frac{\left(q-\rho_{r,S}\lambda_S\right)V_x + \left(1-\rho_{r,S}^2\right)\sigma_r V_{xr}}{\sigma(t,T)xV_{xx}\left(1-\rho_{r,S}^2\right)}$$

Putting the expression of the optimal allocations into (A.1), we obtain the following partial differential equation for the value function:

$$\begin{aligned} 0 = V_t + a(b-r(t))V_r + \frac{1}{2}\sigma_r^2 V_{rr} + r(t)xV_x - \frac{1}{2}\sigma_r^2\frac{V_{xr}^2}{V_{xx}} \\ - q\sigma_r\frac{V_x V_{xr}}{V_{xx}} - \frac{1}{2}\frac{V_x^2}{V_{xx}\left(1-\rho_{r,S}^2\right)}\left(\lambda_S^2 + q^2 - 2\rho_{r,S}q\lambda_S\right) \end{aligned} \tag{A.3}$$

with limit condition $V(X,r,T) = u(X) = -e^{-\gamma X}$. We try a solution inspired by the expression in Young (2004) [26]:

$$V(x,r,t) = -e^{-\gamma x P(t,r) + K(t)}$$

where $P(t,r)$ and $K(t)$ are independent of $X(t)$ and have limit condition $P(T,r) = 1$ and $K(T) = 0$. Then the partial derivatives are:

$$\begin{aligned} &V_r = -\gamma x P_r V; V_{rr} = (\gamma x P_r^2 - P_{rr})\gamma x V \\ &V_x = -\gamma P V; V_{xx} = \gamma^2 P^2 V \\ &V_t = (-\gamma x P_t + K'(t))V; V_{xr} = (\gamma x P - 1)\gamma P_r V \end{aligned}$$

Substituting into (A.3) and after some calculations:

$$\begin{aligned} 0 = -\gamma x\left(P_t + r(t)P + a(\frac{b^*}{a} - r(t))P_r + \frac{1}{2}\sigma_r^2 P_{rr} - \sigma_r^2\frac{P_r^2}{P}\right) \\ + K'(t) - \frac{1}{2}\frac{1}{\left(1-\rho_{r,S}^2\right)}(\lambda_S^2 + q^2 - 2\rho_{r,S}q\lambda_S) - \frac{1}{2}\sigma_r^2\frac{P_r^2}{P^2} - q\sigma_r\frac{P_r}{P} \end{aligned}$$

We try a solution for this SDE depending on $P(t,r)$ inspired by the expression of a zero-coupon bond as previously used in Korn & Kraft (2002) [16] and Young (2004) [26]:

$$P(t,r) = \alpha(t)e^{-\beta(t)r} \tag{A.4}$$

with $\alpha(t)$ and $\beta(t)$ independent of r with limit conditions $\alpha(T) = 1$ and $\beta(T) = 0$. Its partial derivatives are:

$$P_t = \alpha^{'}(t)e^{-\beta(t)r} - P\beta^{'}(t)r$$
$$P_r = -\beta(t)P; P_{rr} = \beta^2(t)P$$

Substituting again in the previous PDE we have:

$$0 = -\gamma x e^{-\beta(t)r}\left(\alpha^{'}(t) - \alpha(t)r(t)\left(\underbrace{\beta^{'}(t) - a\beta(t) - 1}_{\text{Eq.A}}\right) - \alpha(t)\left(b^*\beta(t) + \frac{1}{2}\sigma_r^2\beta^2(t)\right)\right)$$
$$+\left(\underbrace{K^{'}(t) - \frac{1}{2}\frac{1}{\left(1-\rho_{r,S}^2\right)}(\lambda_S^2 + q^2 - 2\rho_{r,S}q\lambda_S) - \frac{1}{2}\sigma_r^2\frac{P_r^2}{P^2} - q\sigma_r\frac{P_r}{P}}_{\text{Eq.B}}\right) \tag{A.5}$$

This transformation would only make sense if the expression above no longer depends on r(t). In this case the solution to Equation A and B are:

$$\beta(t) = -B(t,T) = \frac{e^{-a(T-t)-1}}{a} \tag{A.6}$$
$$K(t) = -\frac{1}{2}\int_t^T (q + \sigma_r B(s,T))^2\, ds - \frac{1}{2}\lambda_S^2(T-t) \tag{A.7}$$

