On the independence between financial and actuarial risks

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Introduction

- Model financial world to describe financial risks via filtered probability space

\[
\left( \Omega^{(1)}, \mathcal{F}^{(1)}, \left( \mathcal{F}_t^{(1)} \right)_{0 \leq t \leq T}, \mathbb{P}^{(1)} \right)
\]

- Model actuarial world to describe actuarial risks via filtered probability space

\[
\left( \Omega^{(2)}, \mathcal{F}^{(2)}, \left( \mathcal{F}_t^{(2)} \right)_{0 \leq t \leq T}, \mathbb{P}^{(2)} \right)
\]

- Model combined financial-actuarial world as product space
Insurance securitization involves financial and actuarial considerations.

Pricing via the no-arbitrage principle: $\mathbb{P} \rightarrow \mathbb{Q}$.

When independence between financial and actuarial risks is assumed in the real world, one often makes the same assumption in the pricing world.

In Black & Scholes setting: independence relations translate from $\mathbb{P}$ to $\mathbb{Q}$ and vice versa. But what happens in a general setting?
Introduction
Black & Scholes setting

- Assume a market with 2 traded assets: a financial asset $S^{(1)}$ and an actuarial asset $S^{(2)}$.
- In the real world, the dynamics are described by
  \[
  \frac{dS^{(i)}(t)}{S^{(i)}(t)} = \mu^{(i)} dt + \sigma^{(i)} dB^{(i)}(t), \quad \text{for } i = 1, 2.
  \]
- Furthermore,
  \[
  \text{cov}_{\mathbb{P}} \left[ B^{(1)}(t), B^{(2)}(t+s) \right] = \rho t, \quad \text{for any } t, s \geq 0.
  \]
- The coefficient $\rho$ is the correlation coefficient of the couple $(B^{(1)}(t), B^{(2)}(t))$. 

For an arbitrage-free market, the pricing dynamics are

\[
\frac{dS^{(i)}(t)}{S^{(i)}(t)} = rdt + \sigma^{(i)}dW^{(i)}(t), \quad \text{for } i = 1, 2.
\]

Also

\[
\text{cov}_Q \left[ W^{(1)}(t), W^{(2)}(t+s) \right] = \rho t = \text{cov}_P \left[ B^{(1)}(t), B^{(2)}(t+s) \right]
\]

for any \( t, s \geq 0 \).
Introduction

Main goal

- **Our goal:**
  - Can we translate an independence assumption from the real world $\mathbb{P}$ to the pricing world $\mathbb{Q}$?
  - What are the conditions needed such that $\mathbb{P}$-independence implies $\mathbb{Q}$-independence?
  - Illustration by some numerical examples.
  - General approach: see paper.
The combined financial-actuarial world: Example

Financial and actuarial risks - The financial world

- The usual set up: Consider the financial world
  \[
  \left( \Omega^{(1)}, \mathcal{F}^{(1)}, \mathbb{P}^{(1)} \right)
  \]
- Assume interest rate \( r = 0 \).
- The stock market:
  - Traded non-dividend paying stock \( S^{(1)} \)
  - \( S^{(1)}(0) = 100 \)
  - At time 1: \( S^{(1)}(1) \) is either 50 or 150
- \( \Omega^{(1)} \) describes all possible evolutions of the financial world:
  \[\Omega^{(1)} = \{50, 150\}.\]
- \( \mathcal{F}^{(1)} \) is the set of all subsets of \( \Omega^{(1)} \).
- Real-world probability measure \( \mathbb{P}^{(1)} \):
  \[
  \left\{
  \begin{array}{ll}
  \mathbb{P}^{(1)}[50] = p_1 \\
  \mathbb{P}^{(1)}[150] = 1 - p_1 \\
  \end{array}
  \right.
  \]
The combined financial-actuarial world: Example

Financial and actuarial risks - The actuarial world

- The usual set up: Consider the *actuarial world*

\[ \left( \Omega^{(2)}, \mathcal{F}^{(2)}, \mathbb{P}^{(2)} \right) \]

- Survival index \( \mathcal{I} \) of a population:
  - 0: 'few' persons survive during period \([0,1]\)
  - 1: 'many' persons survive during period \([0,1]\)
  - At time 1: \( \mathcal{I}(1) \) is 0 or 1

- \( \Omega^{(2)} \) describes all possible evolutions of the actuarial world:

\[ \Omega^{(2)} = \{0, 1\} \]

- \( \mathcal{F}^{(2)} \) is the set of all subsets of \( \Omega^{(2)} \).

