OPTIMAL CONSUMPTION AND INVESTMENT DECISIONS UNDER TIME-VARYING RISK ATTITUDES

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Abstract. As a change in the individual's risk aversion over the life-cycle is observed within several literature, different approaches are made to extend the optimal consumption and investment problem by a time-varying risk aversion. This paper combines the two approaches of including a habit level (see e.g. Constantinides (1990)) and a coefficient of time-varying risk aversion (see e.g. Steffensen (2011)). The optimal consumption and investment rules are derived in a complete market setting. They are examined in a numerical analysis for different magnitudes of habit formation and different shapes of time-varying risk aversion. Our findings show that with a coefficient of time-varying risk aversion, the shape of the decision rules, rather than just their magnitude, depend on the initial wealth of the individual. In the numerical analysis, we find that with an increasing risk aversion and sufficiently large habit formation a hump-shaped consumption pattern is achieved, as it is observed in the literature. Furthermore, in this case the investment into the risky asset is decreasing over the life-cycle, as suggested by financial advisers.

Keywords: Optimal consumption and investment, habit formation, time-varying risk aversion

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1. Introduction

Several literature observes that the risk aversion of an individual is changing over the life-cycle. For example, Riley and Chow (1992) find that the risk aversion decreases with age for individuals younger than 65 and increases for individuals older than 65. Others who find that risk aversion increases with age are e.g. Morin and Suarez (1983), Bakshi and Chen (1994), Bellante and Green (2004), Al-Ajmi (2008), Ho (2009) and Yao et al. (2011). A decreasing risk aversion, on the other hand, is observed by e.g. Bellante and Saba (1986) and Wang and Hanna (1997). The literature thus agrees that the risk aversion is changing over time, however there is no consensus whether it is increasing or decreasing.

Different approaches are used in the literature to account for a changing risk aversion within the optimal consumption and asset allocation problem, formulated in Merton (1969). One of these approaches is to include a habit level for consumption into the model, accounting for the fact that individuals get used to their standard of living. The inclusion of a habit level for consumption is introduced in Sundaresan (1989) and Constantinides (1990) and further analyzed by e.g. Ingersoll Jr (1992), Detemple and Zapatero (1991), Schroder and Skiadas (2002) and Munk (2008). Including a habit level for consumption in the utility function leads to a relative risk aversion that is decreasing in the ratio of actual consumption to the habit level. Since this ratio changes over time, also the risk aversion changes over time. Another technique to account for changing behavior over time, is to directly modify the power utility, such that the coefficient of relative risk aversion is varying over time. This problem is formulated and solved with different approaches in Steffensen (2011) and Aase (2009).

In this paper, we combine the two aforementioned approaches. We consider an additive habit level within a power utility function with a time-varying coefficient of risk aversion. For this combination, the relative risk aversion of the utility function consists of two time-varying parts: the first part is solely the exogenously given coefficient of time-varying risk aversion. Whereas the second part is driven by the consumption to habit ratio and is thus affected by the current and past market participation.

Within a complete market setting, we are able to solve the problem in closed form, using the duality result from Schroder and Skiadas (2002). In a subsequent numerical analysis, we examine the effect of a habit level for consumption and a time-varying risk aversion

on the optimal decision rules. We compare different magnitudes of habit formation and several shapes for the time-varying risk aversion.

Our main findings are: first, by including a coefficient of time-varying risk aversion, the shape of the consumption and investment rules, respectively their expectations, depend on the initial wealth of the individual. In contrast to a time-constant risk aversion, where the investment decision is unaffected by the initial wealth and the consumption decision changes only in absolute values, but not in shape, for different initial wealth. Second, the resulting optimal consumption and investment decisions are more in line with empirical observations. Namely for an increasing risk aversion and a sufficiently large habit formation, the consumption exhibits a hump-shaped pattern. This hump-shaped consumption is observed by e.g. Carroll and Summers (1991), Attanasio et al. (1999), Gourinchas and Parker (2002), Fernandez-Villaverde and Krueger (2007) and Bullard and Feigenbaum (2007). The optimal investment in the risky asset is decreasing over time for an increasing risk aversion. This is consistent with suggestions from financial advisers and literature on investment guides, e.g. Malkiel (1999) and Morris et al. (1998), who suggest that individuals invest a larger proportion into risky assets early in the life-cycle and reduce their investment in risky assets as they age. For a decreasing risk aversion, on the other hand, the asset allocation exhibits a hump-shaped pattern, which is observed by e.g. Bertaut and Starr (2000), Ameriks and Zeldes (2004) and Agnew et al. (2003).

The remainder of the paper is organized as follows. Section 2 introduces the model and the optimization problems: first an optimal consumption and investment problem for a time-varying risk aversion without habit formation, second an optimization problem including a habit level. Optimal solutions in a complete market setting are presented for both problems. Section 3 contains numerical calculations for different shapes of the time-varying risk aversion and a detailed discussion of the results and the effects on the optimal solutions. Section 4 concludes.

2. Model and optimization problem

In this section, we introduce the market model and formulate the individual's optimization problems. We start with the problem with a time-varying risk aversion and without habit formation and review the results of Steffensen (2011) and Aase (2009). Subsequently, we consider the problem including habit formation and, with the previous results, we solve

it by using the dual approach presented in Schroder and Skiadas (2002). For the following, we assume a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$ and finite time horizon $T < \infty$.

2.1. **The model.** The individual has access to a complete financial market consisting of a risk-free asset B_t and a risky asset S_t, evolving according to the following dynamics:

$$dB_t = rB_t dt,$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

The interest rate $r > 0$, the drift of the risky asset $\mu > 0$ and the volatility of the risky asset $\sigma > 0$ are assumed to be constant and W_t is a Brownian motion under $\mathbb{P}$ on the given probability space. The market price of risk is denoted by $\theta = \frac{\mu - r}{\sigma}$. Since the given market is complete, there exists a unique state-price density ξ_t that suffices the following equation:

$$d\xi_t = -\xi_t \left(rdt + \theta dW_t\right), \text{ i.e. } \xi_t = \exp\left\{\left(-r - \frac{1}{2}\theta^2\right)t - \theta W_t\right\}, \text{ with } \xi_0 = 1.$$

The individual has an initial wealth $x_0 > 0$ and at time $t \in [0, T]$ invests a proportion of her wealth π_t into the risky asset and continuously consumes c_t. The controls (π, c) are called admissible, if the consumption process $(c_t)_{t\in[0,T]}$ and the portfolio process $(\pi_t)_{t\in[0,T]}$ are measurable and adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Furthermore, if $c_t \geq 0$ $\mathbb{P} - a.s.$ $\forall t \in [0, T]$ and $E\left[\left(\int_0^T c_t dt\right)^2\right] < \infty$ $\mathbb{P} - a.s.$ By assuming that the remaining fraction of wealth $(1 - \pi_t)$ is invested into the risk-free asset, we guarantee that the portfolio process is self-financing. The set of admissible controls with initial wealth x_0 is denoted by $\mathcal{A}(x_0)$.

The dynamics of the wealth of the investor, X_t, are given by:

$$dX_t = \left(r + \pi_t(\mu - r)\right) X_t dt - c_t dt + \pi_t \sigma X_t dW_t, \text{ with } X_0 = x_0.$$

The individual bases her decisions on expected utility and aims to maximize the expected utility of consumption over the life-cycle. The utility function is denoted by $U(t, c)$ and will be specified later. Her time preferences are represented by her individual discount rate $\rho > 0$. The objective function is hence to maximize the following expected utility over the set of all admissible controls:

$$E\left[\int_0^T e^{-\rho t} U(t, c_t) dt\right].$$

2.2. **Time-varying risk aversion without habit formation.** We start with an individual with a power utility function that exhibits a time-varying risk aversion (RA). The utility function is given by $U(t,c) = \frac{1}{1-\gamma_t} c^{1-\gamma_t}$, with γ_t the time-varying relative risk aversion. The RA $\gamma_t : [0,T] \to \mathbb{R}_+$ is a deterministic, continuous function in time.[1] For notational purpose, we denote its reciprocal by $\Phi_t = \frac{1}{\gamma_t}$.

The optimal investment and consumption problem with time-varying relative risk aversion is solved in Steffensen (2011), by applying the approach of dynamic controls and solving the problem via the Hamilton-Jacobi-Bellman equation. We review his result by alternatively using the martingale approach presented in Cox and Huang (1989) and Karatzas et al. (1987). This is done in a similar way by Aase (2009) and can, in contrast to the dynamic programming approach, also be applied to non-Markov asset dynamics.

For the martingale approach, the problem is rewritten in the following form:

$$\begin{aligned} &\sup_{(\pi,c)\in\mathcal{A}(x_0)} E\left[\int_0^T e^{-\rho t}\frac{1}{1-\gamma_t}c_t^{1-\gamma_t}dt\right], \\ &\text{subject to: } E\left[\int_0^T \xi_t c_t dt\right] \leq x_0. \end{aligned} \tag{2.1}$$

The solution of the optimization problem (2.1) is given in Aase (2009), but since we need its solutions to solve the later problem including the habit, we formulate the solution in the following proposition and review its proof in the Appendix.

