Interest rate model comparisons for participating products under Solvency II

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Abstract

A key aspect of the Solvency II regulatory framework is to compute the best estimate of the liabilities. This best estimate should be the probability-weighted average of future cash-flows, discounted to its present value. Movements in economic variables are often the driving force of changes in liability present values. Hence, many life insurers need stochastic models for producing future paths for e.g. interest rates, equity and bond returns and currencies. The paths should be risk-neutral, meaning that the expected return of all assets should be equal to the risk-neutral rate used for discounting the cash-flows. Hence, the interest rate model is a key component to consider within the Solvency II framework, particularly for life insurers. In this paper we study three interest rate models; the CIR++-model, the G++-model and the Libor Market model. Even when calibrated to the same historical data, the simulations from these models have very different mean value and volatility characteristics, especially far out into the future. However, when using these simulations when computing the best estimate of the liabilities, the differences between the models are surprisingly small, both for a synthetic and for a real-world insurance portfolio.

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1. Introduction

A key aspect of the Solvency II regulatory framework is to compute the best estimate of the liabilities. This best estimate should be the probability-weighted average of future cash-flows, discounted to its present value. Movements in economic variables are often the driving force of changes in liability present values. Hence, life insurers need stochastic models for producing future paths for e.g. interest rates, equity and bond returns and currencies. The paths should be risk-neutral, meaning that the expected return of all assets should be equal to the risk-neutral rate used for discounting the cash-flows. Hence, the interest rate model is a key component to consider within the Solvency II framework, particularly for life insurers. In this paper we study three interest rate models, the CIR++-model, the G++-model and the Libor Market model. The three models are calibrated to the same historical data, and then they are used to compute the best estimate of the liabilities, both for a synthetic and for a real-world insurance portfolio.

The rest of this paper is organised as follows. In Section 2 we review the desirable aspects for an interest rate model to be used in the Solvency II regulatory framework. Section 3 describes the three interest rate models used in this paper, while Section 4 shows the behaviour of these models when used in a Solvency II setting. Finally, Section 5 contains some concluding remarks.

2. Desirable properties of interest rate models

A key issue in interest rate modelling is to define objectives that the model should ideally meet. In reality models often, if not always, have advantages and disadvantages that need to be weighed up against each other. For example, a simpler model could be preferred despite having less adherence to real world data. For insurers, the objectives or principles to strive for regarding interest rate models depends heavily on which products the insurer has, and for what purpose the model will be used for. Guaranteed interest rate or participating (with profits) products sold by life insurers require substantially more consideration given to long term interest rates, than say a typical car or house insurance product with annual renewal.

In this paper we limit our context and purpose to Solvency II Pillar 1 technical provisions for two life insurance traditional with-profits products. The following list then contains the desired (non-exhaustive) properties and objectives of the interest rate model:

- Adherence to data, evidence, judgment and available literature
- The extent to which the model is intuitive for decision makers
- Ability to calibrate to market prices and/or historical data
- Time or cost needed to calibrate and simulate with the model
- Numerical stability of results
- Reasonable results for use in Solvency II

The three interest rate models we study in this paper; the CIR++-model, the G++-model, and the Libor Market model, all seem to be quite compliant with most of these properties and objectives. First, they are all three well-known from the literature. Second, the CIR++-model and G++-model are quite intuitive and easy to calibrate. The Libor Market model is more flexible, but also more complex, and hence more difficult to calibrate. Simulation from all three models is quite straightforward. Concerning numerical stability, the Libor Market model is known to have a relatively high probability of very large interest rates when the time horizon is long, resulting in a large Monte Carlo sampling error. The two other models, on the other hand, produce interest rate simulations more in line with what one would expect.

Property	CIR++	G++	Libor
Type of model	Short-rate	Short-rate	Forward rate
No.of.factors	1	2	Many
Negative rates	No*	Yes	No
Mean-reverting	Yes	Yes	No**
Distribution	Shifted non-central χ^2	Normal	Lognormal**

Table 1: Properties of the interest rate models studied in this paper. (*) The CIR++ may in principle produce negative interest rates for some yield curves. (**) These properties are valid for the forward rates and not the spot rates.

When it comes to the last objective listed above, we will later show that the differences between the best estimates of the liabilities computed using the different models are surprisingly small. Hopefully, this indicates that they all produce reasonable results.

The main properties of the three models are summarised in Table 1. In Section 3 we will give a more thorough description of each model.

3. Interest rate models used in this paper

Two of the models studied in this paper; the one-factor CIR++-model and the two-factor G++-model, are short-rate models, meaning that the short rate is first simulated, and then simulations of the spot rates with different maturities are derived from the short rate simulations. However, with the Libor Market model, one first generates forward rate simulations, and then one utilizes the relationship between forward and spot rates to obtain simulations of the latter.

3.1. CIR++-model

In the this model (Brigo and Mercurio, 2001a), the short rate dynamics are given by:

$$\begin{aligned} &dx(t) = \beta(\mu - x(t))dt + \sigma\sqrt{x(t)}dW(t), \quad x(0) = x_0 \\ &r(t) = x(t) + \varphi(t), \end{aligned} \tag{3.1}$$

where x_0, β, μ and σ are positive constants, $W(t)$ denotes a standard Brownian motion, and $\varphi(t)$ is a function chosen to fit the initial term structure. This function is given by:

$$\begin{aligned} &\varphi(t) = \varphi^{CIR}(t) = f^M(0,t) - f^{CIR}(0,t), \\ &f^{CIR}(0,t) = \tfrac{2\beta\mu(e^{th}-1)}{2h+(\beta+h)(e^{th}-1)} + x_0\tfrac{4h^2e^{th}}{[2h+(\beta+h)(e^{th}-1)]^2}. \end{aligned} \tag{3.2}$$

Here, $h = \sqrt{\beta^2 + 2\sigma^2}$ and $f^M(0,t)$ is the instantaneous forward rate.

Parameter estimation: When estimating the parameters of the CIR++-model one utilizes the fact that the increments of the short-rate follows a non-central chi-square distribution. The parameters are usually estimated by the maximum likelihood method, numerically maximizing the likelihood function.