- Real-world probability measure \( \mathbb{P}^{(2)} \):

\[
\begin{align*}
\mathbb{P}^{(2)}[0] &= p_2 \\
\mathbb{P}^{(2)}[1] &= 1 - p_2
\end{align*}
\]
Consider the combined \textit{financial-actuarial world}

\[(\Omega, \mathcal{F}, \mathbb{P})\]

- Cartesian product of both spaces.

- Universe \(\Omega\):

\[\Omega = \Omega^{(1)} \times \Omega^{(2)} = \{(50,0), (50,1), (150,0), (150,1)\}\]
The combined financial-actuarial world: Example

Financial and actuarial risks - The combined world

- $\mathbb{P}$ is the 'real-world' probability measure.
- Assumption: \[ \mathbb{P} \equiv \mathbb{P}^{(1)} \times \mathbb{P}^{(2)} \]
- In this case:

\[
\begin{align*}
\mathbb{P} [(50, 0)] &= p_1 \times p_2 \\
\mathbb{P} [(50, 1)] &= p_1 \times (1 - p_2) \\
\mathbb{P} [(150, 0)] &= (1 - p_1) \times p_2 \\
\mathbb{P} [(150, 1)] &= (1 - p_1) \times (1 - p_2)
\end{align*}
\]
The combined financial-actuarial world: Example
Financial and actuarial risks - Pricing

- Prices of (non-dividend paying) traded securities in the combined world follow from no-arbitrage principle.
- Replace $\mathbb{P}$ by an 'equivalent martingale measure' $\mathbb{Q}$.
- Price reciprocity:
  For a pay-off at time 1 equal to $X$, the price at time 0 under this measure $\mathbb{Q}$ is given by
  \[ e^{-rT} E^Q[X]. \]
The combined financial-actuarial world: Example
Financial and actuarial risks - Pricing

Given Q, we can construct the projections $Q^{(1)}$ and $Q^{(2)}$ on the financial and the actuarial world, respectively.

Introduce the measure $Q^{(1)} \times Q^{(2)}$.

**Definition**
Independence in pricing world between financial and actuarial risks if

$$Q \equiv Q^{(1)} \times Q^{(2)}$$
The combined financial-actuarial world: Example

A market with 1 actuarial and 2 financial securities

- **Traded securities:**
  - Risk-free bank account (assume \( r = 0 \))
  - Stock with price \( S^{(1)}(1) \) at time 1 and \( S^{(1)}(0) = 100 \)
  - Actuarial security:
    
    \[
    S^{(2)}(1) = 100 \times I(1), \text{ with } S^{(2)}(0) = 70
    \]

- **Determining \( Q \):**

  \[
  \begin{align*}
  \mathbb{E}^Q \left[ S^{(1)}(1) \right] &= 100 \\
  \mathbb{E}^Q \left[ S^{(2)}(1) \right] &= 70 \\
  Q \left[ (50, 0) \right] + Q \left[ (150, 0) \right] + Q \left[ (50, 1) \right] + Q \left[ (150, 1) \right] &= 1
  \end{align*}
  \]
The combined financial-actuarial world: Example

A market with 1 actuarial and 2 financial securities

- Equivalent with

\[
\begin{align*}
Q^{(1)}[50] &= 0.5 \\
Q^{(1)}[150] &= 0.5 \\
Q^{(2)}[0] &= 0.3 \\
Q^{(2)}[1] &= 0.7
\end{align*}
\]

- Particular solutions: \(Q\) and \(Q^{(1)} \times Q^{(2)}\)

\[
\begin{align*}
\overline{Q}[(50, 0)] &= 0.2 \\
\overline{Q}[(150, 0)] &= 0.1 \\
\overline{Q}[(50, 1)] &= 0.3 \\
\overline{Q}[(150, 1)] &= 0.4
\end{align*}
\] and

\[
\begin{align*}
Q^{(1)}[50] \times Q^{(2)}[0] &= 0.15 \\
Q^{(1)}[150] \times Q^{(2)}[0] &= 0.15 \\
Q^{(1)}[50] \times Q^{(2)}[1] &= 0.35 \\
Q^{(1)}[150] \times Q^{(2)}[1] &= 0.35
\end{align*}
\]

- The combined market is arbitrage-free but incomplete.
- Under \(Q\), the independence relation is not maintained.
- There is only one solution with the independence property, namely \(Q^{(1)} \times Q^{(2)}\).
The combined financial-actuarial world: Example
A market with a combined product