Proposition 2.1. *The solution to the optimization problem (2.1) is given by*

$$\begin{aligned} c_t^* =& e^{-\rho t\Phi_t}\left(y\xi_t\right)^{-\Phi_t}, \\ \pi_t^* =& \frac{\mu-r}{\sigma^2}\frac{1}{X_t^*}\int_t^T \Phi_s e^{-\rho s\Phi_s} e^{-(1-\Phi_s)\left(r+\frac{1}{2}\theta^2\Phi_s\right)(s-t)}\left(y\xi_t\right)^{-\Phi_s}ds, \\ X_t^* =& \int_t^T e^{-\rho s\Phi_s} e^{-(1-\Phi_s)\left(r+\frac{1}{2}\theta^2\Phi_s\right)(s-t)}\left(y\xi_t\right)^{-\Phi_s}ds \end{aligned}$$

and y is the optimal Lagrangian-multiplier, which is implicitly given by

$$\int_0^T e^{-\rho t\Phi_t} y^{-\Phi_t} e^{(1-\Phi_t)(-r-\frac{1}{2}\theta^2\Phi_t)t}dt = x_0.$$

Proof: Given in the Appendix.

[1]For the case of $\gamma_t = 1$, we define the utility as the limiting case $U(t,c) = \log(c)$. This does not affect the later results, since the solution only requires the derivative of the utility function.

For the properties of the optimal consumption, Steffensen (2011) shows that, for a fixed time point, the consumption is increasing in wealth and. Furthermore, in the case of an increasing γ_t, it is a convex function in wealth.

In the following, we examine further properties of the optimal solution. First, we calculate the derivative of the consumption with respect to wealth, for a fixed time point, using the representation of the solution from in Proposition 2.1. For the calculation, we take the state-price-density at time t as a function of the wealth x which is implicitly defined by $x = \int_t^T e^{-\rho s \Phi_s} e^{-(1-\Phi_s)\left(r+\frac{1}{2}\theta^2\Phi_s\right)(s-t)} y^{-\Phi_s} \xi_t^{-\Phi_s} ds$. Applying the implicit function theorem, we get the following derivative of the consumption with respect to wealth for a fixed time point t:

$$\frac{\partial}{\partial x} c_t^* = \left(\int_t^T \Phi_s e^{-\rho s \Phi_s} e^{-(1-\Phi_s)\left(r+\frac{1}{2}\theta^2\Phi_t\right)(s-t)} (y\xi_t)^{-\Phi_s} ds \right)^{-1} \left(\Phi_t e^{-\rho t \Phi_t} (y\xi_t)^{-\Phi_t} \right).$$

We recognize that this derivative corresponds to the one from Steffensen (2011). But with this representation in the complete market setting, we see that for a constant RA $\gamma_t \equiv \gamma$, the Lagrangian multiplier y in the derivative cancels out. This does not happen for a time-varying RA which is not constant. Therefore the effect of wealth on the consumption does depend on the Lagrangian multiplier y, and thus on the initial wealth x_0, when a time-varying RA is taken into account.

Next, we examine the average development of the consumption over time. Therefore, we consider its expected value:

$$E[c_t^*] = e^{(r-\rho)t\Phi_t} y^{-\Phi_t} e^{(1+\Phi_t)\frac{1}{2}\theta^2\Phi_t t} = e^{B(t)}, \text{ with } B(t) := (r-\rho)t\Phi_t - \Phi_t \log(y) + (1+\Phi_t)\frac{1}{2}\theta^2\Phi_t t.$$

The derivative of the expected value with respect to time is given by

$$\frac{\partial}{\partial t} E[c_t^*] = \frac{\partial}{\partial t} B(t) \cdot e^{B(t)} = \left[-\Phi_t' \cdot \log(y) + \left(r - \rho + \frac{1}{2}\theta^2 \right) (\Phi_t + t\Phi_t') + \frac{1}{2}\theta^2 \left(2\Phi_t'\Phi_t t + \Phi_t^2 \right) \right] \cdot E[c_t^*].$$

Since $e^{B(t)}$ is positive, the sign of the derivative only depends on $\frac{\partial}{\partial t} B(t)$. For the special case of a constant RA $\gamma_t \equiv \gamma$, we have $\Phi_t' = 0$. Given $\rho \leq r$, this gives that $\frac{\partial}{\partial t} E[c_t^*] > 0$. So if the individual discount is less than or equal to the risk-free rate on the market, the consumption is increasing in expectation. Especially, for a constant γ, the derivative does not depend on the Lagrangian multiplier y and hence not on the initial wealth. By including a time-varying RA, we have that the shape of the development of the expected consumption over time does depend on the initial wealth x_0. For the case of a time-varying RA, we can formulate a condition for the Lagrangian multiplier y and thus implicitly on the

initial wealth x_0, for which the expectation of the consumption is either strictly increasing or strictly decreasing over time. Therefore, define for $\Phi_t' \neq 0$:

$$z_t := \exp\left\{\frac{1}{\Phi_t'}\left[\left(r-\rho+\frac{1}{2}\theta^2\right)(\Phi_t+t\Phi_t')+\frac{1}{2}\theta^2\left(2\Phi_t'\Phi_t t+\Phi_t^2\right)\right]\right\},\ t\in[0,T].$$

Then we get the following conditions for the sign of the derivative of the expectation:

$$\begin{aligned}\text{For } \Phi_t'<0, \text{ i.e. } \gamma_t'>0: &\frac{\partial}{\partial t}E[c_t^*]>0 \Leftrightarrow \max_t z_t < y \\ &\frac{\partial}{\partial t}E[c_t^*]<0 \Leftrightarrow y<\min_t z_t\end{aligned} \tag{2.2}$$

$$\begin{aligned}\text{For } \Phi_t'>0, \text{ i.e. } \gamma_t'<0: &\frac{\partial}{\partial t}E[c_t^*]>0 \Leftrightarrow y<\min_t z_t \\ &\frac{\partial}{\partial t}E[c_t^*]<0 \Leftrightarrow \max_t z_t < y\end{aligned} \tag{2.3}$$

From condition (2.2), we get that for an increasing risk aversion, the consumption is in expectation strictly decreasing for sufficiently small Lagrangian multiplier y, i.e. for sufficiently large initial endowment x_0. On the other hand, if the initial endowment is small enough, i.e. y is big enough, the consumption is strictly increasing in expectation. For a decreasing risk aversion, i.e. condition (2.3), the relation is the other way round.

Since y is decreasing in x_0, given an initial wealth x_0 for which the Lagrangian multiplier y fulfills $y<\min_t z_t$, the condition is also fulfilled for larger values of x_0, i.e. the shape remains the same for larger x_0.[2]

For the investment, we can not calculate the derivatives in a similar closed form. Its behavior is considered in the later numerical analysis.

2.3. **Time-varying risk aversion with habit formation.** After describing the optimization problem for a time-varying RA, we now extend the problem by including a habit level for the consumption in the utility function. The habit level accounts for the fact that

[2]To show that y is decreasing in x_0, define $G(x_0,y) := \int_0^T e^{-\rho t\Phi_t}y^{-\Phi_t}e^{(1-\Phi_t)(-r-\frac{1}{2}\theta^2\Phi_t)t}dt - x_0$. Then the Lagrangian multiplier is implicitly defined by $G(x_0,y)=0$. Since G is differentiable, we can apply the implicit function theorem and get for the derivative of y with respect to x_0:

$$\frac{\partial y}{\partial x_0} = -\left(\frac{\partial}{\partial y}G(x_0,y)\right)^{-1}\frac{\partial}{\partial x_0}G(x_0,y) = -\left(\int_0^T \Phi_t e^{-\rho t\Phi_t}y^{-\Phi_t-1}e^{(1-\Phi_t)(-r-\frac{1}{2}\theta^2\Phi_t)t}dt\right)^{-1} < 0$$

Hence we have that the Lagrangian multiplier is strictly decreasing in x_0.

individuals are assumed to get used to their standard of living. Following Constantinides (1990) we define the habit level h_t in the following way:

$$h_t = h_0 e^{-\beta t} + \alpha \int_0^t e^{-\beta(t-s)} c_s ds, \text{ i.e. } dh_t = -(\beta h_t - \alpha c_t)dt.$$

The parameters α, β and h_0 are required to be non-negative. Furthermore, $\alpha < \beta$ is assumed to ensure that the habit level is decreasing when the consumption is equal to the habit level. With this definition, the habit level is an exponentially smoothed average of past consumption. The exponential weights imply that recent consumption gets a higher weight than earlier consumption in the life-cycle.

