Longevity assets and pre-retirement consumption/portfolio decisions *

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24th March 2015

Abstract

We derive a closed form solution for the optimal consumption/investment problem of an agent whose force of mortality is stochastic and whose financial horizon coincides with a fixed retirement date. The complete financial market allows for investment in a risky asset, a zero-coupon bond and a longevity asset, which we model as a zero-coupon longevity bond. We explore the optimal demand for these assets by a representative agent having Hyperbolic Absolute Risk Aversion preferences on both consumption and final wealth. Our numerical analysis shows that individuals should optimally invest a large fraction of their wealth in the longevity asset, unless its risk premium is (excessively) unattractive. In our base scenario, calibrated on real world data, a 60-year old male retiring after 5 years should invest around 88% of his wealth in the longevity asset. Such a percentage decreases as time to retirement decreases. We explore sensitivity of our results to market and individual characteristics.

*The Authors would like to thank conference participants to the PARTY 2015 Conference. Luca Regis gratefully acknowledges financial support from the CrisisLab project funded by the Italian government.

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1 Introduction

Despite the relevant and increasing interest of pension funds and annuity providers seeking hedging opportunities, the market for longevity risk, i.e. the risk of unexpected changes in the mortality of a group of individuals, has not yet reached a sufficient level of liquidity.

Many reasons may have contributed to undermine a rapid development of the market, such as the lack of standardisation, informational asymmetries, and basis risk. Nevertheless, many researchers have developed a sound technology for designing and evaluating products. From the seminal paper of Lee and Carter (1992), the actuarial literature has devoted increasing attention to: (i) contract design (Blake et al., 006b), (ii) derivative pricing (Denuit et al., 2007, Barrieu et al., 2012, Wills and Sherris, 2010), and (iii) informational asymmetries (Biffis and Blake, 2010).

Furthermore, the transfer and hedging of longevity risk from pension funds to reinsurers has become more and more common, although on an over-the-counter basis. For instance, the volume of outstanding UK longevity swaps has reached 50 billion pounds as of the end of 2014, with a prevalence of very large deals, such as the 16 billion pounds swap between BT Pension Scheme and Prudential and the 12 billion euros Delta Lloyd/RGA Re index-based transaction. Also investment banks have been actively participating in the transactions. Between 2008 and 2014, alongside reinsurance specialists, JP Morgan, Credit Suisse, Goldman Sachs, Deutsche Bank and Société Générale were involved in longevity deals (Luciano and Regis, 2014).

The longevity-linked products could be of interest to asset managers for at least two reasons: their low correlation to other asset classes (at least in the short run – Loeys et al., 2007), and their effectiveness in hedging individual investors against the unexpected fluctuations of their subjective discount factors (Yaari, 1965, Merton, 1971, Huang et al., 2012).

The aim of this paper is to shed some light on the relevance of these aspects, analysing the optimal consumption and portfolio choices of an investor subject to longevity risk prior to retirement. The agent can invest in a friction-less, arbitrage free, and complete financial market where both traditional assets (bonds and stocks) and a longevity bond are listed.

We consider a fixed deterministic retirement age, which coincides with the time horizon of the investor, and obtain the optimal solution in quasi-explicit form. Contrary to the life-cycle analysis, our focus on pre-retirement allows us to reproduce the point of view of asset managers or pension funds whose investment horizon coincides with the time of annuitisation of individual's wealth.[1]

[1]The impact of endogenous retirement on optimal consumption and portfolio choices of finitely

An extensive literature has explored consumption and investment decisions when mortality contingent claims are present. In particular, Huang and Milevsky (2008) analyse the decisions of families in the presence of income risk and life insurance, while Pirvu and Zhang (2012) and Kwak and Lim (2014) obtain explicit solutions when, respectively, stochastic drifts to asset prices and inflation risk are taken into account and constant relative risk aversion (CRRA) preferences are assumed. All these papers consider a deterministic force of mortality, while we model it as a stochastic process. We describe longevity risk by means of a doubly stochastic process whose intensity follows a continuous-time diffusion (like in Dahl, 2004 and Biffis, 2005). This process may be correlated with the other state variables.

The optimal investment problem of pension funds in the accumulation phase with longevity risk has been studied for instance by Battocchio et al. (2007) and Delong et al. (2008). When individual mortality is stochastic, annuity and life insurance sellers are exposed to unexpected changes in the survival of their insureds, implying under or over reserving. As hinted above, longevity-linked derivatives can help mitigating such a problem. Nonetheless, individuals themselves are exposed to unexpected fluctuations in their own life expectancy. This uncertainty can play a significant role in their consumption/portfolio decisions. The role of longevity-linked assets in investor's optimal portfolio has been addressed first by Menoncin (2008). Maurer et al. (2013), solving a life-cycle portfolio investment problem with longevity risk, assess the importance of variable annuities to smooth consumption, while Horneff et al. (2010) analyse the role of deferred annuities. They find that this product should optimally account for 78% of the financial wealth of a retiree.

While insurance products are by definition non-marketable, we explore the role of a longevity asset, whose presence in the market allows individuals to freely adjust their hedges against mortality fluctuations (in our analysis, we abstract from transaction costs). Cocco and Gomes (2012) analyse, in the context of a life-cycle model, the demand for a perfect hedge against shocks to the life expectancy of a CRRA agent. They study the optimal investment in a longevity bond, which is akin to our zero-coupon longevity asset. In their numerical simulations, they find that individuals – at old ages and especially approaching retirement – should invest a relevant fraction of their wealth in the longevity asset. Our contribution with respect to this stream of literature consists in providing a closed form solution to the (finite-horizon) problem of an agent prior to retirement, endowed with a general Hyperbolic Absolute Risk Aversion (HARA) class of preferences. Our calibrated application provides support to the findings of this stream of literature.

Under reasonable stochastic models and calibration for both mortality intensity

lived agents has been studied, for instance, by Farhi and Panageas (2007) and Dybvig and Liu (2010).

and interest rate, we find that individuals should optimally invest a relevant fraction of their wealth in a longevity bond. A 60-year old US male, who wants to retire at age 65 should optimally invest around 88% of his portfolio in the longevity bond and then progressively decrease this share approaching retirement. We explore the sensitivity of our results to both individual and market characteristics, finding that the optimal demand for longevity bonds: (i) is higher for 60-year old US females than for 60-year old US males; (ii) reduces (but very slightly) when the agent displays a more conservative behaviour (a higher risk aversion, a higher minimum consumption or a higher final wealth minimum level); (iii) remains positive over the whole horizon unless the risk premium is unreasonably low. These last two results are robust to a lower initial age.

The outline of the paper is the following. Section 2 describes the modelling setup, while Section 3 describes the individual preferences and the maximisation problem. The optimal consumption and portfolio are found in closed form. Section 4 provides a calibrated application based on US data. Finally, Section 5 concludes, and some technical derivations are left to two appendices.

2 The modelling setup

2.1 State variables

On a continuously open and friction-less financial market over the time set $[t_0, +\infty[$, the economic framework is described by a set of s state variables $z(t) \in \mathbb{R}^s$ which solve the following (matrix) stochastic differential equation:

$$\underset{s\times 1}{dz(t)} = \underset{s\times 1}{\mu_z(t,z)}dt + \underset{s\times n}{\Omega(t,z)'}\underset{n\times 1}{dW(t)}, \tag{1}$$

where $z(t_0)$ is a deterministic vector that defines the initial state of the system, $W(t)$ is a vector of n independent Wiener processes,[2] and the prime denotes transposition. The usual properties for guaranteeing the existence of a strong solution to Equation (1) are assumed to hold. At this stage, we make no other assumptions on the behaviour of either the drift or the diffusion term of (1). The vector $z(t)$ can be divided into two components: the financial state variables $z_f(t)$ and the mortality intensity of a group of individuals, which, as customary in the actuarial literature, are assumed to be

[2]The case with dependent Wiener processes can be easily obtained through the Cholesky's decomposition.

homogeneous by cohort. The state variables evolve over time as follows

$$\underbrace{\begin{bmatrix} \underset{(s-1)\times 1}{dz_f(t)} \\ d\lambda(t) \end{bmatrix}}_{dz(t)} = \underbrace{\begin{bmatrix} \underset{(s-1)\times 1}{\mu_{zf}(t,z)} \\ \mu_\lambda(t,z) \end{bmatrix}}_{\mu_z(t,z)} dt + \underbrace{\begin{bmatrix} \underset{(s-1)\times(n-1)}{\Omega_f(t,z)'} & \underset{(s-1)\times 1}{\mathbf{0}} \\ \underset{1\times(n-1)}{\sigma_{f\lambda}(t,z)'} & \sigma_\lambda(t,z) \end{bmatrix}}_{\Omega(t,z)'} \underbrace{\begin{bmatrix} \underset{(n-1)\times 1}{dW_f(t)} \\ dW_\lambda(t) \end{bmatrix}}_{dW(t)}, \tag{2}$$

where $\mathbf{0}$ is a vector of zeros. The diffusion vector $\sigma_{f\lambda}(t,z)$ captures the correlation between the financial state variables $z_f(t)$ and the mortality intensity $\lambda(t)$.