To determine the instantaneous forward rate we use the Svensson model (Svensson, 1994):

$$f(0,\tau) = \beta_0 + \beta_1\exp(-\frac{\tau}{\lambda_1}) + \beta_2\frac{\tau}{\lambda_1}\exp(-\frac{\tau}{\lambda_1}) + \beta_3\frac{\tau}{\lambda_2}\exp(-\frac{\tau}{\lambda_2}). \tag{3.3}$$

When determining the parameters $\beta_0, \beta_1, \beta_2, \beta_3, \lambda_1$ and λ_2 in this model, one utilizes the following relationship between the yield curve and the forward rates

$$R(t,T) = \frac{\int_{\tau=t}^{T} f(t,\tau)d\tau}{T-t},$$

which means that the current yield curve may be represented by the same parameters as the forward rate:

$$\begin{aligned} R(0,\tau) &= \beta_0 + \beta_1 \frac{1-\exp(-\frac{\tau}{\lambda_1})}{\frac{\tau}{\lambda_1}} + \beta_2 \left[\frac{1-\exp(-\frac{\tau}{\lambda_1})}{\frac{\tau}{\lambda_1}} - \exp(-\frac{\tau}{\lambda_1}) \right] \\ &+ \beta_3 \left[\frac{1-\exp(-\frac{\tau}{\lambda_2})}{\frac{\tau}{\lambda_2}} - \exp(-\frac{\tau}{\lambda_2}) \right]. \end{aligned}$$

Hence, the parameters in Equation 3.3 may be estimated by minimizing the squared differences between observed and theoretical yields to maturity.

Simulation: Simulations of spot rates $R(t,T)$ are obtained by first generating simulations of the short rate, and then using the following two relationships:

$$R(t,T) = -\frac{\ln P(t,T)}{T-t}. \tag{3.4}$$

and

$$P(t,T) = \bar{A}(t,T)e^{-B(t,T)r(t)}, \tag{3.5}$$

Here, $P(t,T)$ is the price at time t of a zero-coupon bond with maturity T and

$$\begin{aligned} \bar{A}(t,T) &= \tfrac{P^M(0,T)A(0,t)e^{-B(0,t)x_0}}{P^M(0,t)A(0,T)e^{-B(0,T)x_0}} A(t,T)e^{B(t,T)\varphi^{CIR}(t;\alpha)}, \\ A(t,T) &= \left[\tfrac{2he^{(\beta+h)(T-t)/2}}{2h+(\beta+h)(e^{(T-t)h}-1)}\right]^{2\beta\mu/\sigma^2}, \\ B(t,T) &= \tfrac{2(e^{(T-t)/h}-1)}{2h+(\beta+h)(e^{(T-t)/h}-1)}, \\ h &= \sqrt{\beta^2+2\sigma^2}. \end{aligned}$$

Finally, $P^M(0,T)$ is the market discount factor for the maturity T.

3.2. G++-model

The CIR++-model is a one-factor model. This means that at every time point, instant rates for all maturities in the yield curve are perfectly correlated. If the product to be priced does not depend on the correlations of different rates, such a model can still be acceptable. However, interest rates are known to exhibit non-perfect correlation. Hence, whenever correlation plays a more relevant role, one has to use models allowing for more realistic correlation patterns. Historical analysis of the yield curve, based on principal component analysis, indicates that while one factor explains 68% to 78% of the total variation in the yield curve, two factors explain as much as 85%

to 90% (Brigo and Mercurio, 2001a). Hence, by moving from one-factor to two-factor models a more realistic evolution of the interest rates is obtained.

Here, the focus is on the two-additive-factor Gaussian (G++)-model (Brigo and Mercurio, 2001b), where the short rate process is given by the sum of two correlated Gaussian factors plus a deterministic function that is properly chosen so as to exactly fit the current term structure. The G++-model bears many similarities with the two-factor Hull-White model. However, the G++-model leads to less complicated formulae and is easier to implement in practice.

In this model, the short rate dynamics are given by

$$\begin{aligned} dx(t) &= -\alpha\, x(t)\, dt + \gamma\, dW_1(t), \quad x(0) = 0 \\ dy(t) &= -\beta\, y(t)\, dt + \eta\, dW_2(t), \quad y(0) = 0 \\ r(t) &= x(t) + y(t) + \varphi(t), \end{aligned} \tag{3.6}$$

where α and β are constants reflecting the rate of mean reversion, γ and η are the volatility constants, $\varphi(t)$ is given by

$$\varphi(t) = f^M(0,t) + \frac{\gamma^2}{2\,\alpha^2}(1-e^{-\alpha\, t})^2 + \frac{\eta^2}{2\beta^2}(1-e^{-\beta\, t})^2 + \kappa\, \frac{\gamma\eta}{\alpha\beta}(1-e^{-\alpha\, t})(1-e^{-\beta\, t}).$$

and $W_1(t)$ and $W_2(t)$ are standard Brownian motions with correlation κ.

Parameter estimation: The G++-model has five parameters that need to be estimated; α, β, γ, η and κ. In this paper we have estimated these parameters based on historical data. More specifically, we follow Park (2004) and minimize the sum of squared differences between theoretical and empirical volatilities of monthly absolute spot rate changes. Other approaches might be used, e.g. calibration via Kalman filtering, see Park (2004).

Simulation: Simulations of spot rates $R(t,T)$ are obtained by first generating simulations of the short rate, and then using the following two relationships:

$$R(t,T) = -\frac{\ln P(t,T)}{T-t}. \tag{3.7}$$

and

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \exp\{A(t,T)\}, \tag{3.8}$$

where $P^M(0,T)$ as for the CIR++-model is the market discount factor for the maturity T and

$$A(t,T) = \frac{1}{2}\left[V(t,T) - V(0,T) + V(0,t)\right] - \frac{1-e^{-\alpha(T-t)}}{\alpha}\, x(t) - \frac{1-e^{-\beta(T-t)}}{\beta}\, y(t).$$

Moreover

$$
\begin{aligned}
V(t,T) &= \frac{\gamma^2}{\alpha^2}\left[T-t+\frac{2}{\alpha}\,e^{-\alpha(T-t)}-\frac{1}{2\,\alpha}\,e^{-2\,\alpha\,(T-t)}-\frac{3}{2\,\alpha}\right] \\
&+ \frac{\eta^2}{\beta^2}\left[T-t+\frac{2}{\beta}\,e^{-\beta(T-t)}-\frac{1}{2\,\beta}\,e^{-2\beta(T-t)}-\frac{3}{2\beta}\right] \\
&+ 2\,\kappa\,\frac{\gamma\,\eta}{\alpha\,\beta}\left[T-t+\frac{e^{-\alpha(T-t)}-1}{\alpha}+\frac{e^{-\beta(T-t)}-1}{\beta}\right. \\
&- \left.\frac{e^{-(\alpha+\beta)\,(T-t)}-1}{\alpha+\beta}\right]
\end{aligned}
\tag{3.9}
$$

3.3. Libor Market model

To capture all variations in the yield curve, not even a two-factor model might be sufficient. In this paper we therefore also consider the Libor Market model (Brace et al., 1997; Miltersen et al., 1997). This model is intrinsically multi-factor, meaning that it captures various aspects of the curve dynamics: parallel shifts, steepenings/flattenings, butterflies, etc. While in the CIR++-model and G++-model spot rates were simulated by first simulating the short rate, in the Libor Market model they are derived from simulated forward rates. Let $F_i(t)$ be the forward rate which is alive up to time i given that the current time point is $t < i$. We have the following relationship between forward rates and spot rates:

$$
\begin{aligned}
F_t(t) &= R(t,t+1) \\
F_{t+i}(t) &= \frac{[1+R(t,t+i+1)]^{i+1}}{[1+R(t,t+i)]^{i}}-1,
\end{aligned}
\tag{3.10}
$$

where $R(t,t+i)$ is the spot rate prevailing at time t for the maturity $t+i$.