We assume that consumption and habit are additive and that the individual has a power utility with a time-varying coefficient of RA γ_t, as introduced before. The utility of consumption is then given by $U(t, c, h) = \frac{1}{1-\gamma_t}(c-h)^{1-\gamma_t}$. The Arrow-Pratt measure of relative risk aversion (RRA) for this utility function is given by:

$$RRA = -\frac{c_t \cdot \frac{\partial^2 U(t,c_t,h_t)}{\partial c_t^2}}{\frac{\partial U(t,c_t,h_t)}{\partial c_t}} = \gamma_t \frac{c_t}{c_t - h_t} = \gamma_t + \gamma_t \cdot \frac{1}{\frac{c_t}{h_t} - 1}. \tag{2.4}$$

Hence, the change of the RRA over time is driven by two parts: first the time-varying γ_t and second the relation between current consumption c_t and the habit h_t, which contains the past consumption. The first summand, γ_t, does solely depend on exogenous changes in the attitude towards risk over the life-cycle. It is independent of the consumption and the habit and thus of market developments. The second summand of the RRA, $\gamma_t \cdot \frac{1}{\frac{c_t}{h_t}-1}$, depends on the time-varying RA and also and the consumption to habit ratio. Hence, it is driven by the current and past consumption of the individual and thus implicitly by her current and past market participation. The magnitude of the impact of the ratio is either amplified or weakened, depending if γ_t is increasing or decreasing, respectively. Independent of the time-varying RA, this part goes to infinity, if the consumption approaches the habit level. In this case the RRA goes to infinity, what ensures that the consumption remains above the habit level. If the consumption to habit ratio is large enough, e.g. the consumption is twice the habit level, the effect of the second summand diminishes and the RRA is mainly driven by the time-varying RA.

As before, the individual's aim is to maximize the expected utility of the consumption over the life-cycle. Using the static formulation from Cox and Huang (1989) and Karatzas

et al. (1987) the optimization problem is given by

$$\sup_{(\pi,c)\in\mathcal{A}(x_0)} E\left[\int_0^T e^{-\rho t}\frac{1}{1-\gamma_t}(c_t-h_t)^{1-\gamma_t}dt\right]$$
$$\text{subject to: } E\left[\int_0^T \xi_t c_t dt\right] \leq x_0. \tag{2.5}$$

To solve the problem, we use the results from Schroder and Skiadas (2002) and rewrite the problem (2.5), called the *primal* problem, into a *dual* problem, by defining a *dual* consumption $\hat{c}_t := c_t - h_t$.[3] We then solve the dual problem and transform its solution to get the solution to the primal problem.

The dual consumption $\hat{c}_t$ consists of current consumption c_t reduced by the habit level h_t, which is some fictitious consumption derived from past consumption. In this dual market, an increase in current consumption, leads to an increase in the current instant utility of the individual. But it also decreases future utility through a larger habit level, see e.g. Detemple and Zapatero (1991).

This has to be taken into account, when pricing the consumption in this dual market and is reflected in the dual state price density given by $\hat{\xi}_t = \xi_t(1+\alpha F_t)$. The function F_t results from calculating the costs of keeping the consumption at the habit level. This can be seen when assuming that from time t on, we consume exactly at the habit level, i.e. $c_s = h_s$, $s > t$. Then the consumption is determined by $dc_s = -(\beta-\alpha)c_s ds$, with $c_t = h_t$, i.e. $c_s = e^{-(\beta-\alpha)(s-t)}h_t$. The costs for this consumption are given by $E_t\left[\int_t^T e^{-(\beta-\alpha)(s-t)}h_t\frac{\xi_s}{\xi_t}ds\right]$, where E_t denotes the expectation conditional on $\mathcal{F}_t$. With

$$F_t := \int_t^T e^{-(\beta-\alpha)(s-t)}E_t\left[\frac{\xi_s}{\xi_t}\right]ds = \int_t^T e^{-(\beta-\alpha+r)(s-t)}ds = \frac{1}{\alpha-\beta-r}\left(e^{-(\beta-\alpha+r)(T-t)}-1\right),$$

the costs to keep the consumption at the habit level can be written as $h_t F_t$.

This function F_t can also be interpreted as the price of a coupon bond, for which the coupon is exponentially decreasing over time, see e.g. Munk (2008).

With $h_t F_t$ the cost to keep the consumption at the habit level h_t, we get a condition for the initial wealth at $t = 0$. Namely, the initial wealth needs to be sufficient to cover the initial habit h_0, i.e. to finance a consumption at least at the initial habit. Thus we require $x_0 > h_0 F_0$.

[3]These *primal* and *dual* problems are not to be confused with the dual problem formulated for the martingale approach in Cox and Huang (1989) and Karatzas et al. (1987), but rather denote a formulation with and without habit.

In the dual market, we get for the dual initial wealth $\hat{x}_0 = \frac{x_0 - h_0 F_0}{1+\alpha F_0}$. This definition accounts for the fact that $h_0 F_0$ is required to finance the minimal consumption at the habit level and the adjusted pricing of the consumption.[4]

With the above considerations, we have completely defined the dual market and can formulate the following dual problem:[5]

$$\begin{aligned} &\sup_{(\hat{\pi},\hat{c})\in\mathcal{A}(\hat{x}_0)} E\left[\int_0^T e^{-\rho t}\frac{1}{1-\gamma_t}\hat{c}_t^{1-\gamma_t}dt\right] \\ &\text{subject to: } E\left[\int_0^T \hat{\xi}_t\hat{c}_t dt\right] \leq \hat{x}_0. \end{aligned} \tag{2.6}$$

This dual problem corresponds to the problem (2.1) and its solution is presented in Proposition 2.1. Hence, we get the solution to the dual problem from the previously presented results and can transform them to the solution for the primal problem (2.5). The following proposition gives the optimal solution to the primal problem.

Proposition 2.2. *Assuming $x_0 > h_0 F_0$, the solution to the optimization problem (2.5) with time-varying γ_t and habit level h_t is given by*

$$\begin{aligned} c_t^* &= e^{-\rho t\Phi_t} y^{-\Phi_t}(1+\alpha F_t)^{-\Phi_t}\xi_t^{-\Phi_t} + h_t^*, \\ h_t^* &= e^{(\alpha-\beta)t}h_0 + \alpha\int_0^t e^{(\alpha-\beta)(t-s)}e^{-\rho s\Phi_s}(y\xi_s)^{-\Phi_s}(1+\alpha F_s)^{-\Phi_s}ds, \\ \pi_t^* &= \frac{\theta}{\sigma}\frac{1}{X_t^*}\int_t^T \Phi_s e^{-\rho s\Phi_s}(1+\alpha F_s)^{1-\Phi_s}e^{(1-\Phi_s)\left(-r-\frac{1}{2}\theta^2\Phi_s\right)(s-t)}(y\xi_t)^{-\Phi_s}ds, \\ X_t^* &= h_t^* F_t + \int_t^T e^{-\rho s\Phi_s}(1+\alpha F_s)^{1-\Phi_s}e^{(1-\Phi_s)\left(-r-\frac{1}{2}\theta^2\Phi_s\right)(s-t)}(y\xi_t)^{-\Phi_s}ds, \end{aligned}$$

where the Lagrangian multiplier y is implicitly defined by

$$\int_0^T e^{-\rho t\Phi_t}y^{-\Phi_t}(1+\alpha F_t)^{1-\Phi_t}e^{(1-\Phi_t)\left(-r-\frac{1}{2}\theta^2\Phi_t\right)t}dt = x_0 - h_0 F_0.$$

In the following, we use the duality result from Schroder and Skiadas (2002) to prove the Proposition.

[4]The special case of a constant habit equal to the initial habit h_0, is achieved by setting $\alpha = 0$. The only difference between the primal and dual market is then the adjustment of the initial wealth by $h_0 F_0$, which is required to finance the constant habit.

[5]In Schroder and Skiadas (2002) a dual market price of risk $\hat{\theta}$ is used. Here we have $\hat{\theta} = \theta$, since our market parameters are constant.