The expression (A.5) becomes:

$$0 = \alpha^{'}(t) - \alpha(t)\left(b^*\beta(t) + \frac{1}{2}\sigma_r^2\beta^2(t)\right) = \alpha^{'}(t) - \alpha(t)h(t) \tag{A.8}$$

which has as solution:

$$\alpha(t) = \exp\left\{\left(-\frac{b^*}{a} + \frac{\sigma_r^2}{2a^2}\right)[B(t,T) - T + t] + \frac{\sigma_r^2}{4a}B(t,T)^2\right\}$$
$$= A_Q(t,T)^{-1} \tag{A.9}$$

The expression of $P(t,r)$ is then given by the inverse of the zero-coupon bond under the Q-dynamics $P_Q(t,T)$. The value function (2.25) becomes (3.1).

A.2 Proof of Theorem 2

The dependence of y, t will be not taken into account in order to ease the notation. Furthermore, we set:

$$A_r = \rho_{r,L}\sigma_1 + \rho_{r,P}\sigma_2$$
$$A_S = \rho_{S,L}\sigma_1 + \rho_{S,P}\sigma_2$$

The maximum principle applied to the value function (2.26) gives the following HJB relation:

$$
\begin{aligned}
&u_t + a(b-r)u_r + \frac{1}{2}\sigma_r^2 u_{rr} + \mu u_y + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + 2\rho_{L,P}\sigma_1\sigma_2)u_{yy} + +rxu_x + A_r\sigma_r u_{yr} \\
&+ \max_{\vartheta(t)} \left\{ \frac{1}{2}u_{xx}\vartheta_P^2\sigma^2(t,T)x^2 - \vartheta_P\sigma(t,T)x\left(qu_x + \sigma_r u_{xr} + A_r u_{xy}\right) \right. \\
&\left. - \vartheta_S(t)\vartheta_P(t)\rho_{r,S}\sigma_S\sigma(t,T)x^2 + \frac{1}{2}\sigma_S^2 x^2\vartheta_S^2 u_{xx} + \vartheta_S\sigma_S x\left(\lambda_S u_x + \rho_{r,S}\sigma_r u_{xr} + A_s u_{xy}\right) \right\}
\end{aligned}
$$

Here we face the same problem as in Proposition 1. Because of the presence of $\rho_{r,S}$ produces that the optimal $\vartheta(t)$ has to be calculated as an equation system. The HJB can be written as the maximum of a functional $\psi(\vartheta(t))$. At the optimal control $\vartheta^*(t)$ we must have simultaneously (A.1) and (A.2). The second condition (A.2) gives the following system of equations:

$$
\begin{cases}
\vartheta_S(t)\sigma_S x u_{xx} - \rho_{r,S}\vartheta_P(t)\sigma(t,T)xu_{xx} = -(\lambda_S u_x + \rho_{r,S}\sigma_r u_{xr} + A_S u_{xy}) \\
-\rho_{r,S}\vartheta_S(t)\sigma_S x u_{xx} + \vartheta_P(t)\sigma(t,T)xu_{xx} = qV_x + \sigma_r u_{xr} + A_r u_{xy}
\end{cases}
$$

The optimal investment strategy $\vartheta^*(t)$ is then:

$$
\begin{aligned}
\vartheta_S^*(t) &= \frac{\left(-\lambda_S + \rho_{r,S}q\right)u_x + \left(\rho_{r,S}A_r - A_S\right)u_{xy}}{\sigma_S x u_{xx}\left(1-\rho_{r,S}^2\right)} \\
\vartheta_P^*(t) &= \frac{\left(q - \rho_{r,S}\lambda_S\right)u_x + \left(1-\rho_{r,S}^2\right)\sigma_r u_{xr} + \left(A_r - \rho_{r,S}A_S\right)u_{xy}}{\sigma(t,T)xV_{xx}\left(1-\rho_{r,S}^2\right)}
\end{aligned}
$$