2.2 Financial market

On the financial market n risky assets are traded. Their prices $S(t) \in \mathbb{R}_+^n$ solve the (matrix) stochastic differential equation

$$\underset{n\times 1}{I_S^{-1}} \underset{n\times 1}{dS(t)} = \underset{n\times 1}{\mu(t,z)dt} + \underset{n\times n}{\Sigma(t,z)'} \underset{n\times 1}{dW(t)}, \tag{3}$$

where I_S is a diagonal matrix containing the elements of vector $S(t)$. The initial asset prices $S(t_0)$ are deterministic. Finally, a risk-less asset exists, whose price $G(t) \in \mathbb{R}_+$ solves the ordinary differential equation

$$G(t)^{-1} dG(t) = r(t,z)\, dt, \tag{4}$$

where $r(t,z) \in \mathbb{R}_+$ is the instantaneously risk-less interest rate. We consider the initial price of the risk-less asset to be $G(t_0) = 1$, i.e. the risk-less asset is the numéraire of the economy. The financial market is assumed to be arbitrage free and complete. In other words, a unique vector of market prices of risk $\xi(t,z) \in \mathbb{R}^n$ exists, such that

$$\Sigma(t,z)'\, \xi(t,z) = \mu(t,z) - r(t,z)\, \mathbf{1}, \tag{5}$$

where $\mathbf{1}$ is a vector of ones. The uniqueness of $\xi(t,z)$ implies that the matrix $\Sigma(t,z)$ can be inverted.

Girsanov's theorem allows us to switch from the historical ($\mathbb{P}$) to the risk-neutral probability $\mathbb{Q}$ according to the following relationship:

$$dW^{\mathbb{Q}}(t) = \xi(t,z)\, dt + dW(t). \tag{6}$$

The value in t_0 of any cash flow $\Xi(t)$ available at time t can be written as

$$\Xi(t_0) = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\Xi(t)\frac{G(t_0)}{G(t)}\right] = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\Xi(t)\, e^{-\int_{t_0}^{t} r(u,z)du}\right] = \mathbb{E}_{t_0}\left[\Xi(t)\, m(t_0,t)\, e^{-\int_{t_0}^{t} r(u,z)du}\right], \tag{7}$$

where $\mathbb{E}_{t_0}[\bullet]$ is the expected value operator (under either the historical $\mathbb{P}$ or the risk neutral probability $\mathbb{Q}$), conditional to the information set at time t_0, and the martingale $m(t_0, t)$, such that $m(t_0, t_0) = 1$, solves

$$m(t_0, t)^{-1} dm(t_0, t) = -\xi(t, z) dW(t).$$

2.3 Longevity bonds market

The mortality intensity (or force of mortality) $\lambda(t, z) \in \mathbb{R}_+$ of a homogeneous group of individuals, which the investor belongs to, is an element of $z(t)$. Following the stochastic mortality approach initiated by Milevsky and Promislow (2001) and Dahl (2004), the death event is modelled as a Poisson process with stochastic intensity. The probability to be alive at time t, given that an agent is alive in t_0, is given by $\mathbb{E}_{t_0}\left[e^{-\int_{t_0}^{t} \lambda(u,z)du}\right]$ (under the historical probability $\mathbb{P}$).

The value in t_0 of a financial flow $\Xi(t)$ available in t if an agent is still alive can be written as

$$\Xi(t_0) = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\Xi(t) e^{-\int_{t_0}^{t} r(u,z)+\lambda(u,z)du}\right],$$

while the value of the same cash flow available at the death time of an agent is given by

$$\Xi(t_0) = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\infty} \lambda(s) \Xi(s) e^{-\int_{t_0}^{s} r(u,z)+\lambda(u,z)du} ds\right],$$

where we have assumed that the death time is defined on the interval $[t_0, +\infty[$.

The financial market described in the previous section is assumed to be complete even with respect to the force of mortality. In other words, we assume that there exists a derivative on $\lambda(t)$, which we will refer to hereafter as the "longevity asset", whose price $\Lambda(t)$ solves

$$\Lambda(t)^{-1} d\Lambda(t) = \mu_\Lambda(t, z) dt + \sigma_{\Lambda f}(t, z)' dW_f(t) + \sigma_{\Lambda\lambda}(t, z)' dW_\lambda(t),$$

where the drift and the diffusion terms can be obtained by applying Ito's lemma. The dynamic equation of asset prices (3) can be rewritten by disentangling the (independent) Wiener processes $W_f(t)$ and $W_\lambda(t)$:

$$\underbrace{\begin{bmatrix} I_{S_f}^{-1} & \underset{(n-1)\times 1}{\mathbf{0}} \\ \underset{(n-1)\times(n-1)}{} & \\ \underset{1\times(n-1)}{\mathbf{0}'} & \Lambda^{-1} \end{bmatrix}}_{I_S^{-1}} \underbrace{\begin{bmatrix} dS_f(t) \\ \underset{(n-1)\times 1}{} \\ d\Lambda(t) \end{bmatrix}}_{dS(t)} = \underbrace{\begin{bmatrix} \mu_f(t, z) \\ \underset{(n-1)\times 1}{} \\ \mu_\Lambda(t, z) \end{bmatrix}}_{\mu(t,z)} dt + \underbrace{\begin{bmatrix} \Sigma_f(t, z)' & \underset{(n-1)\times 1}{\mathbf{0}} \\ \underset{(n-1)\times(n-1)}{} & \\ \underset{1\times(n-1)}{\sigma_{\Lambda f}(t, z)'} & \sigma_{\Lambda\lambda}(t, z) \end{bmatrix}}_{\Sigma(t,z)'} \underbrace{\begin{bmatrix} dW_f(t) \\ \underset{(n-1)\times 1}{} \\ dW_\lambda(t) \end{bmatrix}}_{dW(t)}. \tag{8}$$

3 Investor's maximisation problem

3.1 Investor's wealth, consumption and revenue

The investor holds $\theta_S(t) \in \mathbb{R}^n$ units of the risky assets (whose prices are $S(t)$), and $\theta_G(t) \in \mathbb{R}$ units of the risk-less asset. Thus, at any instant in time, the investor's wealth $R(t)$ is given by the static budget constraint

$$R(t) = \theta_S(t)' S(t) + \theta_G(t) G(t), \tag{9}$$

whose differential is

$$dR(t) = \theta_S(t)' dS(t) + \theta_G(t) dG(t) + \underbrace{d\theta_S(t)' (S(t) + dS(t)) + d\theta_G(t) G(t)}_{dR_a(t)}, \tag{10}$$

which we call the dynamic budget constraint. The first two components on the right hand side of the equation account for the changes in prices. The $dR_a(t)$ component, which accounts for the dynamic adjustment of the portfolio allocation, changes because:

- it finances the instantaneous consumption $c(t)\,dt$;
- it discounts the probability of dying between t and $t + dt$, which is measured by $\lambda(t,z)\,dt$;
- it is financed by the investor's labour income.

We define the accumulated labour income from t_0 up to time t as $L(t)$. Labour income is stochastic, and its dynamic behaviour is

$$dL(t) = w(t,z)\,dt + \underbrace{\begin{bmatrix} \underset{1\times(n-1)}{\sigma_{Lf}(t,z)'} & \sigma_{L\lambda}(t,z) \end{bmatrix}}_{\sigma_L(t,z)'} \underbrace{\begin{bmatrix} \underset{(n-1)\times 1}{dW_f(t)} \\ dW_\lambda(t) \end{bmatrix}}_{dW(t)}, \tag{11}$$

where $w(t,z)$ is the (instantaneous) labour income (or wage) of the agent. Given the above comments, we can write the dynamic behaviour of investor's wealth as

$$dR(t) = \theta_S(t)' dS(t) + \theta_G(t) dG(t) - c(t)\,dt + dL(t) + \lambda(t,z) R(t)\,dt.$$

Once the static budget constraint (9) and the asset differentials (3), (4) and (11) are suitably taken into account, $dR(t)$ becomes

$$\begin{aligned} dR(t) &= \big(R(t)(r(t,z) + \lambda(t,z)) + \theta_S(t)' I_S(\mu(t,z) - r(t,z)\,1) + w(t,z) - c(t)\big)\,dt \\ &\quad + \big(\theta_S(t)' I_S \Sigma(t,z)' + \sigma_L(t,z)'\big)\,dW(t). \end{aligned} \tag{12}$$

3.2 Investor's preferences and objective

The investor obtains utility from both the inter-temporal consumption $U_c(c(t))$ and the wealth at the end of the financial horizon $U_R(R(T))$.

The investor chooses the consumption and investment policy $(c(t), \theta_S(t))$ which maximises the inter-temporal utility of his/her wealth and consumption up to time T. The individual solves the following program

$$\max_{\pi(t),c(t)} \mathbb{E}_{t_0}\left[\int_{t_0}^{T} U_c(c(t))\, e^{-\int_{t_0}^{s}\rho(u,z)+\lambda(u,z)du} dt + U_R(R(T))\, e^{-\int_{t_0}^{T}\rho(u,z)+\lambda(u,z)du}\right], \quad (13)$$

where $\rho(t,z)$ is a (possibly stochastic) subjective discount rate. We assume HARA preferences, defined by the following utility functions:

$$U_c(c(t)) = \frac{(c(t)-c_m)^{1-\delta}}{1-\delta}, \quad U_R(R(T)) = \frac{(R(T)-R_m)^{1-\delta}}{1-\delta},$$

where $\delta > 1$ and the constants c_m and R_m can be interpreted as the minimum subsistence value of consumption and final wealth respectively. The Arrow-Pratt absolute risk aversion indexes are

$$-\frac{\frac{\partial^2 U_c(c(t))}{\partial c(t)^2}}{\frac{\partial U_c(c(t))}{\partial c(t)}} = \frac{\delta}{c(t)-c_m}, \quad -\frac{\frac{\partial^2 U_R(R(T))}{\partial R(T)^2}}{\frac{\partial U_R(R(T))}{\partial R(T)}} = \frac{\delta}{R(T)-R_m}.$$

Accordingly, the higher c_m (or R_m), the higher the risk aversion: an agent who has to guarantee a higher minimum level of consumption (or final wealth) will choose a safer investment.