PROOF: Using Proposition 2.1, $\hat{\xi}_t = \xi_t(1+\alpha F_t)$ and $\hat{x}_0 = \frac{x_0 - h_0 F_0}{1+\alpha F_0}$, the optimal consumption $\hat{c}_t^*$ for the dual problem (2.6) is given by

$$\hat{c}_t^* = e^{-\rho t \Phi_t} y^{-\Phi_t} \hat{\xi}_t^{-\Phi_t} = e^{-\rho t \Phi_t} y^{-\Phi_t} \xi_t^{-\Phi_t} (1+\alpha F_t)^{-\Phi_t}.$$

With $\hat{c}_t = c_t - h_t$, the optimal consumption for the primal problem is given by

$$c_t^* = e^{-\rho t \Phi_t} y^{-\Phi_t} (1+\alpha F_t)^{-\Phi_t} \xi_t^{-\Phi_t} + h_t^*,$$

where h_t^* is the habit level of the optimal consumption. We know that h_t fulfills the differential equation $dh_t = -(\beta h_t - \alpha c_t)dt$. Following the idea from Proposition 1 in Schroder and Skiadas (2002) and plugging $c_t = \widehat{c}_t + h_t$ into the differential equation, we get $dh_t = ((\alpha - \beta)h_t + \alpha \widehat{c}_t)dt$. This equation is solved by

$$\widehat{h}_t := e^{(\alpha-\beta)t} h_0 + \alpha \int_0^t e^{(\alpha-\beta)(t-s)} \widehat{c}_s ds.$$

By the uniqueness of the solution, we get that $h_t(c_t) = \widehat{h}_t(\widehat{c}_t)$ and can express h_t^* in the following form:

$$h_t^* = e^{(\alpha-\beta)t} h_0 + \alpha \int_0^t e^{(\alpha-\beta)(t-s)} e^{-\rho s \Phi_s} (y \xi_s)^{-\Phi_s} (1+\alpha F_s)^{-\Phi_s} ds.$$

For the Lagrangian multiplier y, we transform the constraint from the dual problem and get that it is implicitly given by

$$\int_0^T e^{-\rho t \Phi_t} y^{-\Phi_t} (1+\alpha F_t)^{1-\Phi_t} e^{(1-\Phi_t)\left(-r - \frac{1}{2}\theta^2 \Phi_t\right)t} dt = x_0 - h_0 F_0.$$

The condition "$\alpha < \beta$" ensures that F_t is non-negative in $[0, T]$. Hence, the left-hand-side of the above equation is continuous and strictly decreasing in y. Furthermore, it goes to infinity for y approaching zero and goes to zero for y approaching infinity. With the condition that $x_0 > h_0 F_0$, the solution to y exists.

The optimal wealth of the dual problem is given by

$$\begin{aligned}
\hat{X}_t^* &= \int_t^T e^{-\rho s \Phi_s} y^{-\Phi_s} \hat{\xi}_t^{-\Phi_s} E_t \left[\left(\frac{\hat{\xi}_s}{\hat{\xi}_t} \right)^{1-\Phi_s} \right] ds \\
&= \int_t^T e^{-\rho s \Phi_s} y^{-\Phi_s} \xi_t^{-\Phi_s} \frac{(1+\alpha F_s)^{1-\Phi_s}}{1+\alpha F_t} e^{(1-\Phi_s)\left(-r - \frac{1}{2}\theta^2 \Phi_s\right)(s-t)} ds.
\end{aligned}$$

Following Schroder and Skiadas (2002), we obtain the following optimal wealth of the primal problem:

$$\begin{aligned} X_t^* &= h_t^* F_t + (1+\alpha F_t)\hat{X}_t^* \\ &= h_t^* F_t + \int_t^T e^{-\rho s \Phi_s} y^{-\Phi_s} \xi_t^{-\Phi_s} (1+\alpha F_s)^{1-\Phi_s} e^{(1-\Phi_s)\left(-r-\frac{1}{2}\theta^2\Phi_s\right)(s-t)} ds. \end{aligned}$$

From Proposition 2.1, we get the optimal dual investment:

$$\begin{aligned} \hat{\pi}_t^* &= \frac{\hat{\theta}}{\sigma}\frac{1}{\hat{X}_t^*}\int_t^T \Phi_s e^{-\rho s\Phi_s} y^{-\Phi_s} \hat{\xi}_t^{-\Phi_s} E_t\left[\left(\frac{\hat{\xi}_s}{\hat{\xi}_t}\right)^{1-\Phi_s}\right] ds \\ &= \frac{\theta}{\sigma}\frac{1}{\hat{X}_t^*}\int_t^T \Phi_s e^{-\rho s\Phi_s} y^{-\Phi_s} \xi_t^{-\Phi_s} \frac{(1+\alpha F_s)^{1-\Phi_s}}{1+\alpha F_t} e^{(1-\Phi_s)\left(-r-\frac{1}{2}\theta^2\Phi_s\right)(s-t)} ds. \end{aligned}$$

Following Schroder and Skiadas (2002) and the representation in Munk (2008), the optimal investment for the primal problem is given by $\pi_t^* = \left(1 - \frac{h_t^* F_t}{X_t^*}\right)\hat{\pi}_t^*$. With $\hat{X}_t^* = \frac{X_t^* - h_t^* F_t}{1+\alpha F_t}$ we get the optimal portfolio process for the primal problem:

$$\begin{aligned} \pi_t^* &= \left(\frac{X_t^* - h_t^* F_t}{X_t^*}\right)\frac{\theta}{\sigma}\frac{1}{\frac{X_t^* - h_t^* F_t}{1+\alpha F_t}}\int_t^T \Phi_s e^{-\rho s\Phi_s} y^{-\Phi_s} \xi_t^{-\Phi_s} \frac{(1+\alpha F_s)^{1-\Phi_s}}{1+\alpha F_t} e^{(1-\Phi_s)\left(-r-\frac{1}{2}\theta^2\Phi_s\right)(s-t)} ds \\ &= \frac{\theta}{\sigma}\frac{1}{X_t^*}\int_t^T \Phi_s e^{-\rho s\Phi_s} y^{-\Phi_s} \xi_t^{-\Phi_s} (1+\alpha F_s)^{1-\Phi_s} e^{(1-\Phi_s)\left(-r-\frac{1}{2}\theta^2\Phi_s\right)(s-t)} ds. \end{aligned}$$

□

To compare our results to the previous literature, we set $\gamma_t = \gamma$ (i.e. $\Phi_t = \frac{1}{\gamma}$) and hence get

$$\begin{aligned} X_t^* - h_t^* F_t &= \int_t^T e^{-\rho s\frac{1}{\gamma}} (y\xi_t)^{-\frac{1}{\gamma}} (1+\alpha F_s)^{1-\frac{1}{\gamma}} e^{(1-\frac{1}{\gamma})\left(-r-\frac{1}{2}\theta^2\frac{1}{\gamma}\right)(s-t)} ds \\ \Leftrightarrow (y\xi_t)^{-\frac{1}{\gamma}} &= \frac{X_t^* - h_t^* F_t}{\int_t^T e^{-\rho s\frac{1}{\gamma}} (1+\alpha F_s)^{1-\frac{1}{\gamma}} e^{(1-\frac{1}{\gamma})\left(-r-\frac{1}{2}\theta^2\frac{1}{\gamma}\right)(s-t)} ds}. \end{aligned}$$

This gives the following results for the optimal controls

$$\begin{aligned} \pi_t^* &= \frac{\theta}{\sigma}\frac{1}{\gamma}\frac{X_t^* - h_t^* F_t}{X_t^*}, \\ c_t^* &= e^{-\rho t\frac{1}{\gamma}} (y\xi_t)^{-\frac{1}{\gamma}} (1+\alpha F_t)^{-\frac{1}{\gamma}} + h_t^* \\ &= (1+\alpha F_t)^{-\frac{1}{\gamma}} \frac{X_t^* - h_t^* F_t}{\int_t^T e^{-\rho(s-t)\frac{1}{\gamma}} (1+\alpha F_s)^{1-\frac{1}{\gamma}} e^{(1-\frac{1}{\gamma})\left(-r-\frac{1}{2}\theta^2\frac{1}{\gamma}\right)(s-t)} ds} + h_t^*, \end{aligned}$$

which correspond to the solutions presented in Munk (2008) for constant θ and constant r.

Similar as before, we calculate the expected value of the optimal consumption, to examine the change over time of the average consumption:

$$
\begin{aligned}
E[c_t^*] =& e^{-\rho t \Phi_t} y^{-\Phi_t} (1+\alpha F_t)^{-\Phi_t} E\left[\xi_t^{-\Phi_t}\right] \\
&+ e^{(\alpha-\beta)t} h_0 + \alpha \int_0^t e^{(\alpha-\beta)(t-s)} e^{-\rho s \Phi_s} y^{-\Phi_s} (1+\alpha F_s)^{-\Phi_s} E\left[\xi_s^{-\Phi_s}\right] ds \\
=& e^{(r-\rho)t\Phi_t} y^{-\Phi_t} (1+\alpha F_t)^{-\Phi_t} e^{(1+\Phi_t)\frac{1}{2}\theta^2 \Phi_t t} \\
&+ e^{(\alpha-\beta)t} h_0 + \alpha \int_0^t e^{(\alpha-\beta)(t-s)} e^{(r-\rho)s\Phi_s} y^{-\Phi_s} (1+\alpha F_s)^{-\Phi_s} e^{(1+\Phi_s)\frac{1}{2}\theta^2 \Phi_s s} ds \\
=& e^{B(t)} + e^{(\alpha-\beta)t} h_0 + \alpha \int_0^t e^{(\alpha-\beta)(t-s)} e^{B(s)} ds \\
&\text{with } B(t) := (r-\rho)\Phi_t t + (1+\Phi_t)\frac{1}{2}\theta^2 \Phi_t t - \Phi_t \log(y) - \Phi_t \log(1+\alpha F_t)
\end{aligned}
$$

