THE CAPITAL-ASSET PRICING MODEL RISES FROM THE DEAD:
ELLiptical Symmetry of REAL RETURNS IN SOUTH AFRICA AND ITS
IMPLICATIONS FOR LONG-TERM ACTUARIAL MODELLING

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ABSTRACT
Previous work of the author and a co-author found that, on the assumption that the joint distribution of
real quarterly returns on equities and bonds in South Africa over the past 56 years is multivariate
normal, and on the rational-expectations hypothesis, the CAPM generally fails to explain those returns.
For the zero-beta version of the CAPM with in-period betas it appeared from that paper that the CAPM
could be accepted for more recent years (from 1986 onwards). In this paper the joint distribution of
these returns is tested for elliptical symmetry. The author is unable to reject the hypothesis that this
distribution is elliptically symmetric. On the basis of these results he argues that, notwithstanding the
findings of their previous work, the CAPM may be used in the modelling of real returns for the market-
consistent pricing of liabilities, for investment-management benchmarking and for the determination of
capital adequacy.

KEYWORDS
Elliptical symmetry, capital-asset pricing model; South Africa; long-term actuarial modelling
1. INTRODUCTION

1.1 As explained in a previous paper (Reddy & Thomson, unpublished), from an actuarial point of view, the CAPM provides a useful market-consistent pricing model for the stochastic modelling of investment returns:

“Typical actuarial applications of such modelling are for the market-consistent pricing of the liabilities of a financial institution, the determination of investment-performance benchmarks that are consistent both with its own liabilities and with market prices, and the determination of capital adequacy requirements. In the case of life assurers and retirement funds, the time horizons involved are long-term and the monitoring intervals may be triennial, annual or quarterly. This study has been designed so as to address the needs of such applications.”

1.2 That paper found that, on the assumption that the joint distribution of real quarterly returns on equities and bonds in South Africa over the past 56 years is multivariate normal, and on the rational-expectations hypothesis, the CAPM fails to explain those returns. Bearing in mind the principles cited in the preceding paragraph, this paper returns to the basic assumptions underlying the CAPM and reconsiders it in the light of those assumptions. It addresses the question whether, despite the findings of Reddy & Thomson (op. cit.), there is a role for the CAPM in actuarial modelling. From the discussion in section 2 it emerges that the main requirement for such a role is that the joint distribution of returns be elliptically symmetric.

1.3 The concept of an ‘elliptically symmetric’ distribution may be heuristically explained with reference to the bivariate case. In this case suppose that the variates are $X$ and $Y$ and that the joint probability density function is represented in a third dimension over the $X–Y$ surface. The intercept between the typically bell-shaped surface described by the density function and the level surface described by a constant value in the third dimension may be described as a ‘contour’ of the density
function. The constant value may be described as the ‘altitude’ of that contour. In the case of the bivariate normal distribution the contours are ellipses centred about the point on the \( X-Y \) surface representing the means of the two variates. The shape and orientation of the ellipses will depend on the variances and the covariance of the variates. The area within the contour will diminish as the altitude of the contour increases, tending to zero as the altitude tends to its maximum value over the point in \( X-Y \) space representing the mean. The centre, the shape and the orientation of each contour will be the same as every other contour. This is described as ‘elliptical symmetry’. Similar properties apply to multivariate distributions, though the heuristic visualisation is not possible. It is not only the multivariate normal distribution that is elliptically symmetric; the multivariate \( t \)-distribution (which has fat tails) is also elliptically symmetric. Many other distributions also have this property. Whilst they do not include skew distributions, they do include distributions with any level of kurtosis that may be necessary. In the literature there are numerous descriptions of generic tests of elliptical symmetry, i.e. tests that do not depend on any formulation of the joint probability density function. Some of these are described in section 2.

1.4 As explained in Thomson & Reddy (op. cit.), this paper considers returns in real terms. The data used are the same as for that paper and, as in that paper, returns are measured as forces of return.

1.5 The rest of the paper is organised as follows. In section 2, some literature on the requirement of elliptical symmetry in mean–variance analysis in general and the CAPM in particular, and on tests of elliptical symmetry, is briefly reviewed. In section 3 the test used in this study is described and the results of the test are presented and discussed. The results are summarised, and conclusions are drawn, in section 4.

2. LITERATURE REVIEW
2.1 This section extends the literature of the previous study, dealing only with the literature on the tests considered in this study. For more general literature relevant to this paper, the reader is referred to the literature review in Thomson & Reddy (2011; unpublished).