The case of CRRA preferences is obtained when $c_m = R_m = 0$. Notice that the budget constraint equalises the initial wealth increased by the expected value of all the future revenues to the sum between the final wealth and the whole consumption stream:

$$\begin{aligned} & R(t_0) + \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{T} e^{-\int_{t_0}^{s} r(u,z)+\lambda(u,z)du} dL(s,z)\right] \\ = \; & \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{T} c(s)\, e^{-\int_{t_0}^{s} r(u,z)+\lambda(u,z)du} ds + R(T)\, e^{-\int_{t_0}^{T} r(u,z)+\lambda(u,z)du}\right], \end{aligned}$$

or

$$\begin{aligned} R(t_0) \;=\; & \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{T} \left(c(s) - w(s,z) + \sigma_L(s,z)'\,\xi(s,z)\right) e^{-\int_{t_0}^{s} r(u,z)+\lambda(u,z)du} ds \right. \quad (14) \\ & \left. + R(T)\, e^{-\int_{t_0}^{T} r(u,z)+\lambda(u,z)du}\right]. \end{aligned}$$

Indeed, the difference between the discounted value of final wealth at T and the initial wealth of the investor coincides with the expected value of the discounted flow of risk-adjusted consumption net of labour income.

3.3 The optimal consumption and portfolio

The maximisation problem of the investor (13) under the constraint (14) can be solved either through dynamic programming (via the so-called Hamilton-Jacobi-Bellman equation) or through the so-called martingale approach. This last method is viable in our framework because of market completeness. The following proposition characterises the optimal solution.

Proposition 1. *The optimal consumption and portfolio solving problem (13) are*

$$c^*(t) = c_m + \frac{R(t) - H(t,z)}{F(t,z)}, \tag{15}$$

$$\begin{aligned} I_S \theta_S^*(t) &= -\Sigma(t,z)^{-1} \sigma_L(t,z) + \frac{R(t) - H(t,z)}{\delta} \Sigma(t,z)^{-1} \xi(t,z) \\ &+ \frac{R(t) - H(t,z)}{F(t,z)} \Sigma(t,z)^{-1} \Omega(t,z) \frac{\partial F(t,z)}{\partial z} + \Sigma(t,z)^{-1} \Omega(t,z) \frac{\partial H(t,z)}{\partial z}, \end{aligned} \tag{16}$$

where

$$H(t,z) = \mathbb{E}_t^{\mathbb{Q}} \left[\begin{array}{c} \int_t^T \left(c_m - w(s,z) + \sigma_L(s,z)' \xi(s,z) \right) e^{-\int_t^s r(u,z)+\lambda(u,z)du} ds \\ + R_m e^{-\int_t^T r(u,z)+\lambda(u,z)du} \end{array} \right],$$

$$F(t,z) = \mathbb{E}_t^{\mathbb{Q}_\delta} \left[\begin{array}{c} \int_t^T e^{-\int_t^s \left(\frac{\delta-1}{\delta} r(u,z) + \frac{1}{\delta}\rho(u,z) + \lambda(u,z) + \frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi(u,z)'\xi(u,z) \right) du} ds \\ + e^{-\int_t^T \left(\frac{\delta-1}{\delta} r(u,z) + \frac{1}{\delta}\rho(u,z) + \lambda(u,z) + \frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi(u,z)'\xi(u,z) \right) du} \end{array} \right],$$

and

$$dW(t)^{\mathbb{Q}_\delta} = \frac{\delta - 1}{\delta} \xi(t,z)\, dt + dW(t). \tag{17}$$

Notice that, in order to solve the problem analytically, we used the probability measure $\mathbb{Q}_\delta$ defined in (17). This new probability measure has two relevant properties: (i) when the agent has a log utility function, i.e. $\delta = 1$, the probability $\mathbb{Q}_\delta$ coincides with the historical probability; (ii) when the agent is infinitely risk averse, i.e. $\delta \to +\infty$, the probability $\mathbb{Q}_\delta$ coincides with the risk neutral probability. In fact, we can think of

the Wiener processes under $\mathbb{Q}_\delta$ as a weighted mean of the Wiener processes under the risk neutral and the historical probabilities (the weight is given by the inverse of δ):

$$dW(t)^{\mathbb{Q}_\delta} = \left(1 - \frac{1}{\delta}\right) dW(t)^{\mathbb{Q}} + \frac{1}{\delta} dW(t).$$

The function $H(t,z)$ appearing in the optimal solutions is the expected value at time t, under the risk neutral measure, of the minimum final wealth R_m and of the risk-adjusted minimum consumption level net of wage, appropriately discounted in both actuarial and financial terms. Thus, it represents the net expected balance after financing the minimum consumption and final wealth.

The function $F(t,z)$ is the expected value (under the preference-adjusted measure $\mathbb{Q}_\delta$) at time t of the sum of all the discount factors for both the consumption stream (first term) and the final wealth (second term). We can interpret $F(t,z)$ as a sort of "global" discount factor.

Optimal consumption at time t is equal to the sum of the minimum consumption level c_m and the difference between actual wealth $R(t)$ and the expected value of all the discounted subsistence levels of consumption and terminal wealth($H(t,z)$), divided by the "global" risk and preference-adjusted discount factor $F(t,z)$. We remark that the difference $R(t) - H(t,z)$ is also relevant for computing the optimal portfolio, which depends also on the sensitivities of $H(t,z)$ and $F(t,z)$ with respect to the state variables z.

The role of the longevity asset in the optimal portfolio (16) can be identified using the decomposition of matrices Ω and Σ as shown in (2) and (8), respectively. The optimal portfolio of longevity asset is

$$\Lambda\theta^*_\Lambda = -\frac{\sigma_{L\lambda}}{\sigma_{\Lambda\lambda}} + \frac{R-H}{\delta}\frac{1}{\sigma_{\Lambda\lambda}}\left(\xi_\lambda + \frac{\delta}{F}\sigma'_{f\lambda}\frac{\partial F}{\partial z_f} + \sigma_\lambda \frac{\delta}{F}\frac{\partial F}{\partial \lambda}\right) - \frac{1}{\sigma_{\Lambda\lambda}}\left(\sigma'_{f\lambda}\frac{\partial H}{\partial z_f} + \sigma_\lambda \frac{\partial H}{\partial \lambda}\right),$$

while the optimal portfolio of financial assets is

$$I_{S_f}\theta^*_{S_f} = -\Sigma_f^{-1}\sigma_{Lf} + \frac{R-H}{\delta}\Sigma_f^{-1}\left(\xi_f + \frac{\delta}{F}\Omega'_f\frac{\partial F}{\partial z_f}\right) - \Sigma_f^{-1}\Omega'_f\frac{\partial H}{\partial z_f} - \Sigma_f^{-1}\sigma_{\Lambda f}\Lambda\theta^*_\Lambda.$$

We can identify three components in the demand for the longevity asset:

- a speculative component, related to the risk premium ξ_λ;
- a hedging component against labour income fluctuations;
- a hedging component against the fluctuations of the global risk factor $F(t,z)$;

- a hedging component against the fluctuations of the expected imbalance to finance minimum consumption and wealth $H(t,z)$.

The last two components depend on the risk aversion of the individual, on the variance-covariance matrix of the state variables and on the sensitivities of $F(t,z)$ and $H(t,z)$ with respect to changes in the state variables.

The presence of the longevity asset modifies the individual demand of both the risk-less and the other financial assets. We can interpret the amount of wealth invested in the longevity asset as taken partly from the wealth invested in the risk-less asset and partly from the wealth invested in the financial assets. The proportion taken from the financial assets is given by the ratio between the covariance of the longevity and the financial assets and the variance of the financial assets (in fact, $\left(\Sigma_f'\Sigma_f\right)^{-1}\Sigma_f'\sigma_{\Lambda f} = \Sigma_f^{-1}\sigma_{\Lambda f}$). This means that the higher the correlation between a financial asset and the longevity asset, the higher the amount of wealth taken from the former to be invested in the latter. If the longevity asset is not correlated to financial assets (i.e. $\sigma_{\Lambda f} = 0$), then the amount of money to be invested in it is fully taken from what is invested in the risk-less asset.

4 A numerical application

4.1 The state variables

In order to present a numerical application, a simplified market structure is taken into account. We consider two uncorrelated state variables: the instantaneously risk-less interest rate $r(t)$ and the force of mortality $\lambda(t)$. They solve the following differential equations

$$dr(t) = \alpha_r\left(\beta_r - r(t)\right)dt + \sigma_r\sqrt{r(t)}dW_r(t), \tag{18}$$

$$d\lambda(t) = \alpha_\lambda\left(\beta_\lambda(t) - \lambda(t)\right)dt + \sigma_\lambda\sqrt{\lambda(t)}dW_\lambda(t), \tag{19}$$

where $\alpha_r, \alpha_\lambda > 0$ describe the strength of the mean reversion effect and $\beta_r, \beta(t) > 0$ are the long-term means. $\beta_\lambda(t)$ is the long-term mean of the mortality intensity process, which we set to be equal to the Gompertz law. If the Feller conditions $2a_r\beta_r > \sigma_r^2$ and $2\alpha_\lambda\beta_\lambda(t) > \sigma_\lambda^2$ are satisfied for any t, the two processes are always positive.