The first derivative with respect to time is given by

$$
\frac{d}{dt} E[c_t^*] = \left(\alpha + \frac{d}{dt} B(t)\right) e^{B(t)} + (\alpha-\beta)\left(e^{(\alpha-\beta)t} h_0 + \alpha \int_0^t e^{(\alpha-\beta)(t-s)} e^{B(s)} ds\right),
$$

with

$$
\frac{dB(t)}{dt} = \left(r - \rho + \frac{1}{2}\theta^2\right)(\Phi_t + t\Phi_t') + \frac{1}{2}\theta^2 (2\Phi_t' \Phi_t t + \Phi_t^2) - \Phi_t' \log(y) - \left(\Phi_t' \log(1+\alpha F_t) + \Phi_t \frac{\alpha F_t'}{1+\alpha F_t}\right).
$$

Here we have, as in the case without habit, that the shape of the average consumption over time depends on the initial wealth, implicitly through the Lagrangian multiplier y. For a constant RA γ, we have that $\Phi_t' = 0$ and thus get $\frac{dB(t)}{dt} > 0$, since $F_t' < 0$. Hence, the derivative would be positive, as long as

$$
\left(\alpha + \frac{d}{dt} B(t)\right) e^{B(t)} > -(\alpha-\beta)\left(e^{(\alpha-\beta)t} h_0 + \alpha \int_0^t e^{(\alpha-\beta)(t-s)} e^{B(s)} ds\right),
$$

where the right hand side is positive, since $\alpha - \beta < 0$. This means that the expected value would be increasing, as long as the current expected consumption is large enough compared to the expected habit level. Especially the sign of the derivative would not depend on the initial wealth.

By including the time-varying RA, the average consumption may also decrease depending on the RA and the Lagrangian multiplier, i.e. the initial wealth x_0.

As before, we can not calculate a closed form for the expectation of the optimal investment. In the following numerical section we calculate it numerically and examine the effects on the decision rules for specific choices of time-varying RA. We see that the effects may change for different initial wealth.

3. Numerical results

In this section, we analyze the effect of specific choices for the time-varying RA and the combination with habit formation on the optimal solutions. First, we examine the changes over time by numerically calculating some descriptive statistics. Second, we analyze the effect of wealth, by numerically calculating the decision rules as functions of wealth.

For the financial market, we fix the parameters similar to the ones used in Munk (2008):

$$r = 2\%,\ \mu = 8\%,\ \sigma = 20\%,\ \theta = 30\%$$

and take an individual discount of $\rho = r$, a time horizon of $T = 30$ years and an initial wealth of $x_0 = 1000$. Comparing the results for different initial endowments x_0, we find that the results in case of $x_0 = 1000$ are similar for even bigger endowments. Therefore, we decide for this value and afterwards discuss the effect of smaller initial endowments.

For the habit formation, we use parameters similar to the ones in Munk (2008) and choose the initial habit h_0 proportional to our larger initial wealth. In total, we consider four different parameterizations of the habit level, shown in Table 1, representing different magnitudes of habit formation.

No habit	Low habit	Medium habit	High habit
$\alpha = 0$	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.4$
$\beta = 0$	$\beta = 0.2$	$\beta = 0.4$	$\beta = 0.5$
$h_0 = 0$	$h_0 = 20$	$h_0 = 30$	$h_0 = 40$

Table 1. Habit formation parameters.

For the changes in risk aversion over the life-cycle, as noted in the introduction, the literature does not agree on the direction the risk aversion is changing. Therefore, we consider both increasing and decreasing risk aversions. As base case, we choose a constant risk aversion of $\gamma = 3$, to compare our observations to previous results in the literature. The other parameterizations are chosen such that the average over the life-cycle of the risk

aversion, i.e. $\int_0^T \gamma_t dt$, is the same for all functions γ_t.[6] Their parameters are given in the following Table 2 and their shapes are shown in Figure 1.

Other parameterizations and possible different shapes are discussed in the following.

Constant	Linear increasing	Convex increasing	Linear decreasing
$\gamma_t = 3$	$\gamma_t = \frac{6}{90} \cdot t + 2$	$\gamma_t = \frac{79.74}{44.3 - t}$	$\gamma_t = -\frac{6}{90} \cdot t + 4$

TABLE 2. Parameterizations for the time-varying risk aversion.

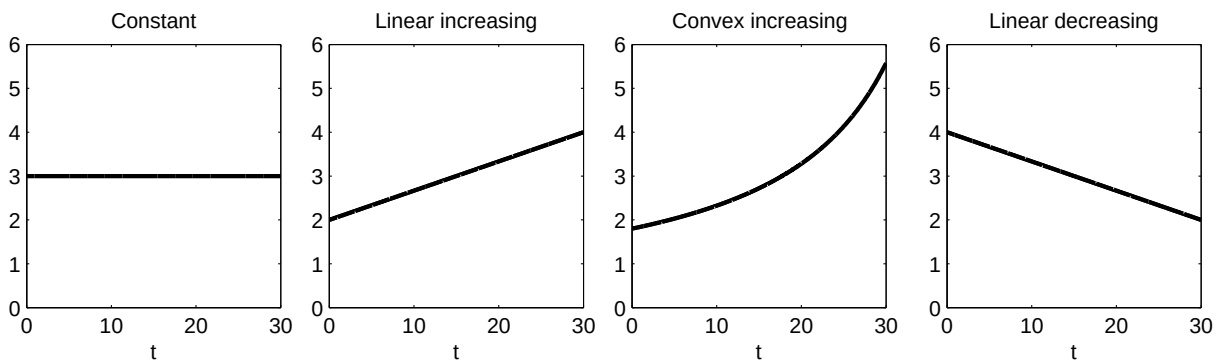

FIGURE 1. Different shapes for RA γ_t over the life-cycle of the individual.

For the cases where we can not find closed forms of the statistics, we perform Monte-Carlo simulations with $N = 10\,000$ paths and partition each year into 50 subintervals.

3.1. **Effect of time on the optimal decisions.** In the following, we examine the changes of the optimal consumption and investment over time. Therefore, we calculate their expected value and variance.

3.1.1. *The base case: constant risk aversion.* We start with the base case of a constant RA of $\gamma_t \equiv 3$, to compare our results to previous literature. Figure 2 shows the corresponding expected value and the variance for the optimal consumption c^* and the expected value for the corresponding optimal habit level h^*. The four images show the four different parameterizations for the habit formation with no habit on the very left and a high habit

[6] As shown in equation (2.4), the relative risk aversion (RRA) is not the same in the case without and with habit. Comparing different RRA values for the case without habit, we find that the shape of the results remains similar, only their absolute values change. Therefore we decide to take the same γ_t for the case without and with habit.

on the right side. The lower row shows the expected value and variance for the optimal investment π^*.

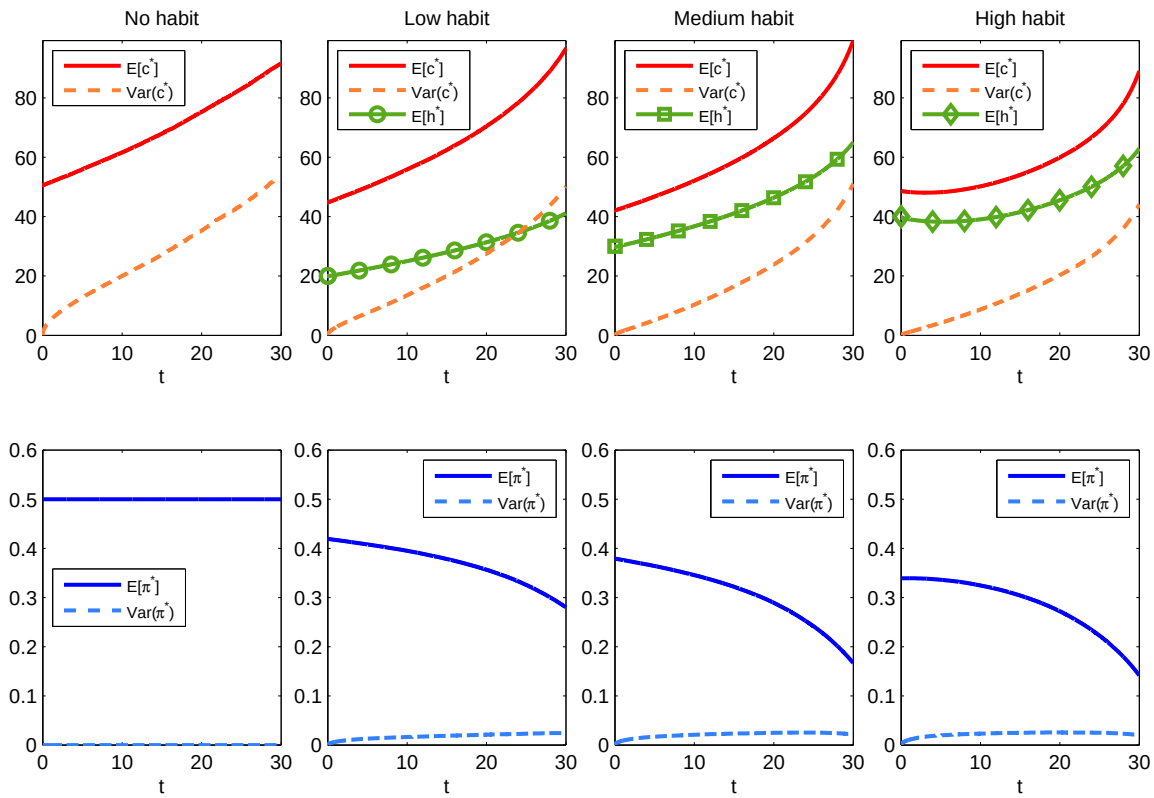

FIGURE 2. Statistics for the optimal consumption, habit and investment for a constant risk aversion over time.