2.2 In the literature, tests of the CAPM are generally based on the assumption that the joint distribution of returns is multivariate normal. In fact at least one standard text (Fabozzi et al., 2011: 71–2) cites fat tails as a reason why the normal distribution cannot be generally accepted and hence why mean–variance analysis should not be used for portfolio selection. No such assumption is made in the statement of the CAPM, nor is such an assumption necessary to justify the use of mean–variance analysis. A more general condition is that the distribution of returns be elliptically symmetric. If the market-participant is an expected-utility maximiser and *ex-ante* distributions are elliptically symmetric then mean–variance analysis is justified.

2.3 In fact even expected-utility maximisation is not necessary; all that is required is that the market participant uses a risk measure that is positive-homogeneous and translation-invariant (McNeil, Frey & Embrechts, 2005: 247). A risk measure is positive-homogeneous if the market participant’s capital required is proportionate to the scale of the distribution of losses. A risk measure is translation-invariant if the extra capital required for a deterministic shift in the distribution of losses is equal to the shift. (ibid.) For normative purposes, these conditions may be regarded as sound and reasonable. They include the use of expected-utility theory but they do not necessitate it. Of course, all such conditions are sufficient, not necessary; if, regardless of the distribution of returns, a market participant wishes to measure her/his preferences in mean–variance space then there is no objective reason why she/he should not do so.

2.4 In summary, therefore, a sufficient condition for the use of mean–variance analysis by a market participant is that she/he:
(a) is an expected-utility maximiser whose utility function is quadratic; or
(b) makes decisions based on an elliptically symmetric *ex-ante* distribution of returns and uses a risk measure that is positive-homogeneous and translation-invariant; or
(c) has preferences that may be expressed in terms of indifference curves in mean–variance space.

2.5 Apart from issues relating to the descriptive validity of expected-utility theory, even for expected-utility maximisers quadratic utility functions are not necessarily typical, nor are they normatively sound (Thomson, unpublished). Condition (b) is therefore of more interest than condition (a).

2.6 In the absence of any explanatory theory, the assumption that a market participant who is not an expected-utility maximiser and who disregards any opinion about the higher moments of the distribution of returns will have preferences that may be expressed in terms of indifference curves in mean–variance space is arbitrary. Condition (b) appears to be of more interest than condition (c).

2.7 McNeil, Frey & Embrechts (op. cit.) observe that elliptically symmetric distributions provided “far superior models to the multivariate normal for daily and weekly US stock-return data.” Not only does condition (b) allow generalisation beyond quadratic utility, it allows generalisation beyond expected-utility theory per se. The cost of this generalisation does not involve the abandonment of fat tails; it still permits any elliptical distribution. Because the requirement that risk measures be positive-homogeneous and translation-invariant is uncontroversial, the main requirement for the use of mean–variance analysis is that the joint distribution of returns be elliptically symmetric.

2.8 As observed above (¶1.3) numerous tests of elliptical symmetry have been developed in the literature. These involve the standardisation of the observed values using:

\[ Z = \hat{\Sigma}^{-1/2} (X - \hat{\mu}) \]
(e.g. McNeil, Frey & Embrechts, 2005: 99) where \( X \) is a \( p \)-component random variable and \( \hat{\mu} \) and \( \hat{\Sigma} \) are estimates of its mean and variance. The distribution of \( X \) is elliptically symmetric if and only if the standardised variable \( Z \) is spherically symmetric. In the bivariate case, circular symmetry exists if the contours of the joint probability distribution function are circular. In particular, any observation from such a distribution may be described by the position on the unit circle of its projection onto that circle and its distance from the centre (i.e. its ‘radial length’). For multivariate distributions, the extension of this concept is referred to as ‘spherical symmetry’. In the trivariate case any observation may be described by its projection onto the unit sphere and its distance from the centre. In the general multivariate case, references to the ‘unit circle’ or the ‘unit sphere’ are replaced by references to the ‘unit hypersphere’. After standardisation, the tests of elliptical symmetry reduce to tests of spherical symmetry.

2.9 It may be shown that another property of a spherically symmetric distribution is that:

- \( U = \frac{Z}{|Z|} \) is uniformly distributed on the unit hypersphere; and

- \( U \) and \( |Z| \) are independent.

Here \( |Z| = \sqrt{Z'Z} \) is the Euclidean norm (i.e. the radial length) of the vector \( Z \) and \( U \) is the radial projection of that vector \( Z \) onto the unit hypersphere. The tests exploit this feature.