In the Libor Market model $F_i(t)$ is represented by the following model:

$$
F_i(t) = F_i(t-1)\,\exp\left(\sigma_i(t)\,\mu_i(t)-\frac{1}{2}\sigma_i(t)^2+\sigma_i(t)\,\epsilon_i(t)\right), \tag{3.11}
$$

where $\sigma_i(t)$ is the volatility of the log increment $\log\left(F_i(t)/F_i(t-1)\right)$, the noise vector $\boldsymbol{\epsilon}(t)=\{\epsilon_1(t),\ldots,\epsilon_N(t)\}$ follows a standard normal distribution, and the drift term $\mu_i(t)$ is given by

$$
\mu_i(t) = \sum_{k=t}^{i}\frac{F_k(t-1)\rho_{i,k}(t)\sigma_k(t)}{1+F_k(t-1)}.
$$

Here $\rho_{i,k}(t)$ is the correlation between $\epsilon_i(t)$ and $\epsilon_k(t)$. These correlations introduce auto-correlation in the spot rates. The model in Equation 3.11 implies that the forward rates are correlated random walks.

Parameter estimation There are two kinds of parameters in the Libor Market model; the volatility parameters $\sigma_i(t)$ and the correlation parameters $\rho_{i,j}(t)$. Jong et al. (2000) show that, to avoid overfitting the model, a time homogeneous form for the volatility functions is preferable. Hence, in this paper we assume that $\sigma_i(t)$ depends only on the time difference $i - t$ rather than on time t and maturity i separately. More specifically we assume the following model for the volatilities:

$$\sigma_{i-t} = a \exp(-b|i-t|).$$

The parameters a and b are estimated by minimizing the sum of squared differences between observed and model-based volatilities.

According to Packham (2005), the correlations $\rho_{i,j}(t)$ should have the following properties:

- $\rho_{i,j}(t)$ should be positive for all i, j and t.
- $\rho_{i,j}(t)$ is assumed to be smaller with increasing distance $|i - j|$.
- $\rho_{i,i+p}(t)$ is assumed to increase when i increases.

Based on this we have chosen the following model for the correlations:

$$\rho_{i,j}(t) = \exp(-c|j-i|).$$

The parameter c is estimated by minimizing the sum of squared differences between observed and model-based correlations.

Simulation: Simulations of spot rates $R(t, t+i)$ are obtained by first generating simulations of the forward rates and then using the following relationship:

$$R(t, t+i) \quad = \quad \left[\prod_{k=0}^{i-1}(1 + F_{t+k}(t))\right]^{1/i} - 1.$$

4. Performance of models in Solvency II applications

4.1. Data and parameter estimation

The interest rate model used in the Solvency II framework should be calibrated to the current risk free term structure. Moreover it should be calibrated to an appropriate volatility measure. There is no general market consensus on the methodology for determining the volatility measure. There are two alternatives; calibration to market prices of different interest rate derivatives and calibration based on historical interest rate data. A problem with the first alternative, is that derivatives with long maturities are often not available. Another important advantage of the latter alternative is that volatilities calibrated from historical data often are more stable than those implied by derivative prices. It would be useful to compare the two alternatives. However, this lies outside the scope of this paper. For the reasons discussed above, we have chosen to use historical interest rate data for calibration.

The short rate, which is a key ingredient in the CIR++ and G++ models, cannot be directly observed. Since the overnight rate is not considered to be a good proxy for the short rate, the more liquid 1-month or 3-month rates are usually used. In this paper we use the latter.

In Norway, the only available FRA-rates are 3-month rates. Hence, we have to calculate implied forward rates on the basis of spot interest rates with different maturities. More specifically, we construct the forward rates using the relationship between the forward and spot rates shown in Equation 3.10.

The same time period is used to estimate the parameters of all three interest rate models. Our data set consists of monthly data for the 3-month rate and swap rates with maturities 1 to 10 years from the period March, 2001 to March, 2011. Figure 1 shows the swap rates. Here, the dark blue line corresponds to the 1-year rate, the red to the 2-years rate and so on. The green line with the smallest volatility corresponds to the 10-year maturity rate. The estimated parameter values are given in Table 2. For the Libor Market model we also show the estimated volatility and correlation functions in Figures 2 and 3.

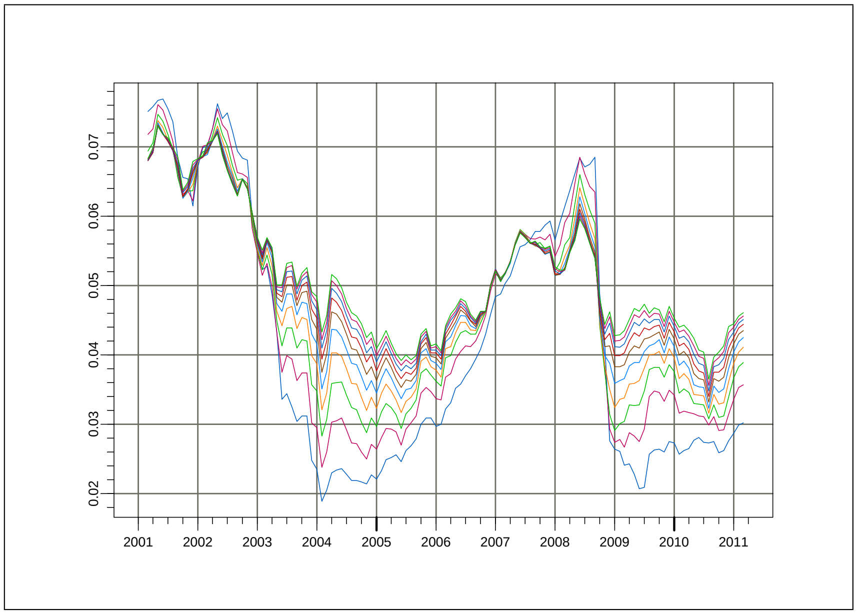

Figure 1: Norwegian swap rates for the period March 30, 2001 to March 29, 2011.