The optimal consumption increases in expectation over time for all presented parameterizations. The inclusion of a habit level leads to a smoother consumption, as seen by the smaller variance of c^*. For larger habit parameters, the average consumption sticks closer to the habit level. Therefore, the path of the expected consumption and habit level are more similar as for the lower habit parameters. The optimal investment is constant and equal to $\frac{\mu - r}{\sigma^2 \gamma} = 50\%$ for the case of no habit. Including the habit level, leads to a decrease of the expected investment over time. It decreases stronger towards the end of the life-cycle, to ensure sufficient wealth for a consumption above the habit. We thus observe that by the inclusion of a habit level, we achieve a decreasing shape of the average optimal investment. The optimal consumption is smoother when a habit level is included, but is increasing in expectation.

3.1.2. *Linear increasing risk aversion.* Next, we examine the case of an increasing RA and present the results for the linear increasing γ_t in Figure 3.

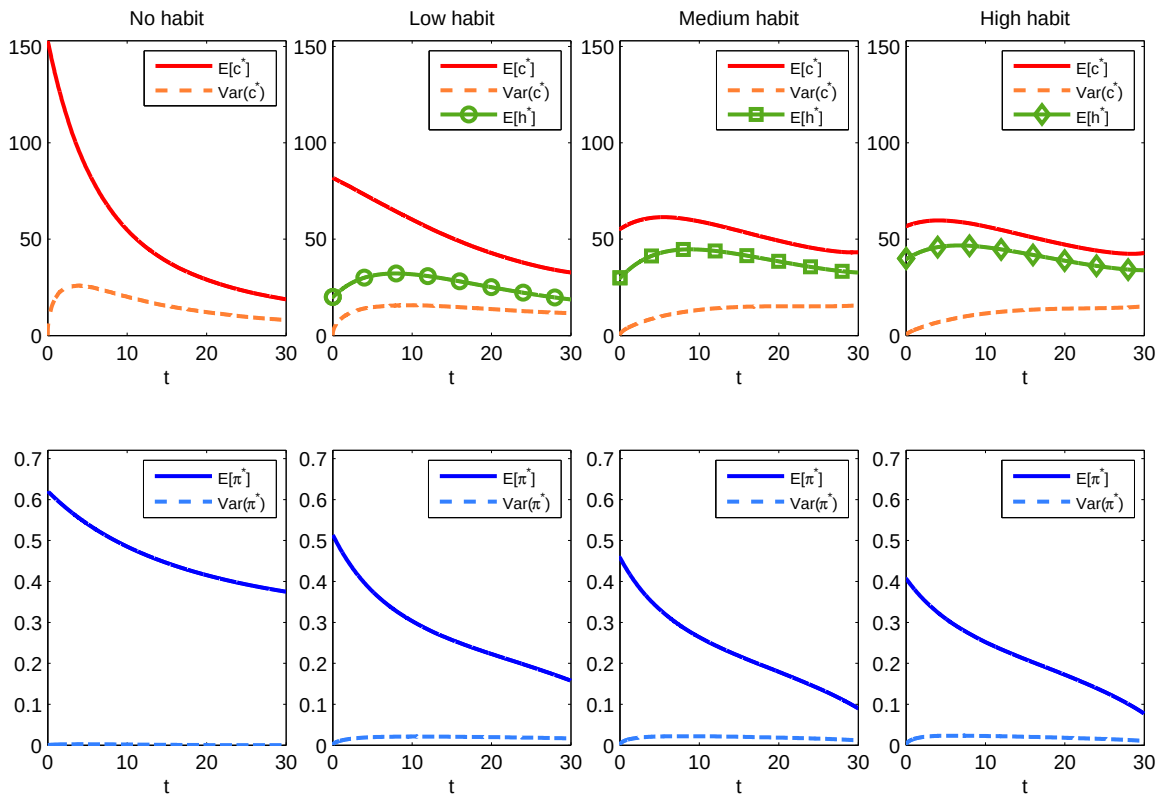

FIGURE 3. Statistics for the optimal consumption, habit and investment for a linear increasing risk aversion over time.

For the case of no habit, the optimal consumption is increasing over time in expectation. That follows directly from condition (2.2) for these parameter combinations. When a habit level is included, the consumption starts at smaller values and its changes over time are much smoother. The average habit level exhibits a hump-shaped form in all three cases. But for the low habit, the expected consumption is far above the habit level and does not adapt its shape. Unlike for the two cases of larger habit formation, where the consumption follows the shape of the habit level and thus exhibits a hump-shaped pattern.

The optimal investment, in the case without any habit, is decreasing in expectation, since the RA is increasing. For the cases with habit, the increasing RA leads to a strong decrease of the investment in the first years. This is different to the case of a time-constant RA, where the investment decreases very slow in the first years. As before, we can see that by including the habit level, we start at a smaller initial investment. This follows from the

higher relative risk aversion, resulting from the desire to ensure a consumption above the habit.

For different parameters of the linear increasing γ_t the overall shape remains similar, only the magnitude of the results changes. Different shapes for γ_t would be a convex or a concave increasing one. For the first one, we get a more distinct hump in the shape of the expected consumption. The average investment in this case has a convex shape and is decreasing even stronger. The second one, a concave increasing shape, leads to consumption and investment decisions which start in expectation at very large values, heavily drop down in the first years and then slowly decrease in the later years. Since the results for the first case are similar to the linear shape and for the concave case seem not in line with empirical observations, we do not present these two cases in more detail.

The combination of a habit level with an increasing RA thus leads to a further smoothed consumption. The average consumption eventually exhibits a hump-shaped pattern, depending on the magnitude of the habit formation. The investment is strictly decreasing in expectation, which is necessary to ensure the smooth consumption above the habit level.

3.1.3. *Linear decreasing risk aversion.* Last, we consider the case of a decreasing RA over time. The results are presented in Figure 4.

In contrast to the previous results, we observe a very strong increase over time in the expectation of the consumption and habit level. This increase is reduced for higher habit formations. Considering the difference between the consumption and the habit level, they are closest at the beginning and then drift apart at the end. This follows from the smaller RA at the end of the life-cycle, compared to the beginning and is the other way round for the increasing RA. In addition to the expected value, also the variance is increasing, resulting in a highly volatile consumption at the end of the life-cycle.

The optimal investment is increasing in expectation. But in the case of a habit level, begins to decrease again towards the end and thus exhibits a hump-shaped pattern. This results from the need to ensure that the large and towards the end highly volatile consumption remains above the habit level.

Regarding different parameterizations for γ_t, we get similar shapes for the expected consumption and investment, only the absolute values are different. Also for different shapes of γ_t, e.g. convex or concave decreasing, we get similar results. For the first one the results are even more extreme, whereas for the latter one, they are more moderate.