2.10 Some of these tests (e.g. Huffer & Park, 2007) involve segmenting the hypersphere into cells comprising the intersection of shells and sectors. In two-dimensional space the shells are concentric rings with boundaries defined by quantiles of the empirical distribution of \( |Z| \) and the sectors are slices, from the centre, of equal angular width. A chi-squared test statistic is then defined with respect to the actual and expected numbers of observations in each cell.
2.11 Most of the tests (e.g. Baringhaus, 1991) involve the determination of statistics that have various desirable properties specified by the authors. The literature on such tests requires an understanding of spherical harmonics. In the absence of such understanding the test statistics developed lack intuitive meaning.

2.12 An alternative approach (Diks & Tong, 1999) involves the determination of a test statistic that represents in some sense the ‘distance’ between the empirical distribution of $X$ and the projection of a spherically symmetric distribution to the length of $X$ to give:

$$Z_i = \frac{Y_i}{|Y_i|} |X_i|.$$ 

Here $Y_i$ may be any spherically symmetric distribution; typically $Y_i \sim N(\theta, I)$.

2.13 All these tests involve the standardisation of $X$. They assume that $\mu$ and $\Sigma$ are equal to their sample values; i.e. that these values are known.

2.14 Other literature of relevance is reviewed in Reddy & Thomson (2011; unpublished); that review is not repeated here.

3. **EMPIRICAL TEST OF ELLIPTICAL SYMMETRY**

3.1 The distribution of the returns was tested for the null hypotheses of elliptical symmetry. For the purposes of discussion of these results in the light of those of Reddy & Thomson (op. cit.), the distribution was also tested for the null hypothesis of normality. In this section those tests are described and the results of some of those tests are reported.

3.2 For a small data set such as the one under consideration in this paper, and for dimensions of four or seven, the chi-squared test proposed by Huffer & Park (op. cit.; ¶2.10 above) is problematic. For
example, during the period [153, 185] if the sectors comprised orthants (i.e. the analogue of quadrants in higher dimensions) and three sectors were used there would be $2^7 \times 3 = 378$ cells. (In this paper, as in Reddy & Thomson (op. cit.), quarters are numbered from 1 for the quarter ended 31/12/1964 to 185 for the quarter ended 31/12/2011; periods are defined with reference to these numbers.) These cells would be populated by only 33 observations, so that in most cells there would be no observation. This would not permit the use of the proposed test.

3.3 The tests statistics described by Baringhaus (op. cit.; ¶2.11) were computed and pseudo-random samples of the test statistics were generated by means of Monte Carlo simulation. For this purpose a bootstrapping method was used. Data were sampled from the observed sample with replacement. For each such sample the test statistics were computed. This gave a pseudo-random sample of each of the test statistics. From this sample the probability that the test statistic lay in the rejection region was calculated. These probabilities were checked against those determined by Baringhaus (ibid.). It was found, however, that, for the purposes of this study, those tests were insufficiently powerful. This was because, whilst the actual samples had means and variance–covariance matrices equal to those of the empirical population, the pseudo-random sample means and variance–covariance matrices were not necessarily equal to those of the population from which they were supposed to have been drawn. This gave the actual sample an advantage over the pseudo-random sample, which made it extremely unlikely that the test statistic would lie in the rejection region. The results of those tests are therefore not reported.

3.4 The test statistic proposed by Diks & Tong is based on a measure of the closeness of two sample values. The measure of closeness may be expressed as:

$$K(x, y) = \exp \left\{ - \frac{(x - y)'(x - y)}{4d^2} \right\},$$

where the sample values are:
The sample values may be from the same sample or from two different samples. Here $d$ is a bandwidth value, which (in broad terms) influences the ranges over which the closeness of two sample values is to be given relatively substantial weighting.

3.5 Let

$$X_i = \hat{\Sigma}^{-1} (R_i - \hat{\mu});$$

where:

$$\hat{\Sigma} = \frac{1}{n} R'R;$$

$$R = \begin{pmatrix} R'_1 - \hat{\mu} \\ \vdots \\ R'_n - \hat{\mu} \end{pmatrix};$$

$$R_i = \begin{pmatrix} R_{ii} \\ \vdots \\ R_{pi} \end{pmatrix};$$

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} R_i.$$

The test statistic proposed by Diks & Tong is:

$$T = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} K(X_i, X_j).$$
Here, in the measure of closeness, the sample values are from the same sample. Heuristically, the formula for $T$ thus constitutes the sample mean of the measure of the closeness of $T$ to itself. A high value of this statistic indicates that a high proportion of sample values are close together, in a similar sense to that in which, under Huffer & Park’s (op. cit.) test, a high proportion of standardised observations would be in certain cells. The test is therefore a one-tailed test; a high value of $T$ must be rejected.