	CIR++-parameters				
	μ	β	σ		
Value	0.0238	0.2843	0.0433		
	G++-parameters				
	α	β	γ	η	κ
Value	0.6930	0.1203	1e-07	0.0102	-0.896
	Libor-parameters				
	a	b	c		
Value	0.272	0.1	0.15		

Table 2: Estimated parameter values for the different models.

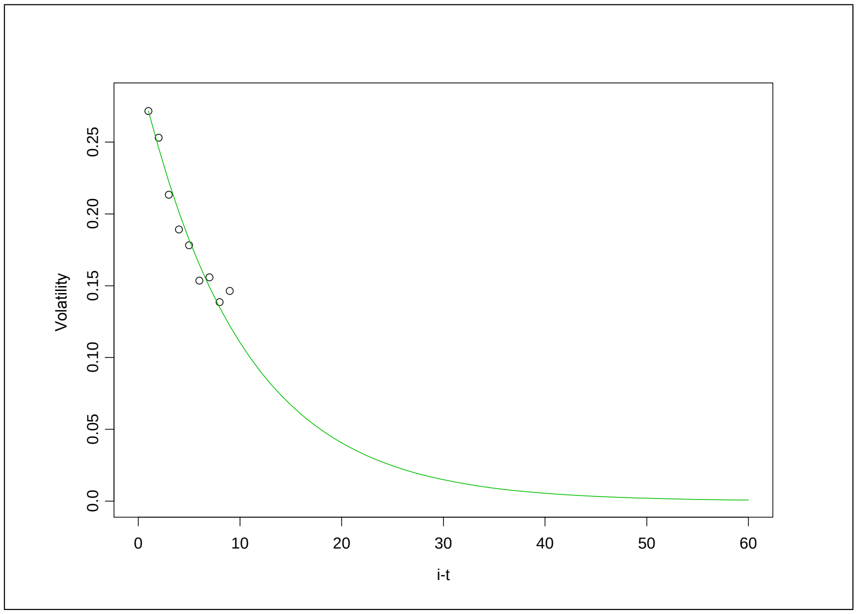

Figure 2: Estimated volatility function with observed volatilities superimposed.

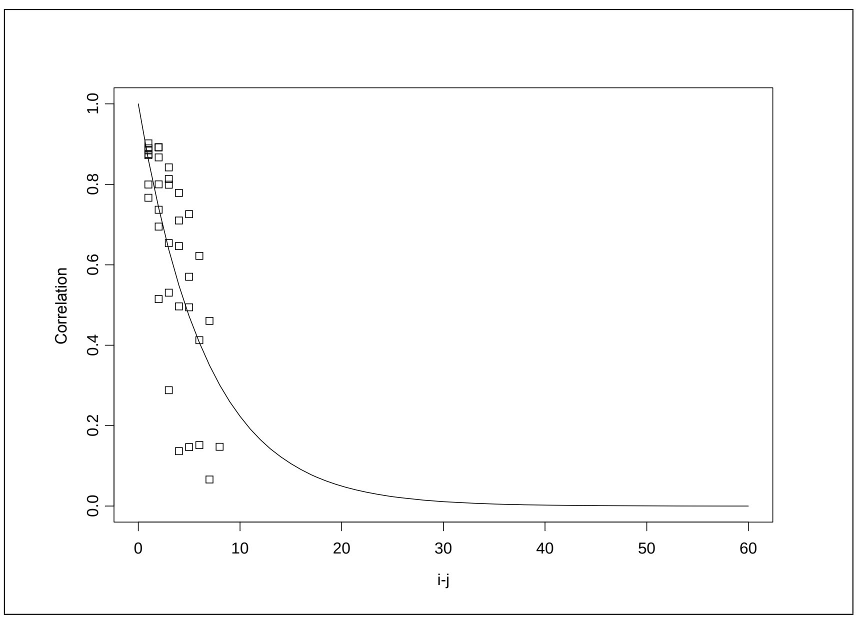

Figure 3: Estimated correlation function with observed correlations superimposed.

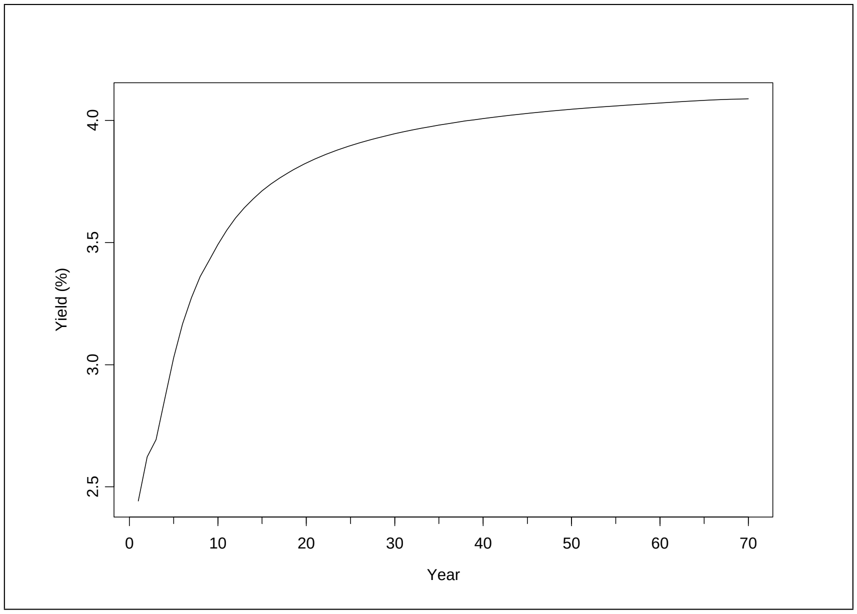

Figure 4: The yield curve specified by EIOPA in December 2011.

4.2. Simulation

When producing interest rate simulations used in the computation of the best estimate of liabilities under Solvency II, we want the expected future values of the interest rates to correspond to the yield curve prescribed by EIOPA. In this paper we use the yield curve shown in Figure 4, which was specified by EIOPA in December 2011. From this curve we derive the quantities $P^M(0,T)$ and $P^M(0,t)$ needed in the CIR++- and G++-models, as well as the starting values $F_i(0)$ in the Libor Market model. The latter are determined by setting $t = 0$ in Equation 3.10.