- **Traded securities:**
  - Risk-free bank account (assume \( r = 0 \))
  - Stock \( S^{(1)} \)
  - Actuarial security \( S^{(2)} \)
  - Combined product:

\[
S(1) = \left( 100 - S^{(1)}(1) \right)_+ \times \mathcal{I}(1), \text{ with current price } S(0)
\]

- **Determining \( Q \):**

\[
\begin{align*}
\mathbb{E}_Q^Q \left[ S^{(1)}(1) \right] &= 100 \\
\mathbb{E}_Q^Q \left[ S^{(2)}(1) \right] &= 70 \\
\mathbb{E}_Q^Q [S(1)] &= S(0) \\
Q [(50, 0)] + Q [(150, 0)] + Q [(50, 1)] + Q [(150, 1)] &= 1
\end{align*}
\]
The combined financial-actuarial world: Example
A market with a combined product

- The single solution $\tilde{Q}$ depends on $S(0)$:

$$
\begin{align*}
\tilde{Q}[(50, 0)] &= \frac{25 - S(0)}{50} \\
\tilde{Q}[(150, 0)] &= \frac{-10 + S(0)}{50} \\
\tilde{Q}[(50, 1)] &= \frac{S(0)}{50} \\
\tilde{Q}[(150, 1)] &= \frac{35 - S(0)}{50}
\end{align*}
$$

- The combined market is now arbitrage-free AND complete.

- $\tilde{Q} = \tilde{Q}^{(1)} \times \tilde{Q}^{(2)} \iff S(0) = 17.5$

- **Conclusion:** The market can choose the measure with the independence property by setting the price of the combined product equal to $S(0) = 17.5$. 
The combined financial-actuarial world: Example
An incomplete market with a combined product

- **Financial world:**
  - Risk-free bank account (assume $r = 0$)
  - Barometer of the economy: *Booming* ($B$), *Moderate growth* ($M$) or *Recession* ($R$)
  - Financial universe $\Omega^{(1)} = \{ B, M, R \}$
  - Real-world probabilities: $\mathbb{P}^{(1)}[B]$, $\mathbb{P}^{(1)}[M]$ and $\mathbb{P}^{(1)}[R]$

- **Actuarial world:**
  - Universe $\Omega^{(2)}$:
    \[
    \Omega^{(2)} = \{ 0, 1 \} .
    \]
  - Real-world probabilities: $\mathbb{P}^{(2)}[0]$ and $\mathbb{P}^{(2)}[1]$
The combined financial-actuarial world: Example

An incomplete market with a combined product

- **Combined financial-actuarial world:**
  - Universe $\Omega$:
    $$\Omega = \{(B, 0), (M, 0), (R, 0), (B, 1), (M, 1), (R, 1)\}.$$
  - **Assumption:** under $\mathbb{P}$, financial and actuarial risks are independent
    $$\mathbb{P} \equiv \mathbb{P}^{(1)} \times \mathbb{P}^{(2)}$$

- **Traded securities:**
  - Risk-free bank account (assume $r = 0$)
  - Financial asset with current price $S^{(1)}(0) = 50$
    $$S^{(1)}(1) = \begin{cases} 100 & \text{if the economy is Booming} \\ 0 & \text{otherwise} \end{cases}$$
  - Actuarial security $S^{(2)}$ with current price $S^{(2)}(0) = 70$
  - Combined product $\bar{S}$ with pay-off
    $$\bar{S}(1) = S^{(1)}(1) \times (1 - I(1))$$ and current price $\bar{S}(0)$
The combined financial-actuarial world: Example

An incomplete market with a combined product

- **Determining \( Q \):**

\[
\begin{align*}
\mathbb{E}^{Q} \left[ \bar{S}^{(1)}(1) \right] &= 50 \\
\mathbb{E}^{Q} \left[ S^{(2)}(1) \right] &= 70 \\
\mathbb{E}^{Q} \left[ \bar{S}(1) \right] &= \bar{S}(0) \\
Q[(B, 0)] + Q[(B, 1)] + Q[(M, 0)] + Q[(M, 1)] + Q[(R, 0)] + Q[(R, 1)] &= 1
\end{align*}
\]

- **Equivalent with:**

\[
\begin{align*}
Q[(B, 0)] &= \frac{\bar{S}(0)}{100} \\
Q[(B, 1)] &= \frac{50 - \bar{S}(0)}{100} \\
Q[(M, 0)] + Q[(R, 0)] &= \frac{30 - \bar{S}(0)}{100} \\
Q[(M, 1)] + Q[(R, 1)] &= \frac{20 + \bar{S}(0)}{100}
\end{align*}
\]

- **More than one solution \( \Rightarrow \) the market is incomplete.**
We find that:

\[ Q^{(1)}[B] = 0.5 \quad \text{and} \quad Q^{(2)}[0] = 0.3. \]

For every \( Q \) satisfying the system of equations, one has that

\[ Q[(B, 0)] = Q^{(1)}[B] \times Q^{(2)}[0] \iff \bar{S}(0) = 15, \]

which is a necessary condition for the independence property to hold.