In order to keep the statistical properties of process r and λ unchanged after switching from probability $\mathbb{P}$ to both probabilities $\mathbb{Q}$ and $\mathbb{Q}_\delta$, we assume that both the market price of the interest rate risk and the market price of the mortality risk are proportional to the square root of the respective variable:

$$\xi_r = \phi_r\sqrt{r(t)}, \tag{20}$$

$$\xi_\lambda = \phi_\lambda \sqrt{\lambda(t)}, \tag{21}$$

where ϕ_r and ϕ_λ are constant.

In this section we are about the present some results which rely on the following proposition.

Proposition 2. *If the variable $y(t)$ solves the stochastic differential equation*

$$dy(t) = \alpha(\beta(t) - y(t))\,dt + \sigma\sqrt{y(t)}dW(t),$$

then, for any constant χ,

$$\mathbb{E}_t\left[e^{-\chi\int_t^T y(u)du}\right] = e^{-\alpha\int_t^T \beta(s)C(s;\chi,\alpha,\sigma,T)ds - C(t;\chi,\alpha,\sigma,T)y(t)},$$

where

$$C(t;\chi,\alpha,\sigma,T) = 2\chi\frac{1-e^{-\Delta(T-t)}}{\Delta+\alpha+(\Delta-\alpha)e^{-\Delta(T-t)}}, \tag{22}$$

$$\Delta \equiv \sqrt{\alpha^2+2\sigma^2\chi}.$$

Proof. See Appendix B. □

In order to simplify the computations, we assume that the labour income is deterministic (i.e. $\sigma_L = 0$) and that the wage w is constant. Under these hypotheses, the value of the function $H(t,z)$ can be written as

$$\begin{aligned} H(t,z) &= (c_m - w)\int_t^T \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s r(u,z)du}\right]\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s \lambda(u,z)du}\right]ds \\ &\quad + R_m\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u,z)du}\right]\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \lambda(u,z)du}\right], \end{aligned}$$

where

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}\right] = e^{-\alpha_r^{\mathbb{Q}}\beta_r^{\mathbb{Q}}\int_t^T C\left(s;1,\alpha_r^{\mathbb{Q}},\sigma_r,T\right)ds - C\left(t;1,\alpha_r^{\mathbb{Q}},\sigma_r,T\right)r(t)},$$

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \lambda(u)du}\right] = e^{-\alpha_\lambda^{\mathbb{Q}}\int_t^T \beta_\lambda^{\mathbb{Q}}(s)C\left(s;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T\right)ds - C\left(t;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T\right)\lambda(t)}.$$

4.2 The traded assets

Four assets are traded on the financial market:

- the risk-less asset, whose price evolves as in (4);

- a risky asset which could be thought of as a stock index, and whose price $A(t)$ follows a GBM:

$$A(t)^{-1} dA(t) = \mu dt + \sigma_A dW_A(t) + \sigma_{Ar} dW_r(t),$$

where σ_{Ar} measures the instantaneous covariance between the stock and the riskless interest rate. We assume ξ_A is constant and thus

$$\begin{aligned} A(t)^{-1} dA(t) &= r(t) dt + \sigma_A dW_A^{\mathbb{Q}}(t) + \sigma_{Ar} dW_r^{\mathbb{Q}}(t) \\ &= \left(r(t) + \sigma_A \xi_A + \sigma_{Ar}\phi_r \sqrt{r(t)}\right) dt + \sigma_A dW_A(t) + \sigma_{Ar} dW_r(t); \end{aligned}$$

- a constant time to maturity (T_B) zero-coupon bond whose price is

$$B(t) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{t+T_B} r(u)du}\right],$$

and whose differential is

$$\begin{aligned} B(t)^{-1} dB(t) &= r(t) dt - C\left(0; 1, \alpha_r^{\mathbb{Q}}, \sigma_r, T_B\right) \sigma_r \sqrt{r(t)} dW_r^{\mathbb{Q}}(t) \\ &= r(t)\left(1 - C\left(0; 1, \alpha_r^{\mathbb{Q}}, \sigma_r, T_B\right) \sigma_r \phi_r\right) dt \\ &\quad -C\left(0; 1, \alpha_r^{\mathbb{Q}}, \sigma_r, T_B\right) \sigma_r \sqrt{r(t)} dW_r(t); \end{aligned}$$

- a constant time to maturity (T_Λ) zero-coupon longevity bond whose price is

$$\Lambda(t) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{t+T_\Lambda} r(u)+\lambda(u)du}\right],$$

and whose differential is

$$\begin{aligned} \Lambda(t)^{-1} d\Lambda(t) &= (r(t) + \lambda(t)) dt - C\left(0; 1, \alpha_r^{\mathbb{Q}}, \sigma_r, T_\Lambda\right) \sigma_r \sqrt{r(t)} dW_r^{\mathbb{Q}}(t) \\ &\quad -C\left(0; 1, \alpha_\lambda^{\mathbb{Q}}, \sigma_\lambda, T_\Lambda\right) \sigma_\lambda \sqrt{\lambda(t)} dW_\lambda^{\mathbb{Q}}(t) \\ &= \left(\left(1 - C\left(0; 1, \alpha_r^{\mathbb{Q}}, \sigma_r, T_\Lambda\right) \sigma_r \phi_r\right) r(t) + \left(1 - C\left(0; 1, \alpha_\lambda^{\mathbb{Q}}, \sigma_\lambda, T_\Lambda\right) \sigma_\lambda \phi_\lambda\right) \lambda(t)\right) dt \\ &\quad -C\left(0; 1, \alpha_r^{\mathbb{Q}}, \sigma_r, T_\Lambda\right) \sigma_r \sqrt{r(t)} dW_r(t) - C\left(0; 1, \alpha_\lambda^{\mathbb{Q}}, \sigma_\lambda, T_\Lambda\right) \sigma_\lambda \sqrt{\lambda(t)} dW_\lambda(t). \end{aligned}$$

Since the market price of the stock has been assumed to be constant (ξ_A) then the function $F(t)$ can be written as

$$\begin{aligned} F(t) &= \int_t^T e^{-\left(\frac{1}{\delta}\rho + \frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi_A^2\right)(s-t)} \mathbb{E}_t^{\mathbb{Q}_\delta}\left[e^{-\frac{\delta-1}{\delta}\left(1+\frac{1}{2}\frac{1}{\delta}\phi_r^2\right)\int_t^s r(u)du}\right] \mathbb{E}_t^{\mathbb{Q}_\delta}\left[e^{-\left(1+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\phi_\lambda^2\right)\int_t^s \lambda(u)du}\right] ds \\ &\quad + e^{-\left(\frac{1}{\delta}\rho + \frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi_A^2\right)(T-t)} \mathbb{E}_t^{\mathbb{Q}_\delta}\left[e^{-\frac{\delta-1}{\delta}\left(1+\frac{1}{2}\frac{1}{\delta}\phi_r^2\right)\int_t^T r(u)du}\right] \mathbb{E}_t^{\mathbb{Q}_\delta}\left[e^{-\left(1+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\phi_\lambda^2\right)\int_t^T \lambda(u)du}\right], \end{aligned}$$

or

$$
\begin{aligned}
F(t) &= \int_t^T \begin{bmatrix} e^{-\left(\frac{1}{\delta}\rho+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi_A^2\right)(s-t)} \\ e^{-\alpha_r^{\mathbb{Q}\delta}\beta_r^{\mathbb{Q}\delta}\int_t^s C\left(u;\frac{\delta-1}{\delta}\left(1+\frac{1}{2}\frac{1}{\delta}\phi_r^2\right),\alpha_r^{\mathbb{Q}\delta},\sigma_r,s\right)du-C\left(t;\frac{\delta-1}{\delta}\left(1+\frac{1}{2}\frac{1}{\delta}\phi_r^2\right),\alpha_r^{\mathbb{Q}\delta},\sigma_r,s\right)r(t)} \\ e^{-\alpha_\lambda^{\mathbb{Q}\delta}\int_t^s \beta_\lambda^{\mathbb{Q}\delta}(s)C\left(u;\left(1+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\phi_\lambda^2\right),\alpha_\lambda^{\mathbb{Q}\delta},\sigma_\lambda,s\right)du-C\left(t;\left(1+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\phi_\lambda^2\right),\alpha_\lambda^{\mathbb{Q}\delta},\sigma_\lambda,s\right)\lambda(t)} \end{bmatrix} ds \\
&\quad + e^{-\left(\frac{1}{\delta}\rho+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi_A^2\right)(T-t)} \\
&\quad \times e^{-\alpha_r^{\mathbb{Q}\delta}\beta_r^{\mathbb{Q}\delta}\int_t^T C\left(s;\frac{\delta-1}{\delta}\left(1+\frac{1}{2}\frac{1}{\delta}\phi_r^2\right),\alpha_r^{\mathbb{Q}\delta},\sigma_r,T\right)ds-C\left(t;\frac{\delta-1}{\delta}\left(1+\frac{1}{2}\frac{1}{\delta}\phi_r^2\right),\alpha_r^{\mathbb{Q}\delta},\sigma_r,T\right)r(t)} \\
&\quad \times e^{-\alpha_\lambda^{\mathbb{Q}\delta}\int_t^T \beta_\lambda^{\mathbb{Q}\delta}(s)C\left(s;\left(1+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\phi_\lambda^2\right),\alpha_\lambda^{\mathbb{Q}\delta},\sigma_\lambda,T\right)ds-C\left(t;\left(1+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\phi_\lambda^2\right),\alpha_\lambda^{\mathbb{Q}\delta},\sigma_\lambda,T\right)\lambda(t)}.
\end{aligned}
$$