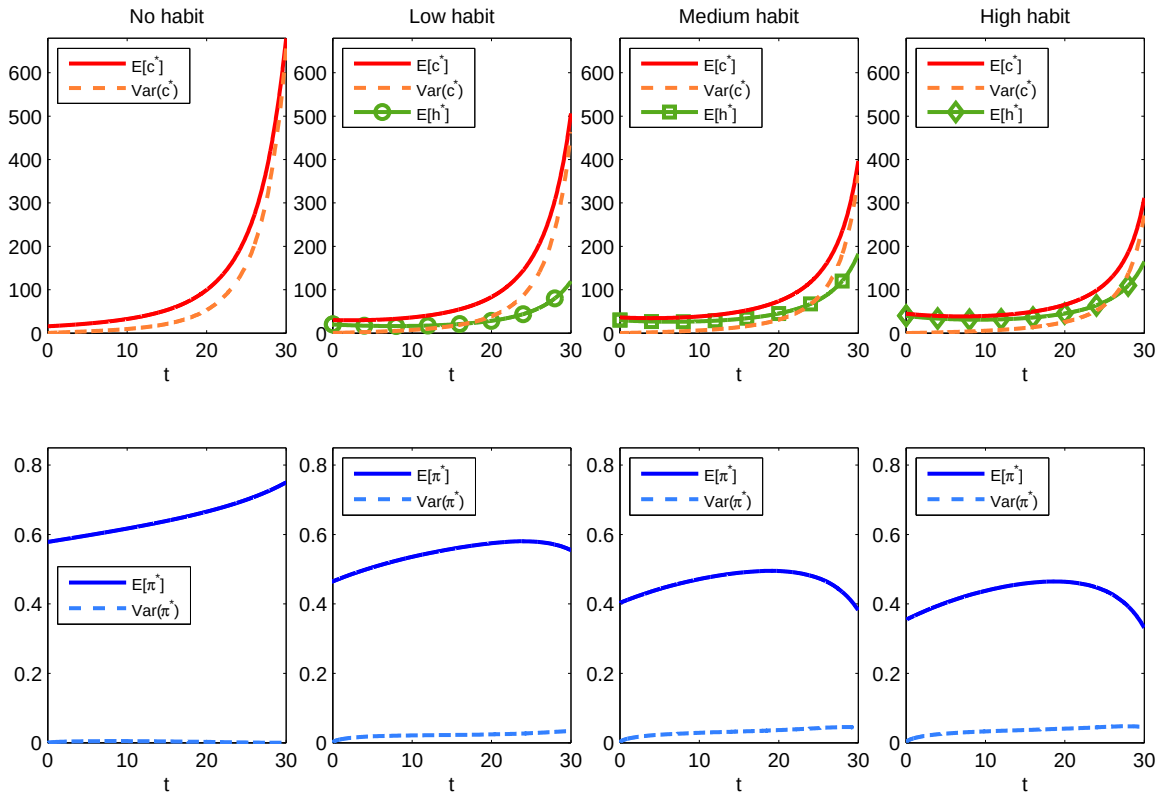

FIGURE 4. Statistics for the optimal consumption, habit and investment for a linear decreasing risk aversion over time.

The average optimal consumption for a decreasing RA is thus different to the observations in the literature. On the other hand, the average investment exhibits a hump-shaped form as observed in some literature presented in the introduction.

3.1.4. *The effect of the initial wealth.* As previously discussed, the shape of the expected consumption depends on the initial wealth. For the case of no habit level, we can even express the dependence in closed form. For example for a linear increasing risk aversion, no habit level and the above mentioned parameters it holds that the expected value of the optimal consumption is increasing in t $\forall x_0 \leq 32.64$ and decreasing in t $\forall x_0 \geq 41.97$. For values of x_0 in between, the expected value is increasing in the first years and decreasing in the later years. We have therefore compared the results for different values of the initial wealth x_0. When the RA is constant, i.e. $\gamma_t \equiv \gamma$, the shape of the decision rules are similar, only the absolute values change proportionally to the change in the initial endowment. But when a time-varying RA is considered, in particular for an initial wealth of $x_0 = 1$, the

results for a linear increasing and linear decreasing γ_t are almost reversed.[7] Namely, when a habit level is included, the initial wealth is $x_0 = 1$ and the RA is linear increasing, the results are similar as for the case of $x_0 = 1000$ and linear decreasing RA. Except that the average consumption increases more moderate and the investment is decreasing stronger towards the end, when $x_0 = 1$. On the other hand, when $x_0 = 1$ and the RA is linear decreasing, the results are similar as for the case of $x_0 = 1000$ and linear increasing RA. Only for the case of $x_0 = 1$ the hump in the expected consumption is much more distinct and the average investment is smaller and hence decreases less strong.

In the situation without a habit level, $x_0 = 1$ and RA linear increasing, the optimal investment is still decreasing in expectation. On the other hand, the optimal consumption is increasing, similar as for a decreasing RA and $x_0 = 1000$, only more moderate. For $x_0 = 1$ and RA linear decreasing, the average investment is increasing as one would expect. But the average consumption is decreasing as for the case of $x_0 = 1000$ and increasing RA.

Thus the results when a habit level is included and linear increasing and decreasing RA is considered are almost opposite for initial endowment of one and 1000. For the case without a habit level, only the consumption is opposite, the investment still has a similar shape.

3.2. **Effect of wealth on the optimal decisions.** Next we examine the effect of wealth on the optimal consumption and investment decision. Since we can not express either c^* nor π^* in dependence of the optimal wealth, we numerically calculate them as functions of the wealth. As shown before, the shapes depend heavily on the initial wealth x_0. Therefore, we consider both an initial wealth of $x_0 = 1$ and $x_0 = 1000$ in the following. Since the results are similar for the three habit levels, we limit the following to the case of a medium habit parameterization. For the RA, we compare the case of a constant RA, linear increasing RA and linear decreasing RA, as before. As fixed time points we choose $t = 5$ and $t = 20$, to compare the results early and late in the life-cycle. For time points closer to maturity, e.g. $t = 25$, the effects are similar as for $t = 20$.

[7] When the initial wealth is reduced, also the initial habit has to be reduced. For $x_0 = 1$, we choose initial habit of $h_0 = 0.02$, $h_0 = 0.03$, $h_0 = 0.04$ for the different habit formations respectively.

3.2.1. *Optimal consumption.* The optimal consumption c^* as function of wealth is shown in Figure 5. The left image shows $t = 5$ and the right one $t = 20$. The top row shows the case of an initial endowment of $x_0 = 1$ and the bottom row of $x_0 = 1000$.

It stands out that the consumption in the case of a habit formation can not go below a certain wealth level. This level corresponds to the minimum wealth $h_t F_t$, which is required to finance the future habit level.

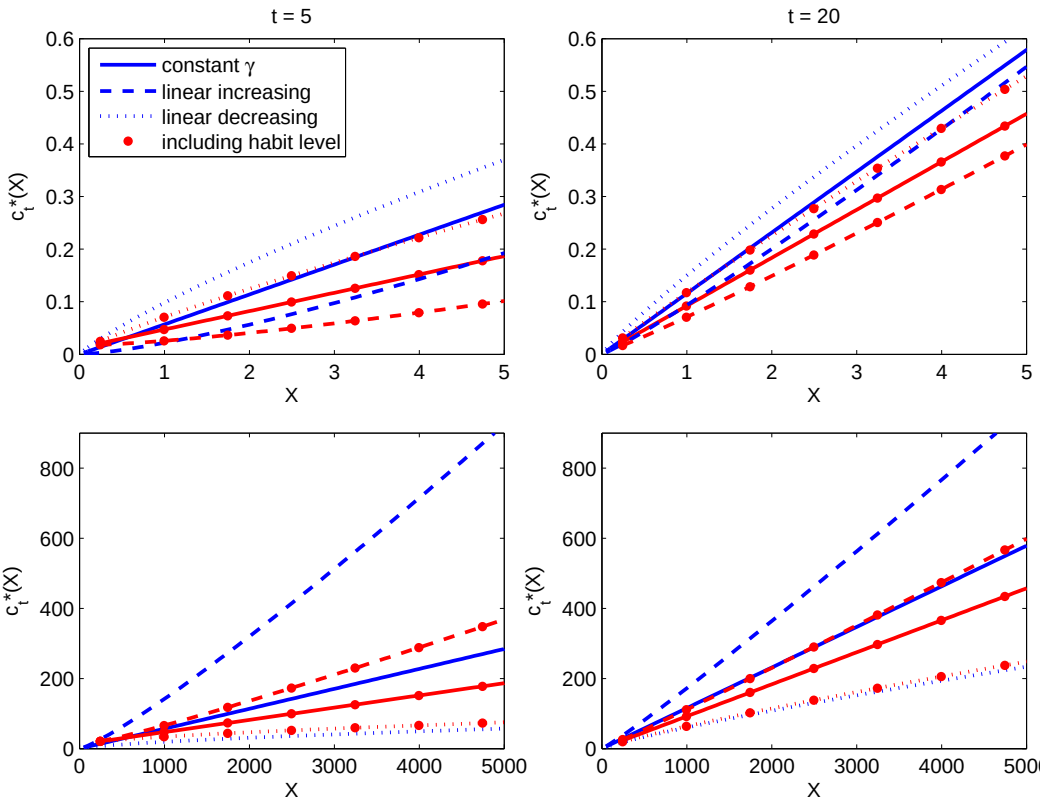

FIGURE 5. Optimal consumption c^* as function of wealth for fixed time points $t = 5$ (left) and $t = 20$ (right) for initial endowment $x_0 = 1$ (top row) and $x_0 = 1000$ (bottom row) for three different time-varying RA. Lines without a round marker represent consumption without a habit level and lines with a round marker the consumption including a habit level.

We observe that for all presented cases the consumption is increasing in wealth and it increases stronger later in the life-cycle. Whereas the increases is less strong when a habit level is included, compared to no habit formation. The later results, since an increase in consumption leads to an instant additional utility, but a decreased future utility through a higher habit level. For the different RA we see that for a constant risk aversion of $\gamma_t \equiv 3$ the consumption is linear in wealth, for an increasing RA convex in wealth and for a

decreasing RA concave in wealth. This was shown for the case without habit in Steffensen (2011) and is also observable for the consumption including a habit level.

So far the observations are similar for the two initial endowments. When considering the differences between the results for different values of x_0, in the case of a constant RA, the consumption changes proportional to the change of the initial wealth. But when considering the order of the lines representing the consumption for different time-varying RA, we see that the order is reversed for the two initial endowments. For $x_0 = 1$ the consumption is highest for a decreasing RA, for $x_0 = 1000$ it is highest for an increasing RA. This is similar to the previous section, where we see that the effect of increasing and decreasing RA is reversed for the two different initial endowments.