With the yield curve described here and the parameter estimates shown in Section 4.1, we generated, for each of the three models, 10 000 simulations with time horizon 60 years, of spot rates with maturities 1, 2, and 3 years. The number of simulations is chosen as a tradeoff between accurracy and execution time in the best estimate computations. It should be noted that in these computations, the derivation of the cash-flows are far more time-consuming than the interest rate simulations. The mean and standard deviation curves corresponding to the 10 0000 simulations are shown in Figures 5 and 6, respectively. As can be seen from the figures, even when calibrated to the same historical data, the simulations from the three models have very different mean value and volatility characteristics, especially far out into the future. It should be noted that these differences does not mean that not all three models fit the initial term structure. This can

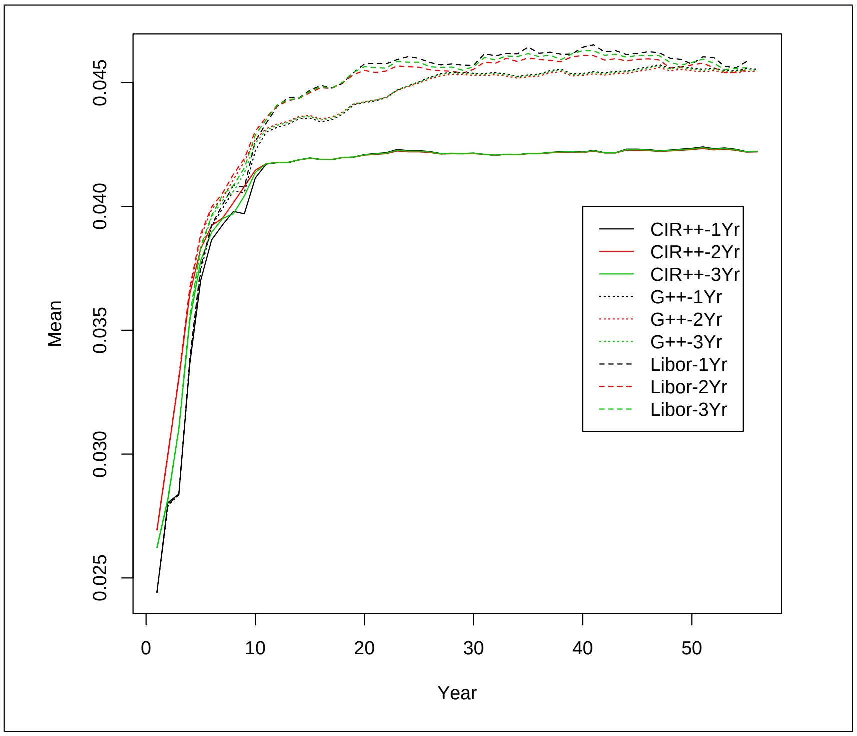

Figure 5: Mean of 10 000 simulations of spot rates with 1, 2 and 3 years maturities generated using the three different interest rate models.

be shown as follows: Let the discount factor in simulation s and year t, $d_{t,s}$ be defined as

$$d_{t,s} = \frac{1}{\prod_{u=1}^{t}(1 + R(u, u+1, s))},$$

where $R(u, u+1, s)$ is the value of the risk-free 1-year interest rate in year u and simulation s. Then for each year t, compute the average, $\bar{d}_t$ of $d_{t,s}$ over all simulations, and finally the yield curve derived from these average values as

$$y_t = (\bar{d}_t)^{-1/t} - 1.$$

For the simulations to be appropriate, the difference between the yield curve defined by $y_t, t = 1, \ldots T$ and the input yield curve in Figure 4 should be small. Figure 7, showing these differences (in basis points) for the three different models, verifies that this is the case.

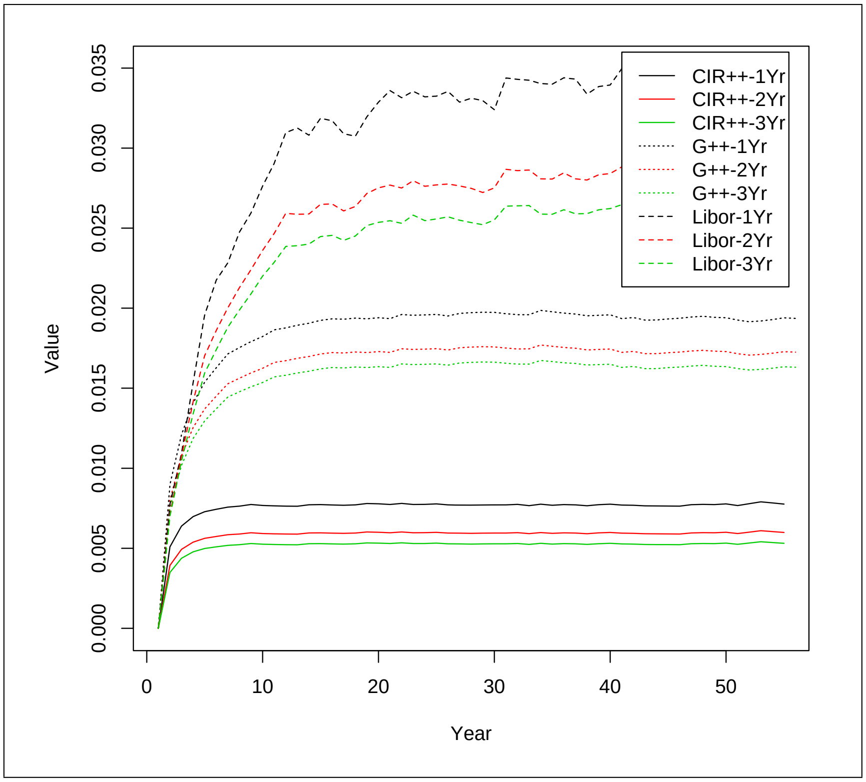

Figure 6: Standard deviation of 10 000 simulations of spot rates with 1, 2 and 3 years maturities generated using the three different interest rate models.