4.3 The optimal portfolio

In order to explicitly compute the optimal portfolio from (16), we define the following matrices: $\theta_S(t) = \begin{bmatrix} \theta_A(t) & \theta_B(t) & \theta_\Lambda(t) \end{bmatrix}$, $dW(t) = \begin{bmatrix} dW_A(t) & dW_r(t) & dW_\lambda(t) \end{bmatrix}$, $z(t) = \begin{bmatrix} r(t) & \lambda(t) \end{bmatrix}$,

$$
I_S = \begin{bmatrix} A(t) & 0 & 0 \\ 0 & B(t) & 0 \\ 0 & 0 & \Lambda(t) \end{bmatrix}, \xi = \begin{bmatrix} \xi_A \\ \phi_r\sqrt{r(t)} \\ \phi_\lambda\sqrt{\lambda(t)} \end{bmatrix}, \Omega' = \begin{bmatrix} 0 & \sigma_r\sqrt{r(t)} & 0 \\ 0 & 0 & \sigma_\lambda\sqrt{\lambda(t)} \end{bmatrix},
$$

$$
\Sigma' = \begin{bmatrix} \sigma_A & \sigma_{Ar} & 0 \\ 0 & -C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_B\right)\sigma_r\sqrt{r(t)} & 0 \\ 0 & -C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_\Lambda\right)\sigma_r\sqrt{r(t)} & -C\left(0;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T_\Lambda\right)\sigma_\lambda\sqrt{\lambda(t)} \end{bmatrix}.
$$

The optimal portfolio is

$$
\begin{aligned}
\begin{bmatrix} A(t)\,\theta_A(t) \\ B(t)\,\theta_B(t) \\ \Lambda(t)\,\theta_\Lambda(t) \end{bmatrix} &= \frac{R(t)-H(t)}{\delta} \begin{bmatrix} \frac{\xi_A}{\sigma_A} \\ \frac{1}{C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_B\right)}\left(\frac{\sigma_{Ar}\xi_A}{\sigma_A\sigma_r\sqrt{r(t)}}-\frac{\phi_r}{\sigma_r}+\frac{\phi_\lambda C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_\Lambda\right)}{\sigma_\lambda C\left(0;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T_\Lambda\right)}\right) \\ -\frac{\phi_\lambda}{\sigma_\lambda C\left(0;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T_\Lambda\right)} \end{bmatrix} \\
&\quad + \frac{R(t)-H(t)}{F(t)} \begin{bmatrix} 0 \\ \frac{1}{C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_B\right)}\left(-\frac{\partial F(t)}{\partial r(t)}+\frac{C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_\Lambda\right)}{C\left(0;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T_\Lambda\right)}\frac{\partial F(t)}{\partial\lambda(t)}\right) \\ -\frac{1}{C\left(0;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T_\Lambda\right)}\frac{\partial F(t)}{\partial\lambda(t)} \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 \\ \frac{1}{C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_B\right)}\left(-\frac{\partial H(t)}{\partial r(t)}+\frac{C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_\Lambda\right)}{C\left(0;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T_\Lambda\right)}\frac{\partial H(t)}{\partial\lambda(t)}\right) \\ -\frac{1}{C\left(0;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T_\Lambda\right)}\frac{\partial H(t)}{\partial\lambda(t)} \end{bmatrix}.
\end{aligned}
$$

The optimal investment in the risk-less asset is obtained as the difference between the total wealth $R(t)$ and the amount invested in the other assets.

4.4 Calibration

All the parameters related to the financial market have been estimated from three time series for the period between January 1st 1962 and January 1st 2007 (thus we take out the turbulence of the sub-prime crisis):

- the 3-month US Treasury Bill interest rate (on secondary market): for calibrating the parameters of $r(t)$ process;
- the 10-year US Bond interest rate (on secondary market): for calibrating the parameters of $B(t)$ process (with $T_B = 10$);
- S&P 500: for calibrating the parameters of $A(t)$.

The parameters of the risk-free interest rate α_r, β_r and σ_r are gathered in Table 1.

The initial value of interest rate is set to its long term equilibrium value (i.e. $r_0 = \beta_r$). The average return on 10-year bonds is about 7.1%. Thus, if we solve

$$\mathbb{E}_t\left[d\ln B(t)\right] = 0.071dt,$$

i.e.

$$\left(1 - C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_B\right)\sigma_r\phi_r - \frac{1}{2}C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_B\right)^2\sigma_r^2\right)r(t) = 0.071,$$

with respect to ϕ_r (recall that $\alpha_r^{\mathbb{Q}}$ is a function of ϕ_r) and where, instead of $r(t)$, we use the long term equilibrium value β_r, we obtain $\phi_r = -0.5590635$, as listed in Table 1.

The variance and the mean of the log-return of S&P 500 are

$$\mathbb{V}\left[d\ln A(t)\right] = \left(\sigma_{Ar}^2 + \sigma_A^2\right)dt = 0.0223dt,$$

$$\mathbb{E}\left[d\ln A(t)\right] = \left(r(t) + \sigma_A\xi_A + \sigma_{Ar}\phi_r\sqrt{r(t)} - \frac{1}{2}\left(\sigma_A^2 + \sigma_{Ar}^2\right)\right)dt = 0.06688dt,$$

where $r(t)$ will be substituted with the long term equilibrium level β_r.

The covariance between the S&P 500 log-return and the return on the 10-year bonds is

$$\mathbb{C}\left[d\ln A(t), d\ln B(t)\right] = -C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_B\right)\sigma_{Ar}\sigma_r\sqrt{r(t)}dt = -0.0004552dt,$$

where the interest rate will be again substituted with its long term mean β_r. Thus, we have to solve the following system

$$\begin{cases} \sigma_{Ar}^2 + \sigma_A^2 = 0.0223, \\ \beta_r + \sigma_A\xi_A + \sigma_{Ar}\phi_r\sqrt{\beta_r} - \frac{1}{2}\left(\sigma_A^2 + \sigma_{Ar}^2\right) = 0.06688, \\ -C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_B\right)\sigma_{Ar}\sigma_r\sqrt{\beta_r} = -0.0004552, \end{cases}$$

Table 1: Parameters for the base scenario, calibrated on the S&P 500, 3-month Treasury Bills, and 10-year Bonds time series (between January 1st 1962 and January 1st 2007).

Interest rate/Bond	Stock	Wealth/Preferences	Mortality/Longevity
$\alpha_r = 0.0904668$	$\sigma_A = 0.14926$	$R_0 = 100$	$\alpha_\lambda = 0.561$
$\beta_r = 0.0621328 = r_0$	$\sigma_{Ar} = 0.0046306$	$w = 10$	$\sigma_\lambda = 0.0352$
$\sigma_r = 0.0543625$	$\xi_A = 0.1108301$	$T = 65$	$\phi_0 = 0.0009944$
$\phi_r = -0.5590635$		$\rho = 0.01$	$b = 12.9374$
$T_B = 10$		$R_m = 100$	$m = 86.4515$
		$c_m = 0$	$t_0 = 60$
		$\delta = 2.5$	$T_L = 10$

which has a positive and a negative solution for σ_A; we take the positive one as shown in Table 1.

Mortality is estimated by fitting the observed survival probabilities for US males born in 1950, who were aged 60 (t_0) at January 1st, 2010. We fix the observation point to January 1st, 1990 and we obtain the observed survival curve from cohort tables available at the Human Mortality Database using 20 data points. For $t \in \{1, 2, ..., 19\}$ we get the survival probability $p(0, t)$. We calibrate the intensity process by minimising the mean squared error between fitted and observed values of the survival probability, imposing at the same time the Feller condition. The initial value of the mortality intensity is

$$\lambda(t_0) = \phi_0 + \frac{1}{b}\left(1 + \frac{1}{\alpha_\lambda}\right) e^{\frac{t_0 - m}{b}}.$$

The survival probability up to T is available in semi-closed form and is equal to

$$\hat{p}(t_0, T) = \mathbb{E}_{t_0}^{\mathbb{P}}\left[e^{-\int_{t_0}^{T} \lambda(u)du}\right] = e^{-\alpha_\lambda \int_{t_0}^{T} \beta_\lambda(s) C(s;1,\alpha_\lambda,\sigma_\lambda,T)ds - C(0;1,\alpha_\lambda,\sigma_\lambda,T)\lambda(t_0)}.$$

We thus determine the parameters of the intensity process, α_λ, σ_λ, ϕ_0, b and m, by minimising the cost function

$$\min_{\alpha_\lambda,\sigma_\lambda,\phi_0,b,m} \frac{1}{n}\sqrt{\sum_{t=1}^{n} \left(\hat{p}(0,t) - p(0,t)\right)^2}.$$

The values of all the parameters are gathered in Table 1.

4.5 Base scenario

We assume the existence of a continuously-rolled over longevity bond with maturity $T_L = 10$ years whose underlying is the mortality intensity of the cohort of US males born in 1950. In our base scenario, we set the risk premium for mortality $\phi_\lambda = 0$. We consider an individual who does not want to deplete his initial wealth R_0, having in mind that he would like to annuitise a part of the financial wealth at time T to face his consumption needs after retirement. Hence, we set $R_m = R_0 = 100$. We set the salary to one tenth of the initial wealth per year, i.e $w = 10$. We fix retirement age to 65 years, and thus $T = 5$. The optimal consumption/investment is computed at daily intervals.