Putting the optimal ϑ into into (A.1), and after some tedious algebra, we obtain the following partial differential equation for the value function:

$$
\begin{aligned}
&u_t + a(b-r)u_r + \frac{1}{2}\sigma_r^2 u_{rr} + rxu_x + A_r\sigma_r u_{yr} + \mu u_y + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + 2\rho_{L,P}\sigma_1\sigma_2)u_{yy} + \\
&- q\sigma_r\frac{u_x u_{xr}}{V_{xx}} - A_r\sigma_r\frac{u_{xr}u_{xy}}{u_{xx}} - \frac{u_x u_{xy}}{u_{xx}}\frac{1}{\left(1-\rho_{r,S}^2\right)}\left(\lambda_S(A_S - \rho_{r,S}A_r) + q(A_r - \rho_{r,S}A_S)\right) \\
&- \frac{1}{2}\sigma_r^2\frac{u_{xr}^2}{u_{xx}} - \frac{1}{2}\frac{1}{\left(1-\rho_{r,S}^2\right)}\frac{u_x^2}{u_{xx}}\left(\lambda_S^2 + q^2 - 2\rho_{r,S}q\lambda_S\right) - \frac{1}{2}\frac{u_{xy}^2}{u_{xx}}\frac{A_S^2 + A_r^2 - 2\rho_{r,S}A_S A_r}{1-\rho_{r,S}^2}
\end{aligned}
\tag{A.10}
$$

Due to the good separability properties of the exponential utility we try a solution inspired by the limit condition:

$$
u(x,y,t) = -e^{-\gamma x P(t,r)}F(y,t) \tag{A.11}
$$

where $F(y,t)$ corresponds to the part of the value function related to the nontraded asset $Y(t)$. The limit condition is $F(y,T) = h(y)$. The partial derivatives are then:

$$
\begin{aligned}
&u_t = -\gamma x u P_t - e^{-\gamma x P}F_t;\ u_r = -\gamma x P_r u;\ u_{rr} = (\gamma x P_r^2 - P_{rr})\gamma x u \\
&u_x = -\gamma P u;\ u_{xx} = \gamma^2 P^2 u;\ u_y = -e^{-\gamma x P}F_y;\ u_{yy} = -e^{-\gamma x P}F_{yy} \\
&u_{xr} = (\gamma x P - 1)\gamma P_r u;\ u_{xy} = e^{-\gamma x P}\gamma P F_y;\ u_{yr} = e^{-\gamma x P}\gamma x P_r F_y
\end{aligned}
$$

Substituting into (A.10) and after some calculations

$$
\begin{aligned}
0 = & -\gamma x u \left(P_t + rP + a(\frac{b^*}{a} - r)P_r + \frac{1}{2}\sigma_r^2 P_{rr} - \sigma_r^2 \frac{P_r^2}{P} \right) \\
& - e^{-\gamma x P} \left(F_t + \left(\mu - A_r \sigma_r \frac{P_r}{P} - \frac{\lambda_S (A_S - \rho_{r,S} A_r) + q(A_r - \rho_{r,S} A_S)}{1 - \rho_{r,S}^2} \right) F_y \right. \\
& + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + 2\rho_{L,P}\sigma_1\sigma_2) F_{yy} - \frac{1}{2} F \left(\frac{\lambda_S^2 + q^2 - 2\rho_{r,S} q \lambda_S}{\left(1 - \rho_{r,S}^2\right)} + \sigma_r^2 \frac{P_r^2}{P^2} + 2q\sigma_r \frac{P_r}{P} \right) \\
& \left. - \frac{1}{2} \frac{A_S^2 + A_r^2 - 2\rho_{r,S} A_S A_r}{1 - \rho_{r,S}^2} \frac{F_y^2}{F} \right)
\end{aligned} \tag{A.12}
$$

We now do the following transformation in order to linearize the PDE as in Zariphopoulou [27]:

$$
F(y,t) = f(y,t)^{\delta} \tag{A.13}
$$

for a δ which has to be determined. The partial differential equations are then:

$$
F_t = \delta f^{\delta-1} f_t; F_y = \delta f^{\delta-1} f_y; F_{yy} = \delta(\delta - 1) f^{\delta-2} f_y^2 + \delta f^{\delta-1} f_{yy}
$$