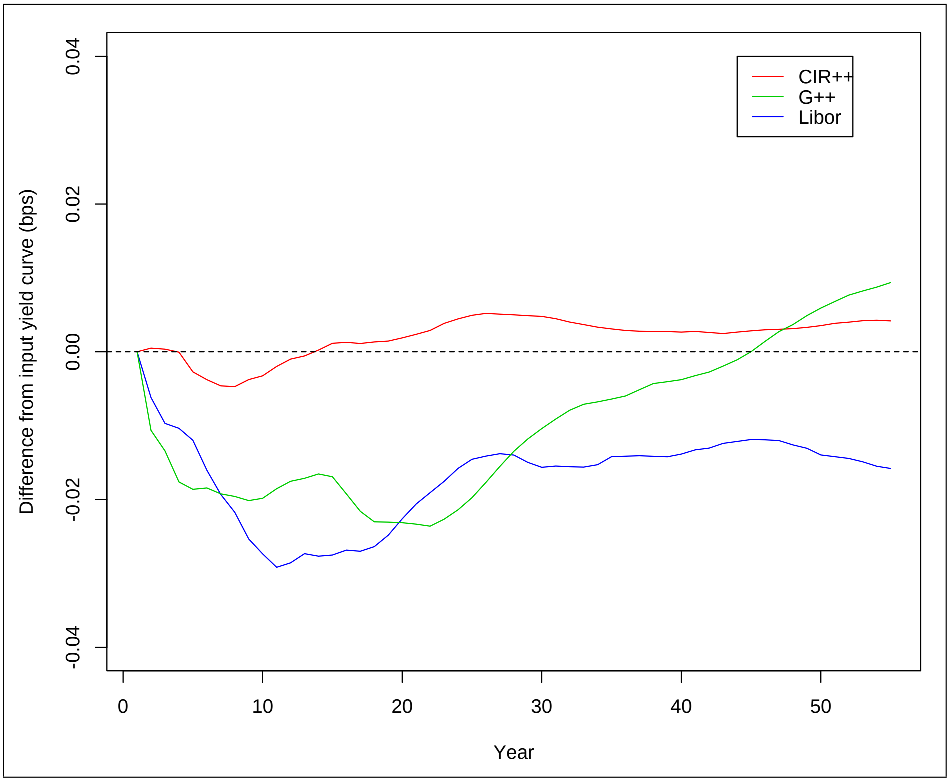

Figure 7: The differences between the implied yield curves and the market yield curve used when calibrating the models.

4.3. A simple example

Before describing the real-world case, we show the effects of the interest rate models for a simple example. In this example we start with a premium reserve of 462.38 MNOK and assume a time horizon of 50 years. The guaranteed benefits the first year are 100 MNOK and the development of the benefits over the time horizon is as shown in Figure 8. The product in question is assumed to be paid-up policies, where the annual guaranteed interest rate is 3.5% and any surplus is divided 20/80 between the company and the policyholders (any deficits must be covered by the company). Finally, we assume that the premium reserve is invested in assets with a volatility of 5%.

Let V_t be the premium reserve at the beginning of year t and z_t and g the achieved financial return in this year and guaranteed interest rate, respectively. Then, the company's financial market related profit, e_t, in year

	CIR++	G++	Libor
Mean	-39.84	-40.93	-41.41
St.dev	25.63	28.17	27.58
1%	8.50	11.39	11.01
99%	-106.68	-114.54	-114.10

Table 3: The characteristics of the distribution of the net present value of owner's result using the different models.

t is given by

$$e_t = \begin{cases} V_t\,(z_t - g) & z_t <= g \\ 0.2\,V_t\,(z_t - g) & z_t > g. \end{cases}$$

Further, the net present value of these results over the whole time horizon is

$$E = \sum_t e_t\, d_t,$$

where d_t is the discount factor in year t, defined as described in Section 4.2. To generate the probability distribution of E, we use the 10 000 interest rate simulations from Section 4.2. Moreover, in this simple example we simulate financial returns by

$$z_{t,s} \sim N(R(t, t+1, s), 0.05),$$

where $R(y, t+1, s)$ is the 1-year spot rate in year t and simulation s. We have thereby simplified the simulation of property, bond and share market returns in a risk neutral world. Table 3 shows the characteristics of the distribution of E for each of the three interest rate models. As can be seen from the table, the differences between the models are surprisingly small in view of the very different mean value and volatility characteristics shown in Section 4.2. This is probably due to the fact that the discounted cash-flows far out in the future are far less important for the best estimate than those in the first 10 years.

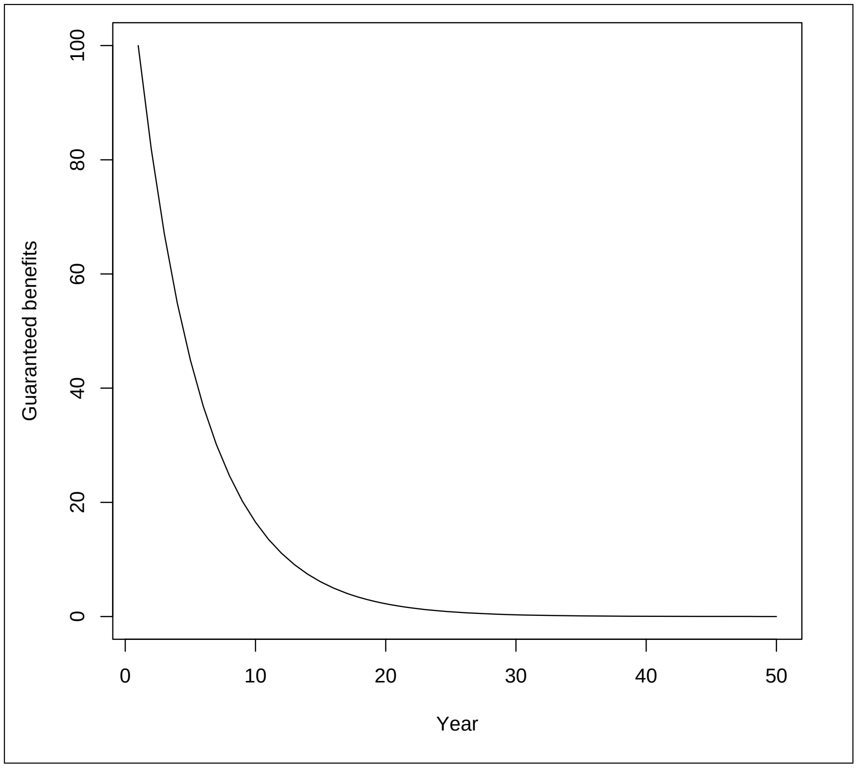

Figure 8: The development of the guaranteed benefits over the time horizon.

4.4. Real example

4.4.1. Asset models and parameters

The asset portfolio of the life insurance company may be divided into 5 main asset classes; *Norwegian stocks* (2%), *International stocks* (10%), *Real estate* (20%), *Credit bonds* (33%) and *Government bonds* (35%), where the approximate portfolio weights are given in parenthesis. For all asset classes we assume that the value develops as follows

$$\log V_t = \log V_{t-1} + \epsilon_t.$$

Moreover, since we are in a risk neutral world for Solvency II purposes, all kinds of assets will earn the risk free return on average, meaning that the expectation of the relative return $(V_t - V_{t-1})/V_{t-1}$ is equal to the discount rate. For the stocks, government bonds, and real estate we assume that

$$\epsilon_t \sim N(\mu_t, \sigma^2),$$

where μ_t is chosen such that the expected yearly relative return of the corresponding asset class equals the risk-free 1-year interest rate. Further, the volatilities used in the experiments described in this paper are 21%, 14%, 8%, and 0.7%, respectively, for Norwegian stocks, international stocks, real estate and government bonds.