A sample consumption/investment profile is presented in Figure 1. The individual's wealth volatility is decreasing over time. At the beginning of the horizon, he invests around 22% in the stock, goes short 9% in the risk-free asset, and 1.5% in the bond. On the other side, he invests 88% of his wealth in the longevity asset. While he approaches retirement, he progressively reduces his investment in equity, in order to reach his final wealth target with a higher degree of certainty, and increases the share invested in bonds and in the risk-free asset, which reaches almost 100% approaching retirement age. For the same reason, he reduces consumption, which drops from around 15% at t_0 to around 7.5% at T. This drop is consistent across simulations, though its intensity varies. Investment in the longevity asset is decreasing over time as well. This is due to the fact that, as time passes, the fluctuations of the discount factors due to changing in mortality likelihood affect less the valuation of wealth of the individual. Indeed, since $\phi_\lambda = 0$ and the correlation between the longevity asset and the financial asset is null, the demand for the longevity asset is entirely due to the hedging component against the fluctuations of the functions F and H. Figure 2, which collects 100 paths with optimal strategies computed at monthly intervals, allows to appreciate the variability in sample paths. Investment profiles are very much stable across different paths, while the consumption profile is more volatile, as hinted above. In this case, we thus confirm the results found in other fields that consumption tends to absorb most of the uncertainty in the model. Thus, after a financial shock, an agent prefers to vary his consumption rather than his portfolio allocation.

4.6 Sensitivity analysis: longevity risk premium

We consider first the case in which the risk premium for the longevity asset is positive and $\phi_\lambda = 0.1$. Given the absence of a liquid market for longevity securities at present, estimating a reasonable value for the risk premium is very difficult, and our choice of $\phi_\lambda = 0.1$ can be considered as a worst-case scenario. Indeed, the risk premium should

be negative, as longevity bonds negatively react to changes in the force of mortality, for which investors would require a compensation (a lower price for the zero-coupon bond). Nonetheless, due to the particular nature of the asset, whose underlying is related to the individual discount factor, we explore this worst-case scenario, to provide a lower bound to the demand for longevity bonds in optimal individual portfolios. Under this parametrisation, Figure 3 reports the optimal consumption and investment profiles. Investment in the longevity asset is lower than in the base case, and has a humped-shaped profile, increasing up to almost 50% from the initial 40.5% before decreasing to reach even negative values (-6.6%) at retirement. While investment in the stock and the risk-less asset remain unchanged w.r.t. the base scenario, now the agent initially invests a positive amount of wealth in the bond. The time profile is then similar, with disinvestment followed by a re-investment while approaching the horizon. Consumption falls as in the base scenario. Obviously, any negative value for the risk premium increases the investment in the longevity asset. We performed the analysis by fixing $\phi_\lambda = -0.25$, a value which was suggested as reasonable by Loeys et al. (2007) and that is close to the tentative estimate of Bisetti et al. (2014). The simulations show a pattern which is similar to the base scenario, but the investment choices are tilted towards the longevity asset, whose "speculative" attractiveness adds to the hedging motive. Initial investment reaches more than 200%, and the additional allocation in the longevity asset is financed by short selling the risk-less asset.

4.7 Sensitivity analysis: individual characteristics

In this subsection we present an analysis of the changes in the consumption/portfolio choices due to: (i) the sex of the consumer/investor, (ii) the time horizon, (iii) the risk aversion (δ), (iv) subsistence final wealth (R_m), and (v) subsistence consumption (c_m).

Sex

We perform first a sensitivity analysis with respect to the sex of the individual. We calibrate the stochastic mortality model to 60-year old US females, using the same procedure we described in Section 4.4. Parameters are reported in Table 2. Women should invest a higher fraction of their wealth in the longevity asset (due to their longer lifetime expectancy). Optimal share invested in this asset lies between 99% and 102% for the first 4 years of the horizon and drops only at the very end to reach 37% in the last month prior to retirement. This is due to the higher predictability of mortality rates, due to the females' higher level of mean reversion to the Gompertz law, which is due to the lower variability of mortality rates displayed by females in the calibrated part of the curve (survival rates from age 40 to 60).

Table 2: Calibrated parameters of the mortality processes of individuals outside the base case scenario.

	60-yr old females	55-year old males
α_λ	0.5686	0.5659
σ_λ	0.0277	0.0243
ϕ_0	$6.12 \cdot 10^{-17}$	0.0002
b	13.0579	17.0836
m	91.9838	90.9065

Time horizon

Analysis of the consumption/investment profile of a younger individual, who is 55 at time t_0 (parameters are reported in Table 2), with same fixed retirement age at 65, reveals that demand of the longevity asset is above around 90% and slightly increasing, from age 55 to 60. Consumption/investment decisions in the second part of the time period (when the agent is aged 60 to 65) are in line with the base scenario. In the first 3-year period, instead, consumption stays on average almost constant, and starts decreasing markedly around age 58. Investment in equity decreases in time, while the share of wealth in the risk-free asset increases, from -55% to almost 100%. Investment in the bond market is positive in the first 5 years, but drops from the initial 34.5% down to around -30% before increasing in the last two years of the horizon.

Risk aversion

A smaller risk aversion ($\delta = 1.5$) obviously increases investment in the stock, up to an initial 37%. Also, it slightly increases investment in the longevity asset over the whole horizon. The initial investment increases to 88.6% (w.r.t. 88.1% in the base scenario) of the initial wealth, as the individual is willing to bear a higher amount of longevity risk. On the contrary, a higher risk aversion ($\delta = 5$), decreases investment in the stock (down to 11% at time t_0) and in the longevity asset (87.7%). Apart from absolute values, investment and consumption profiles approaching retirement are similar to the base scenario.

Subsistence final wealth

In the case the investor has no interest in setting a minimum level of final wealth (i.e. $R_m = 0$) Figure 4 shows that the agent progressively increases consumption as retirement approaches. Also, the investment profile is riskier, as the individual invests

initially 42% in the stock and around 30% in the bond. Investment in the longevity asset is slightly reduced with respect to the base scenario, accounting initially for 71% of wealth. The agent finances his investments short selling the risk-less asset.

Subsistence consumption

When the agent wants to set a minimum consumption level ($c_m = 8$, i.e. 8% of his initial wealth, 2% less than his deterministic labour income flow) he/she assumes a more conservative behaviour. The initial investment in stocks (12%) and bonds (-16%) are both lowered, while the share in the risk-less asset (18%) is increased w.r.t. the base scenario. Investment in the longevity asset is almost insensitive, slightly lowered, but still accounting for 86.5% of the initial portfolio. As time passes, the optimal portfolio qualitatively behaves as in the base scenario.

5 Concluding comments

This paper derives the optimal consumption and investment profiles of an individual prior to a fixed retirement age in the presence of longevity risk. The market is complete and an asset written on the mortality intensity of the individual is present. Investment in such an asset is driven by three motivations: (i) speculation, if the risk premium is attractive, (ii) diversification, if the asset is negatively correlated with other financial assets, and (iii) hedging, as changes in mortality affect the discount factors of the individual. When focusing on this latter factor, we calibrate our model to US real data, and we find that the demand for longevity assets might be substantial. The result is robust across different individual types. We explore sensitivity with respect to mortality risk premium, finding that the demand for the longevity asset is positive even when the risk premium is comparable to the one required for stocks. Our results seem to suggest that, alongside reinsurance of longevity risk, which is currently growing in volume and number of transactions, there is potential room for the involvement of individual investors in the market for longevity. They would benefit from the possibility of adjusting their hedge against mortality fluctuations, at least before annuitisation of their wealth.

An interesting research question, which has been raised by Cocco and Gomes (2012) but needs further and more careful exploration, concerns the optimal design of longevity securities. Further research is indeed needed as well in order to explore the impact of basis risk, i.e. the possibility that the longevity asset is imperfectly correlated with the mortality intensity of the individual.

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A Proof of Proposition 1

We solve the maximisation problem following the martingale approach. The Lagrangian function from problem (13) under constraint (14) is defined as follows (the functional dependencies on z have been omitted for the sake of simplicity):

$$\begin{aligned}\mathcal{L} &= \mathbb{E}_{t_0}\left[\int_{t_0}^{T}\frac{\left(c\left(s\right)-c_m\right)^{1-\delta}}{1-\delta}e^{-\int_{t_0}^{s}\rho(u)+\lambda(u)du}ds+\frac{\left(R\left(T\right)-R_m\right)^{1-\delta}}{1-\delta}e^{-\int_{t_0}^{T}\rho(u)+\lambda(u)du}\right]\\ &\quad+\phi\left(R\left(t_0\right)-\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{T}\left(c\left(s\right)-w\left(s\right)+\sigma_L\left(s\right)'\xi\left(s\right)\right)e^{-\int_{t_0}^{s}r(u)+\lambda(u)du}ds\right.\right.\\ &\quad\left.\left.+\ R\left(T\right)e^{-\int_{t_0}^{T}r(u)+\lambda(u)du}\right]\right)\end{aligned}$$

where ϕ is the (constant) Lagrangian multiplier. Through a change of probability, the function $\mathcal{L}$ can be rewritten as

$$\begin{aligned}\mathcal{L} &= \mathbb{E}_{t_0}\left[\int_{t_0}^{T}\frac{\left(c\left(s\right)-c_m\right)^{1-\delta}}{1-\delta}e^{-\int_{t_0}^{s}\rho(u)+\lambda(u)du}ds+\frac{\left(R\left(T\right)-R_m\right)^{1-\delta}}{1-\delta}e^{-\int_{t_0}^{T}\rho(u)+\lambda(u)du}\right]\\ &\quad+\phi\left(R\left(t_0\right)-\mathbb{E}_{t_0}\left[\int_{t_0}^{T}\left(c\left(s\right)-w\left(s\right)+\sigma_L\left(s\right)'\xi\left(s\right)\right)m\left(t_0,s\right)e^{-\int_{t_0}^{s}r(u)+\lambda(u)du}ds\right.\right.\\ &\quad\left.\left.+\ R\left(T\right)m\left(t_0,T\right)e^{-\int_{t_0}^{T}r(u)+\lambda(u)du}\right]\right).\end{aligned}$$