3.2.2. *Optimal investment.* Next, we show the effect of wealth on the optimal investment π^*. Figure 6 presents the results in the same way as previously for the consumption.

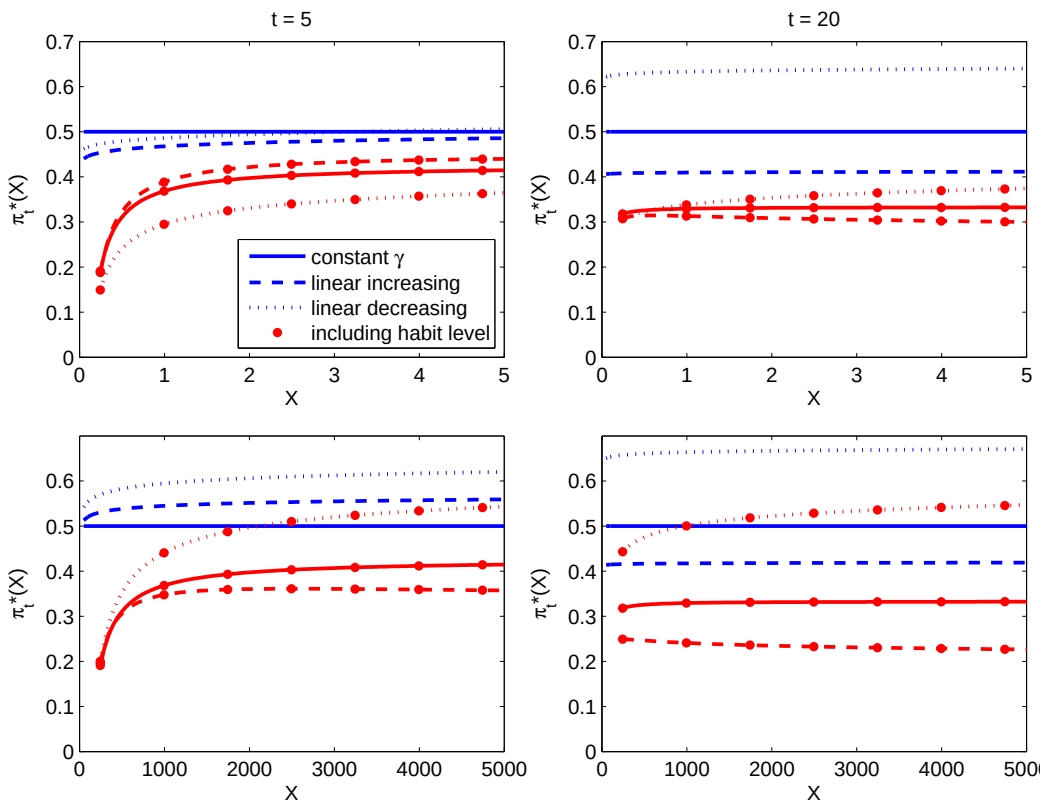

FIGURE 6. Optimal investment π^* as function of wealth for fixed time points $t = 5$ (left) and $t = 20$ (right) for initial endowment $x_0 = 1$ (top row) and $x_0 = 1000$ (bottom row) for three different time-varying RA. Lines without a round marker represent consumption without a habit level and lines with a round marker the consumption including a habit level.

For most cases the optimal investment is increasing in wealth. Only for an increasing RA with a habit level, it is decreasing in wealth late in the life-cycle, i.e. at $t = 20$.

One of the most significant observations is that the investment decision for the case of habit formation goes towards zero early in the life-cycle, when the wealth is close to the minimal wealth level required to finance the future habit. This effect vanishes later in the life-cycle (i.e. at $t = 20$), since the costs of the habit level are much smaller at that time point.

Regarding the case of no habit level and comparing the results for the different initial endowments, we see that the differences are bigger at $t = 5$, than at $t = 20$, showing that the effect of the initial endowment is larger early in the life-cycle than later, where the investment decision is more influenced by the market than by the initial endowment.

For the case including a habit level, the difference between the initial endowments are large early and late in the life-cycle, but have different effects. Early in the life-cycle the order for a decreasing and increasing RA are reversed, as discussed before. Late in the life-cycle, the investment decisions for the different RA are similar when $x_0 = 1$, but are far apart, when $x_0 = 1000$.

4. Conclusion

In this paper, we combine a time-varying risk aversion with a habit level in the optimal consumption and investment problem. In a complete market, we first review the problem without a habit level. Using these results and the dual approach from Schroder and Skiadas (2002), we subsequent find the solution for the problem with a time-varying risk aversion and a habit level. In a numerical analysis, we compare the effects of different shapes of time-varying risk aversion and different magnitudes of habit formation on the optimal solution. We compare both increasing and decreasing shapes for the risk aversion, as is observed in several literature.

Our findings show that the shape of the optimal decision rules depends on the initial wealth, what is not the case for a time-constant risk aversion. Furthermore, with an increasing risk aversion we achieve a hump-shaped consumption pattern and decreasing investment in the risky asset with increasing time. The results are thus more in line with empirical observations.

In our setting, we consider a simple market model and neglect further future endowments by e.g. labor income. Further research might consider a more complex market model or consider the inclusion of labor income into the presented setting. The habit level is considered as additive, considering a non-additive habit level might be of further interest. The presented utility function omits the time separability of utility, but still the concepts of risk aversion and elasticity of intertemporal substitution are mixed. Further research might try to disentangle these effects further and for example transfer the concept of time-varying risk attitudes to recursive utility functions.

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Appendix

Proof of Proposition 2.1. Following the methodology described in Cox and Huang (1989), the first order condition for the optimal solution to c_t is given by $e^{-\rho t}c_t^{-\gamma_t} = y\xi_t$, which gives an optimal consumption of $c_t^* = e^{-\rho t\Phi_t}\left(y\xi_t\right)^{-\Phi_t}$.

Plugging c_t^* into the constraint, we get the following equation implicitly defining the Lagrangian multiplier y:

$$\int_0^T e^{-\rho t\Phi_t} y^{-\Phi_t} e^{(1-\Phi_t)(-r-\frac{1}{2}\theta^2\Phi_t)t} dt = x_0.$$

The optimal wealth at time t is given by

$$\begin{aligned} X_t^* =& \frac{1}{\xi_t} E_t\left[\int_t^T \xi_s c_s^* ds\right] = \int_t^T e^{-\rho s\Phi_s} y^{-\Phi_s} E_t\left[\left(\frac{\xi_s}{\xi_t}\right)^{1-\Phi_s} \xi_t^{-\Phi_s}\right] ds \\ =& \int_t^T e^{-\rho s\Phi_s} e^{-(1-\Phi_s)\left(r+\frac{1}{2}\theta^2\Phi_s\right)(s-t)} \left(y\xi_t\right)^{-\Phi_s} ds, \end{aligned}$$

where we changed integration and conditional expectation as done in e.g. Munk (2008) and used $E\left[\left(\frac{\xi_s}{\xi_t}\right)^{1-\Phi_s}\right] = e^{-(1-\Phi_s)\left(r+\frac{1}{2}\theta^2\Phi_s\right)(s-t)}$.

In order to get the optimal asset allocation, we follow Chen et al. (2011) and calculate the optimal amount invested into the stock $\alpha_t^* = \frac{\partial X_t^*}{\partial S_t}$ and then the optimal proportion

$\pi_t^* = \frac{\alpha_t^* S_t}{X_t^*}$. Therefore, we reformulate the state price density as a function of the stock price process S_t, i.e. $\xi_t =: g(t, S_t)$. Then α_t^* is given as the delta-hedge:

$$\begin{aligned}\alpha_t^* =& \frac{\partial X_t^*}{\partial S_t} = \int_t^T e^{-\rho s \Phi_s} e^{-(1-\Phi_s)\left(r+\frac{1}{2}\theta^2\Phi_s\right)(s-t)} y^{-\Phi_s} \frac{\partial}{\partial S_t} g(t, S_t)^{-\Phi_s} ds \\ =& \frac{\mu - r}{\sigma^2} \frac{1}{S_t} \int_t^T \Phi_s e^{-\rho s \Phi_s} e^{-(1-\Phi_s)\left(r+\frac{1}{2}\theta^2\Phi_s\right)(s-t)} (y\xi_t)^{-\Phi_s} ds.\end{aligned}$$

For the optimal proportion π_t^* follows:

$$\pi_t^* = \frac{\alpha_t S_t}{X_t^*} = \frac{\mu - r}{\sigma^2} \frac{1}{X_t^*} \int_t^T \Phi_s e^{-\rho s \Phi_s} e^{-(1-\Phi_s)\left(r+\frac{1}{2}\theta^2\Phi_s\right)(s-t)} (y\xi_t)^{-\Phi_s} ds.$$

□