For the credit bonds we use another model. We assume that the credit bond portfolio is rebalanced every year to maintain a fixed duration D. Then, the yearly change in the logarithm of the market value of the portfolio is given by

$$\epsilon_t = \log(1 + R(t, t + D) + \gamma_t) + \psi_t,$$

where $R_{D,t}$ is the risk free interest rate with maturity D years at the beginning of year t, γ_t is the change in the market value of the bond portfolio in year t, and ψ_t is the credit spread (we assume that the credit spread is the same for all durations). The credit spread term is assumed to be Gaussian distributed with mean 0 and standard deviation σ_ψ.

When computing the change in the market value we make the simplifying assumption that at the beginning of each year t, the whole bond portfolio is sold and replaced by a new portfolio for which the duration is D. Under this assumption, γ_t is given by

$$\gamma_t = \frac{\exp(-(D-1)\, R(t+1, t+D))}{\exp(-(D-1)\, R(t, t+D))} - 1, \tag{4.1}$$

The nominator in Equation 4.1 is the value of a zero coupon bond with duration $D-1$ issued at the beginning of period $t+1$, while the denominator is the value at the beginning of period $t+1$ of a zero coupon bond issued at the beginning of period t. The latter bond has remaining duration $D-1$ at the beginning of period $t+1$.

In the experiments described in this section we assume that the credit bond portfolio has a fixed duration of 3 years. This means that we only need to simulate three interest rates; the 1-year interest rate, which is used for discounting and for computing the expected return of stocks, government bonds, and real estate, and the 2-year and 3-year rates, that are used to determine the yearly change in the market value of the bond portfolio. The results in this section are based on the 10 000 interest rate simulations from Section 4.2. It should be noted that for the credit bonds, the volatility is dependent on the interest rate behaviour and hence will vary with time.

4.4.2. Liability models and parameters

In this paper we study two different products; *old-age pension for individuals with profit sharing* and *paid-up defined benefit pension policies with profit sharing (paid-up policies).* According to Norwegian legislation one must split these products' profits into their three main elements when allocating profit between the insurance company and policyholders: (i) the risk result, (ii) the administration result, and (iii) the financial market result. Roughly speaking the results are determined monthly and annually as follows:

- *Risk result:* Pure risk premium income minus benefits paid to policyholders and changes in premium reserves
- *Administration result:* Administration fees minus expenses and commissions
- *Financial market result:* Financial market income minus guaranteed interest and changes in market risk related reserves

The market risk related reserves are a conditional benefit. If the financial income is not sufficient to match guaranteed interest rate, the company can use these additional reserves before profit sharing is calculated.

The product *old-age pension for individuals with profit sharing* consists of an old-age pension with the possibility of a payment (lump sum or annuity) if the policyholder dies. The old-age pension is either paid out in a defined number of years or as a lifelong benefit, usually starting at the age 67. The guaranteed interest rate is between 2.5% and 4%, depending on when the policy was first taken out. The dataset used in this analysis has the age distribution shown as a blue line in Figure 9. If the sum of the risk result, administration result and financial market result is positive, then the policyholder will receive a minimum of 65% of the profit. In the analysis performed in this paper, both the risk result and the administration result are set to zero. This is expected to have a negligible impact on the results. These profit sharing rules ("Old profit sharing") are shown in Figure 10.

Paid-up policies are fully paid contracts from a defined benefit plan. The benefits are old-age pensions, spouse pension and disability pension. The old-age pension, with benefits usually starting at the age 67, and spouse pension, is either paid out in a defined number of years, or as a lifelong benefit. A disability pension may be paid until age 67. The guaranteed interest rate is between 2.5% and 4%, depending on when the premiums were paid. The dataset used in this analysis has the age distribution shown as a red line in Figure 9. If the risk result is positive, the policyholders will receive the profit from this element. The company receives the administration result in any case. If the sum of financial market result and risk result is positive, then the policyholder gets a minimum 80% of the profit. In this analysis the administration result is set to zero, and the risk result is slightly positive each year. This is expected to have a negligible impact on the analysis. These profit sharing rules ("Modified profit sharing") are shown in Figure 11.

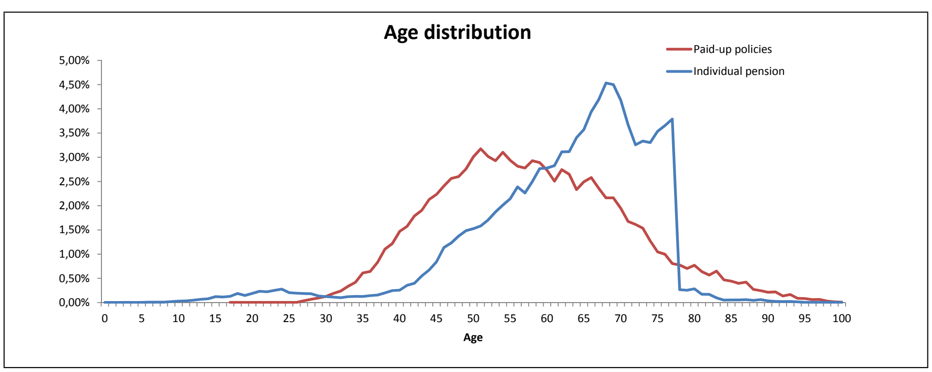

Figure 9: The age distributions for the **individual pension with profit sharing** (blue line) and the **paid-up policies** (red line) portfolios.

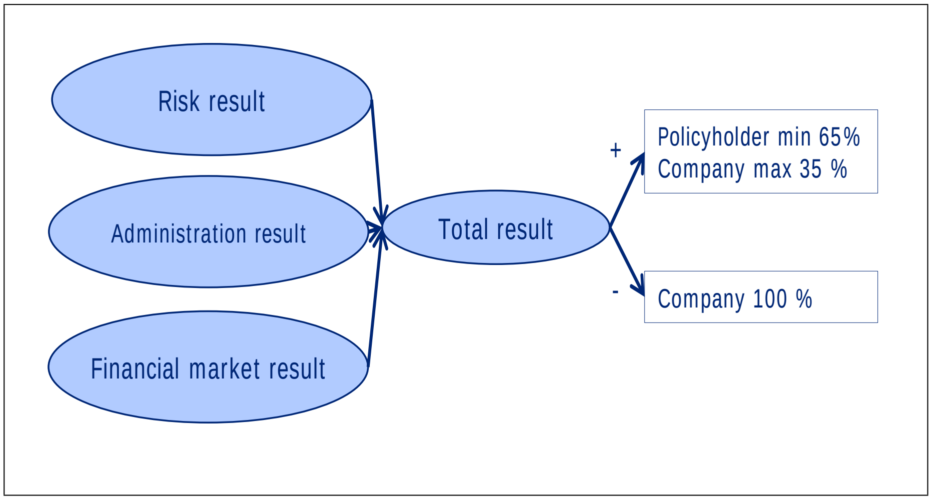

Figure 10: Profit sharing rules for **old-age pension for individuals**.