The first order condition on consumption

$$\frac{\partial\mathcal{L}}{\partial c\left(s\right)} = \mathbb{E}_{t_0}\left[\int_{t_0}^{T}\left(\left(c\left(s\right)-c_0\right)^{-\delta}e^{-\int_{t_0}^{s}\rho(u)+\lambda(u)du}-\phi m\left(t_0,s\right)e^{-\int_{t_0}^{s}r(u)+\lambda(u)du}\right)ds\right]=0,$$

must hold for any state of the world and, accordingly, the optimal consumption at any time s is

$$c^*\left(s\right)=c_m+\left(\phi m\left(t_0,s\right)\frac{e^{-\int_{t_0}^{s}r(u)du}}{e^{-\int_{t_0}^{s}\rho(u)du}}\right)^{-\frac{1}{\delta}}.$$

The same approach on final wealth gives the first order condition

$$\frac{\partial\mathcal{L}}{\partial R\left(T\right)} = \mathbb{E}_{t_0}\left[\left(R\left(T\right)-R_m\right)^{-\delta}e^{-\int_{t_0}^{T}\rho(u)+\lambda(u)du}-\phi m\left(t_0,T\right)e^{-\int_{t_0}^{T}r(u)+\lambda(u)du}\right]=0,$$

and the optimal final wealth

$$R^{*}(T)=R_{m}+\left(\phi m\left(t_{0},T\right)\frac{e^{-\int_{t_{0}}^{T}r(u)du}}{e^{-\int_{t_{0}}^{T}\rho(u)du}}\right)^{-\frac{1}{\delta}}.$$

When the constraint is rewritten at time t (instead of t_0) as follows

$$\begin{aligned}R(t) &= \mathbb{E}_{t}\left[\int_{t}^{T}\left(c(s)-w(s)+\sigma_{L}(s)'\xi(s)\right)m(t,s)\,e^{-\int_{t}^{s}r(u)+\lambda(u)du}ds\right.\\ &\quad\left.+R(T)\,m(t,T)\,e^{-\int_{t}^{T}r(u)+\lambda(u)du}\right],\end{aligned}$$

and the optimal consumption and final wealth are both substituted in it, the following optimal wealth is obtained

$$\begin{aligned}R(t) &= \left(\phi m\left(t_{0},t\right)\frac{e^{-\int_{t_{0}}^{t}r(u)du}}{e^{-\int_{t_{0}}^{t}\rho(u)du}}\right)^{-\frac{1}{\delta}}\mathbb{E}_{t}\left[\int_{t}^{T}m(t,s)^{1-\frac{1}{\delta}}\,e^{-\int_{t}^{s}\left(\frac{\delta-1}{\delta}r(u)+\frac{1}{\delta}\rho(u)+\lambda(u)\right)du}ds\right.\\ &\quad\left.+m(t,T)^{1-\frac{1}{\delta}}\,e^{-\int_{t}^{T}\left(\frac{\delta-1}{\delta}r(u)+\frac{1}{\delta}\rho(u)+\lambda(u)\right)du}\right]\\ &\quad+\mathbb{E}_{t}^{\mathbb{Q}}\left[\int_{t}^{T}\left(c_{m}-w(s)+\sigma_{L}(s)'\xi(s)\right)e^{-\int_{t}^{s}r(u)+\lambda(u)du}ds+R_{m}e^{-\int_{t}^{T}r(u)+\lambda(u)du}\right].\end{aligned}$$

We can define the following functions

$$H(t,z)=\mathbb{E}_{t}^{\mathbb{Q}}\left[\int_{t}^{T}\left(c_{m}-w(s)+\sigma_{L}(s)'\xi(s)\right)e^{-\int_{t}^{s}r(u)+\lambda(u)du}ds+R_{m}e^{-\int_{t}^{T}r(u)+\lambda(u)du}\right],$$

$$F(t,z)=\mathbb{E}_{t}\left[\int_{t}^{T}m(t,s)^{1-\frac{1}{\delta}}\,e^{-\int_{t}^{s}\left(\frac{\delta-1}{\delta}r(u)+\frac{1}{\delta}\rho(u)+\lambda(u)\right)du}ds+m(t,T)^{1-\frac{1}{\delta}}\,e^{-\int_{t}^{T}\left(\frac{\delta-1}{\delta}r(u)+\frac{1}{\delta}\rho(u)+\lambda(u)\right)du}\right].$$

We highlight that $m(t,\dot{s})^{1-\frac{1}{\delta}}$ is not a martingale but $m(t,\dot{s})^{1-\frac{1}{\delta}}e^{\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\int_{t_0}^{t}\xi'\xi ds}$ is, in fact

$$\frac{d\left(m(t,\dot{s})^{1-\frac{1}{\delta}}e^{\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\int_{t_0}^{t}\xi'\xi ds}\right)}{m(t,\dot{s})^{1-\frac{1}{\delta}}e^{\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\int_{t_0}^{t}\xi'\xi ds}}=-\frac{\delta-1}{\delta}\xi(s)\,dW(s).$$

Accordingly, we can define the new probability

$$dW(t)^{\mathbb{Q}_{\delta}}=\frac{\delta-1}{\delta}\xi(t)\,dt+dW(t),$$

and write

$$F(t) = \mathbb{E}_t^{\mathbb{Q}_\delta}\left[\begin{array}{c}\int_t^T e^{-\int_t^s\left(\frac{\delta-1}{\delta}r(u,z)+\frac{1}{\delta}\rho(u,z)+\lambda(u,z)+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi(u,z)'\xi(u,z)\right)du}ds \\ +e^{-\int_t^T\left(\frac{\delta-1}{\delta}r(u,z)+\frac{1}{\delta}\rho(u,z)+\lambda(u,z)+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi(u,z)'\xi(u,z)\right)du}\end{array}\right].$$

The optimal wealth can then be written as

$$R(t) = \left(\phi m(t_0,t)\frac{e^{-\int_{t_0}^t r(u)du}}{e^{-\int_{t_0}^t \rho(u)du}}\right)^{-\frac{1}{\delta}} F(t,z) + H(t,z),$$

whose differential (through Ito's lemma) is (the drift term is neglected since it is immaterial to replication):

$$\begin{aligned} dR(t) &= (...)\,dt + \frac{1}{\delta}\left(\phi m(t_0,t)\frac{e^{-\int_{t_0}^t r(u)du}}{e^{-\int_{t_0}^t \rho(u)du}}\right)^{-\frac{1}{\delta}} F(t,z)\,\xi(t,z)'\,dW(t) \\ &\quad + \left(\phi m(t_0,t)\frac{e^{-\int_{t_0}^t r(u)du}}{e^{-\int_{t_0}^t \rho(u)du}}\right)^{-\frac{1}{\delta}} F_z(t,z)'\,\Omega(t,z)'\,dW(t) \\ &\quad + H_z(t,z)'\,\Omega(t,z)'\,dW(t), \end{aligned}$$

where the subscripts on $F(t,z)$ and $H(t,z)$ indicate partial derivatives. Once the following relationship

$$\frac{R(t)-H(t,z)}{F(t,z)} = \left(\phi m(t_0,t)\frac{e^{-\int_{t_0}^t r(u,z)du}}{e^{-\int_{t_0}^t \rho(u,z)du}}\right)^{-\frac{1}{\delta}},$$

is suitably taken into account, the differential equation becomes

$$dR(t) = (...)\,dt + \left(\frac{R(t)-H(t,z)}{\delta}\xi(t,z)' + \frac{R(t)-H(t,z)}{F(t,z)}F_z(t,z)'\,\Omega(t,z)' + H_z(t,z)'\,\Omega(t,z)'\right)dW(t),$$

which implies that the optimal portfolio is (recall (12))

$$\Sigma(t,z)\,I_S\theta_S(t)+\sigma_L(t,z) = \frac{R(t)-H(t,z)}{\delta}\xi + \frac{R(t)-H(t,z)}{F(t,z)}\Omega(t,z)\,F_z(t,z)+\Omega(t,z)\,H_z(t,z),$$

as shown in Proposition 1.