Conclusion: The market can choose an equivalent martingale measure with the independence property by setting the current price of the combined product equal to \( \bar{S}(0) = 15 \).
A general combined financial-actuarial world

Financial and actuarial risks - The financial world

- Consider the filtered probability space

\[ \left( \Omega^{(1)}, \mathcal{F}^{(1)}, \left( \mathcal{F}_t^{(1)} \right)_{0 \leq t \leq T}, \mathbb{P}^{(1)} \right) \]

- Pricing financial assets by the no-arbitrage principle.

- \( \left( S^{(1)}(t) \right)_{0 \leq t \leq T} \) : financial stochastic process

- \( (r(t))_{0 \leq t \leq T} \) : instantaneous risk-free interest rate process
A general combined financial-actuarial world

Financial and actuarial risks - The actuarial world

Consider the filtered probability space

\[
\left( \Omega^{(2)}, \mathcal{F}^{(2)}, \left( \mathcal{F}_t^{(2)} \right)_{0 \leq t \leq T}, \mathbb{P}^{(2)} \right)
\]

Used to describe the biometrical evolutions in a time interval \([0, T]\). E.g. future mortality of a group of insureds

\[
\left( S_{(x)}^{(2)}(t) \right)_{0 \leq t \leq T} : \text{biometrical stochastic process of } (x)
\]

\[
S_{(x)}^{(2)}(t) = \begin{cases} 
1 & \text{if } (x) \text{ is alive at } t \\
0 & \text{otherwise}
\end{cases}
\]
A general combined financial-actuarial world

Financial and actuarial risks - The combined world

- Consider the filtered probability space

\[
(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}) = \left(\Omega^{(1)} \times \Omega^{(2)}, \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}, \left(\mathcal{F}^{(1)}_t \otimes \mathcal{F}^{(2)}_t\right)_{0 \leq t \leq T}, \mathbb{P}\right)
\]

- To model combined financial - actuarial risks and stochastic processes.

- Pricing securities of which the pay-off depends on financial and actuarial quantities.
  E.g. mortality-linked securities
A general combined financial-actuarial world
Financial and actuarial risks - Pricing

- An equivalent martingale measure $Q$ is needed for pricing purposes.

- **Price recipe:**
  The price of a traded mortality-linked security at time $t$ with pay-off $H$ at fixed term $u \geq t$ is equal to

$$
\mathbb{E}^Q \left[ e^{-\int_t^u r(\tau) \, d\tau} \ H \mid \mathcal{F}_t \right].
$$

- Financial assets can be priced in $Q$ or $Q^{(1)}$.
- Actuarial assets can be priced in $Q^{(2)}$ only if $r$ is deterministic.
Again, we assume

\[ P \equiv P^{(1)} \times P^{(2)} \]

For an equivalent pricing measure \( Q \), we determine the projections \( Q^{(1)} \) and \( Q^{(2)} \) and define

\[ Q' = Q^{(1)} \times Q^{(2)} \]

**Question**: Is \( Q \) independent or equivalently, is \( Q \equiv Q' \)?
A general combined financial-actuarial world
From real-world independence to pricing independence

Consider a combined world where the following assumptions hold:

- In the $\mathbb{P}$-world, financial and actuarial risks are independent,
- The risk-free interest rate is deterministic,
- No combined assets are traded in the market,
- The combined market is arbitrage-free in the sense that there exists a pricing measure.

In such a combined world, it is always possible to construct an equivalent martingale measure $\mathbb{Q}$ under which financial and actuarial risks are independent.
Conclusion

- A real-world independence between both risks may seem reasonable.
- Real-world independence does not imply pricing world independence.
- This wrong implication is sometimes made in the literature.
- Often, the market can choose a martingale measure $Q$ for which the financial and actuarial risks are independent.
Further research

- In a market where combined assets are traded, what are the extra conditions to guarantee the existence of a $Q$-measure with the independence property?
- Can we extend this theory to other copulas, besides the independent copula?
- What conditions are required to maintain a stochastic order relation when moving from $P$ to $Q$?
References


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