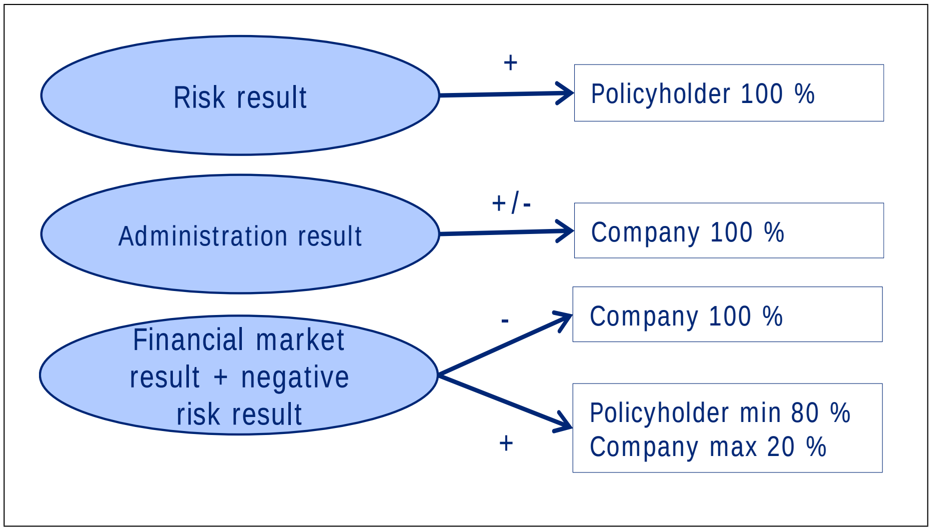

Figure 11: Profit sharing rules for **paid-up policies**.

4.4.3. Results

Based on the information in Sections 4.4.1 and 4.4.2 the best estimate $\hat{L}$ of the liabilities may be computed using Monte Carlo simulations as follows:

$$\hat{L} = \frac{1}{S} \sum_{s=1}^{S} \sum_{t=1}^{T} d_{t,s}\, X_{t,s},$$

where T is the time to ultimate run-off, S is the number of simulations, $X_{t,s}$ are the liability cash flows in year t and simulation s, and $d_{t,s}$ is the discount factor in year t and simulation s, defined as described in Section 4.2.

For this company, X_t is given as the sum of guaranteed and future discretionary benefits (FDB), B_t^G and B_t^F and operating expenses, E_t, less premiums, P_t, i.e.

$$X_t = B_t^G + B_t^F + E_t - P_t. \tag{4.2}$$

The development of the guaranteed benefits and the premiums is assumed to be deterministic. The two other quantities are stochastic variables depending on the behaviour of the capital markets. For the sake of simplicity the specific projections of these variables are omitted here. See Aas et al. (2014) for a more thorough description of how the cash-flows are derived for similar insurance products.

Table 4 shows the best estimate for the portfolio of pensions for individuals with profit sharing computed using the different interest rate models together with the underlying discounted cash flows. In the computations we have used T = 55 years and S =10 000. The corresponding figures for the paid-up policies portfolio are shown in Table 5. As can be seen from the tables, the relative difference between the smallest and the largest best estimate value is as low as 0.2% for the individual pension with profit sharing portfolio and 1% for the paid-up policies. Hence, even like for the simple example, the differences between the models are surprisingly small. It should be mentioned that the Monte Carlo sampling error for the Libor Market model results might be quite large even with 10 000 simulations. This is due to a well-known drawback of the classic Libor Market model. Typically when the time horizon is very long, one observes a relatively high probability of large interest rates. The largest 1-year rate simulated with this model was e.g. 67%, while the corresponding quantities for the CIR++-model and G++-models were 10.3% and 12.5%, respectively.

	CIR++	G++	Libor
Premiums	85.74	85.81	85.81
Guaranteed Benefits	7373.75	7383.09	7384.36
FDB	961.26	969.54	946.57
Expenses	326.03	327.15	326.78
Best estimate	**8575.32**	**8593.97**	**8571.92**

Table 4: **Portfolio of pensions for individuals with profit sharing:** Net present values computed using the different models.

	CIR++	G++	Libor
Premiums	0.00	0.00	0.00
Guaranteed Benefits	9217.96	9237.49	9241.40
FDB	1202.97	1292.65	1290.46
Expenses	650.05	653.41	654.10
Best estimate	**11070.98**	**11183.52**	**11185.96**

Table 5: **Portfolio of paid-up policies:** Net present values computed using the different models.

5. Summary and discussion

A key aspect of the Solvency II regulatory framework is to compute the best estimate of the liabilities. This best estimate should be the probability-weighted average of future cash-flows, discounted to its present value. Movements in economic variables are often the driving force of changes in liability present values. Hence, life insurers need stochastic models for producing future paths for e.g. interest rates, equity and bond returns and currencies. The paths should be risk-neutral, meaning that the expected return of all assets should be equal to the risk-neutral rate used for discounting the cash-flows. Hence, the interest rate model is a key component to consider within the Solvency II framework, particularly for life insurers. In this paper we have studied three interest rate models; the CIR++-model, the G++-model and the Libor Market model. Even when calibrated to the same historical data, the simulations from these models have very different mean value and volatility characteristics, especially far out into the future. However, when using these simulations when computing the best estimate of the liabilities, the differences between the models are surprisingly small, both for a synthetic and for a real-world insurance portfolio.

There might be several reasons for the small differences. First, the discounted cash-flows far out in the future are less important for the best estimate than those in the first 10 years. Second, the duration of the bond portfolio used in the experiments described in this paper is only three years. Hence, we simulate interest rates that have similar maturities. The real correlation between such near rates is likely to be rather high, meaning that the perfect correlation induced by the one-factor model probably is not unacceptable in principle.

More experiments should be performed to investigate whether the small differences we have observed in this paper are valid also for other yield curve shapes and other portfolio weights. However, if the differences really are small in general, our conclusion would be that model transparency and ease of use should be the deciding factors rather than which model is ideal in theory alone.

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