B Proof of Proposition 2

Let us assume that the stochastic variable $y(t)$ solves the following stochastic differential equation

$$dy(t) = \alpha(\beta(t) - y(t))\,ds + \sigma\sqrt{y(t)}dW(t).$$

If χ is constant, then the expected value $\mathbb{E}_t\left[e^{-\chi\int_t^T y(u)du}\right]$ can be written as the exponential of an affine transformation of $y(t)$ as follows

$$\mathbb{E}_t\left[e^{-\chi\int_t^T y(u)du}\right] = e^{-D(t)-C(t)y(t)}.$$

If the Ito's lemma is applied to this exponential, the following differential is obtained

$$\frac{d\left(e^{-D(t)-C(t)y(t)}\right)}{e^{-D(t)-C(t)y(t)}} = \left(-\frac{\partial D(t)}{\partial t} - \frac{\partial C(t)}{\partial t}y(t) - C(t)\,\alpha(\beta(t) - y(t)) + \frac{1}{2}C(t)^2\sigma^2 y(t)\right)dtC(t)\,\sigma\sqrt{y(t)}dW(t),$$

and its drift component must be such that

$$-\frac{\partial D(t)}{\partial t} - \frac{\partial C(t)}{\partial t}y(t) - C(t)\,\alpha(\beta(t) - y(t)) + \frac{1}{2}C(t)^2\sigma^2 y(t) = \chi y(t).$$

This differential equation can be split into two differential equation: one with the terms containing $y(t)$ and one with the terms which do not contain that variable. The result is as follows:

$$0 = -\frac{\partial D(t)}{\partial t} - C(t)\,\alpha\beta(t),$$

$$0 = -\frac{\partial C(t)}{\partial t}y(t) + C(t)\,\alpha y(t) + \frac{1}{2}C(t)^2\sigma^2 y(t) - \chi y(t),$$

whose boundary conditions are $D(T) = C(T) = 0$.

The PDE for $C(t)$ is a Riccati equation with constant coefficients whose solution is well known and given by (22). Once $C(t)$ has been computed, the value of $D(t)$ is obtained from the other PDE (and the corresponding boundary condition):

$$D(t) = \alpha\int_t^T \beta(s)\,C(s)\,ds.$$

Figure 1: Sample path of optimal portfolio with the values gathered in Table 1

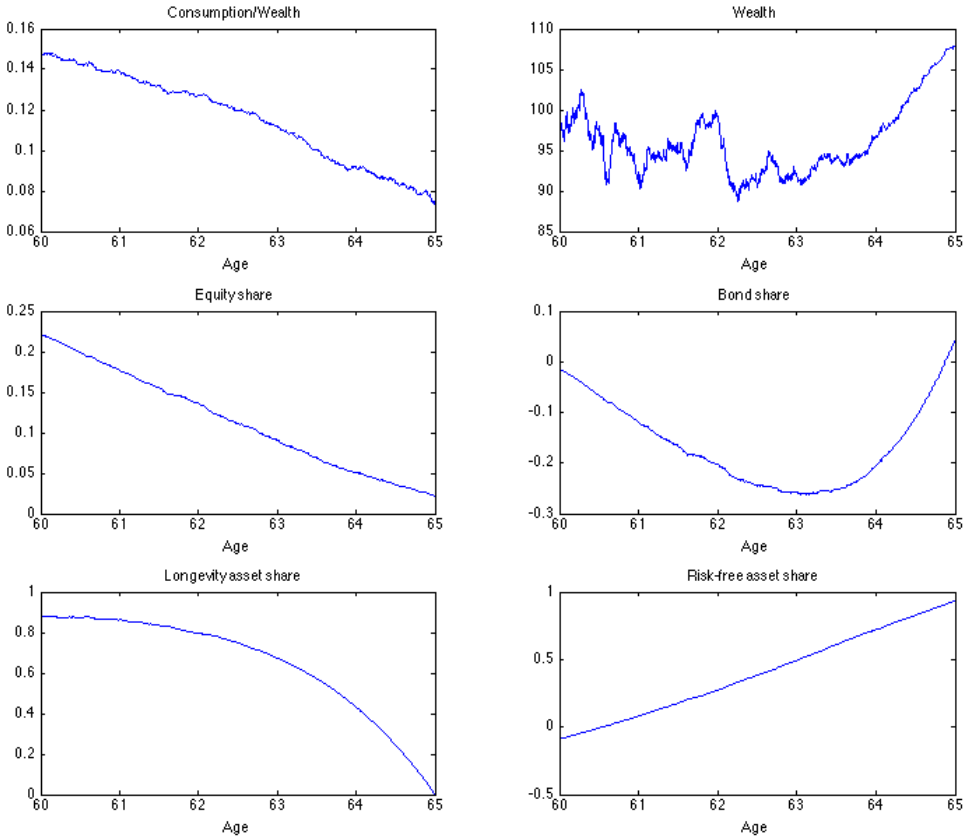

Figure 2: Simulation of optimal portfolio with the values gathered in Table 1, 100 paths

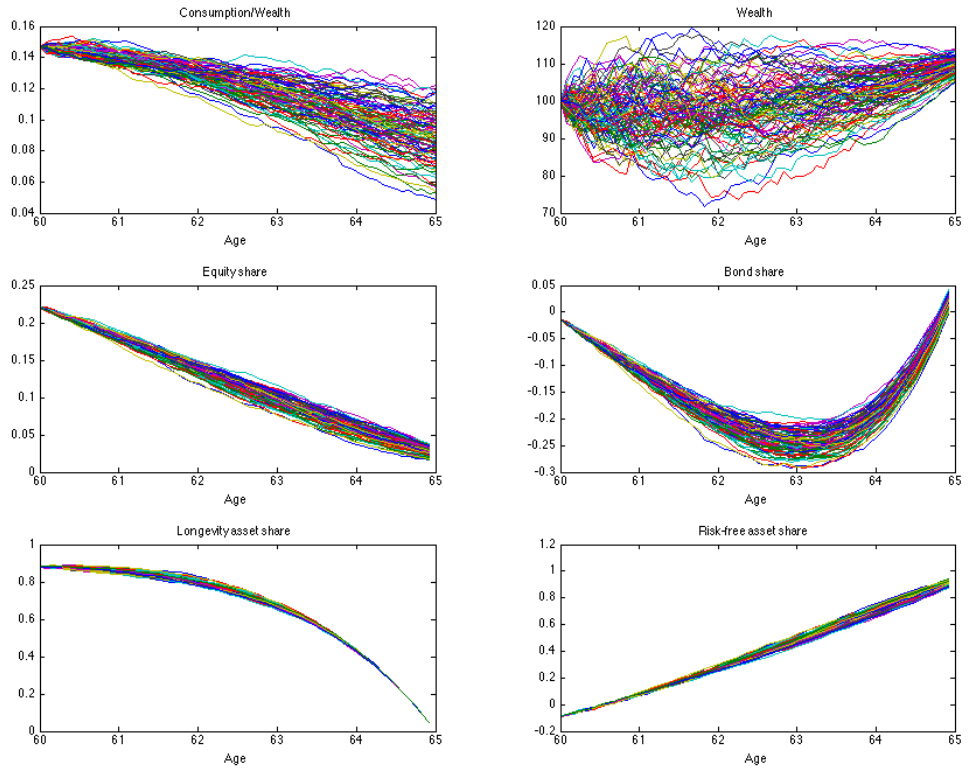

Figure 3: Sample path of optimal portfolio with the values gathered in Table 1, with $\phi_\lambda = 0.1$

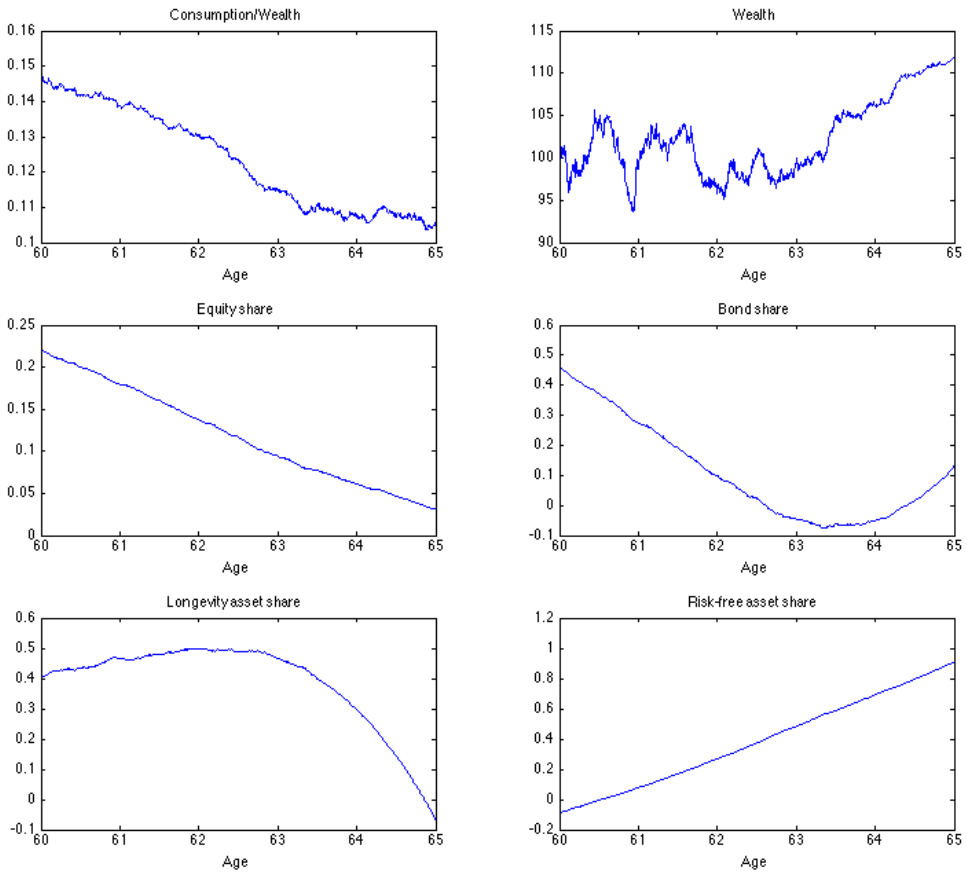

Figure 4: Sample path of optimal portfolio with the values gathered in Table 1, with $R_m = 0$