Substituting in (A.12)

$$
\begin{aligned}
0 = & -\gamma x u \left(P_t + rP + a(\frac{b^*}{a} - r)P_r + \frac{1}{2}\sigma_r^2 P_{rr} - \sigma_r^2 \frac{P_r^2}{P} \right) \\
& - e^{-\gamma x P} \delta f^{\delta-1} \left(f_t + \left(\mu - A_r \sigma_r \frac{P_r}{P} - \frac{\lambda_S (A_S - \rho_{r,S} A_r) + q(A_r - \rho_{r,S} A_S)}{1 - \rho_{r,S}^2} \right) f_y \right. \\
& + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + 2\rho_{L,P}\sigma_1\sigma_2) f_{yy} - \frac{1}{2}\frac{f}{\delta} \left(\frac{\lambda_S^2 + q^2 - 2\rho_{r,S} q \lambda_S}{\left(1 - \rho_{r,S}^2\right)} + \sigma_r^2 \frac{P_r^2}{P^2} + 2q\sigma_r \frac{P_r}{P} \right) \\
& \left. - \frac{1}{2}\frac{f_y^2}{f} \left(\delta \frac{A_S^2 + A_r^2 - 2\rho_{r,S} A_S A_r}{1 - \rho_{r,S}^2} - (\delta - 1)\left(\sigma_1^2 + \sigma_2^2 + 2\rho_{L,P}\sigma_1\sigma_2\right) \right) \right)
\end{aligned} \tag{A.14}
$$

If we choose δ as:

$$
\delta = \frac{\sigma_1^2 + \sigma_2^2 + 2\rho_{L,P}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 + 2\rho_{L,P}\sigma_1\sigma_2 - \frac{A_S^2 + A_r^2 - 2\rho_{r,S} A_S A_r}{1 - \rho_{r,S}^2}} \tag{A.15}
$$

then the PDE becomes a linear parabolic differential equation. We perform finally the same transformation for the function $P(t,r)$ as in (A.4) and obtain the same values for $\beta(t)$ (A.6) and $\alpha(t)$ (A.8). The remaining PDE is thus:

$$
\begin{cases}
f_t + \left(\mu - A_r \sigma_r \frac{P_r}{P} - \frac{\lambda_S (A_S - \rho_{r,S} A_r) + q(A_r - \rho_{r,S} A_S)}{1 - \rho_{r,S}^2} \right) f_y + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + 2\rho_{L,P}\sigma_1\sigma_2) f_{yy} \\
= \frac{1}{2}\frac{f}{\delta} \left(\frac{\lambda_S^2 + q^2 - 2\rho_{r,S} q \lambda_S}{\left(1 - \rho_{r,S}^2\right)} + \sigma_r^2 \frac{P_r^2}{P^2} + 2q\sigma_r \frac{P_r}{P} \right) \\
f(y,T) = h(y) = e^{\frac{\gamma}{\delta} g(Y(T))}
\end{cases}
$$

The drift coincides with the one in Proposition 1 under the pricing measure. This partial differential equation can be rewritten in terms of expectancy under the measure $\mathcal{Q}^{\mathrm{T}}$ according to the Feynman-Kac representation theorem as follows

$$\begin{aligned} f(y,t) =& e^{-\frac{1}{2}\frac{1}{\delta}\int_t^T \left(\frac{\lambda_S^2+q^2-2\rho_{r,S}\lambda_S q}{1-\rho_{r,S}^2}+\sigma_r^2 B(s,T)^2+2q\sigma_r B(s,T)\right)ds} \\ & \times E_{\mathcal{Q}^T}\left[e^{\frac{\gamma}{\delta}g(Y(T))}|Y(t)=y\right] \end{aligned} \tag{A.16}$$

where $Y(t)$ is given by (2.17).Then the writer's value function when the option is taken into account becomes:

$$u(x,y,r,t) = -e^{-\gamma\frac{x}{P_Q(t,T)}}f^{\delta} \tag{A.17}$$

with δ given by (A.15).