3.6 In order to estimate the distribution of the test statistic, and hence the p-value of the observed sample, a pseudo-random sample may be generated by determining:

$$Z_{\nu t} = \frac{Y_{\nu t}}{|Y_{\nu t}|} |X_t| \quad \text{for } \nu = 1, \ldots, N; \, t = 1, \ldots, n;$$

where:

$$|x| = \sqrt{x'x}; \text{ and}$$

$$Y_{\nu t} \sim N(0, I).$$

Here $Y_{\nu t}/|Y_{\nu t}|$ represents a uniform distribution on the unit hypersphere and $|X_t|$ represents the radial distance of the standardised empirical distribution from the centre. $Z_{\nu t}$ thus represents a spherically symmetrical version of the standardised empirical distribution: a radial projection of the pseudo-random sample value on the unit hypersphere onto the hypersphere of radius $|X_t|$.

3.7 For each $\nu$ the test statistic

$$T_{\nu} = \frac{2}{n(n-1)} \sum_{\ell=1}^{n-1} \sum_{i=\ell+1}^{n} K(X_{\ell}, Z_{\nu \ell})$$
may then be calculated to give a pseudo-random sample. Here, in the measure of closeness, the sample values are from different samples: the standardised values from the observed sample and the pseudo-random sample. The p-value of the test statistic may then be calculated as:

\[ P\left( T \geq T_\nu \mid \mu = \hat{\mu}, \Sigma = \hat{\Sigma}, \nu = 1, \ldots, N \right) = \frac{k}{N+1}; \]

where \( k \) is the number of values in \( \bigcup_{\nu=1}^{N} \{T_\nu\} \cup \{T\} \) such that \( k \geq T \).

3.8 A problem with the above test statistics as proposed by Diks & Tong is that, in general, for \( x \neq y \),

\[ \sum_{i=1}^{n-1} \sum_{u=1}^{n} K(x_i, y_u) \neq \sum_{i=1}^{n-1} \sum_{u=1}^{n} K(y_i, x_u). \]

In the expression on the left-hand side, values of \( x \) are compared with values of \( y \) in later years. In the expression on the right-hand side, values of \( x \) are compared with values of \( y \) in earlier years. This does not affect the test statistic calculated from the observed values, since:

\[ \sum_{i=1}^{n-1} \sum_{u=1}^{n} K(X_i, X_u) = \sum_{i=1}^{n-1} \sum_{u=1}^{n} K(X_u, X_i). \]

However, it does affect those based on the closeness of the observed sample and the pseudo-random sample, since, in general:

\[ \sum_{i=1}^{n} \sum_{u=1}^{n} K(X_i, Z_{yu}) \neq \sum_{i=1}^{n} \sum_{u=1}^{n} K(Z_{yu}, X_i). \]

3.9 It does not make heuristic sense to limit the test so that standardised values of the observed sample are compared with values from the pseudo-random sample in later quarters but not from earlier quarters. In fact this limitation introduces a bias in that, for later quarters, there are fewer terms from the standardised values of the observed sample and more from the pseudo-random sample; stronger emphasis
is placed on the former in earlier quarters and on the latter in later quarters. For this reason the test statistic $T$ was redefined as:

$$T = \frac{1}{n(n-1)} \sum_{i,j=1}^{n} K(X_i, X_u);$$

and the test statistic $T_v$ as:

$$T_v = \frac{1}{n(n-1)} \sum_{i,j=1}^{n-1} K(X_i, Z_{uv}).$$

The redefined value not only includes those values for which $u > t$ but also those for which $t > u$. The value of $T$ is unaltered by this redefinition, but the value of $T_v$ is. The redefinition of $T$ is merely for consistency with that of $T_v$. The averaging factor is changed from $\frac{2}{n(n-1)}$ to $\frac{1}{n(n-1)}$ because we now have twice as many terms in the summation. This will not affect the test but it makes the statistic more meaningful.

3.10 As Diks & Tong (op. cit.) observe, their test assumes that the parameters of the distribution of the variable concerned are known. In practice, the mean vector and the variance–covariance matrix of returns on the asset categories of interest are unknown. This means that the tests are biased. The tests are consistent, so the p-value will tend to a limit as the sample size increases. However, for some periods the sample sizes being tested here are quite small. For those periods in particular, the null hypothesis may have been falsely accepted. It was therefore decided to determine test statistics based on those of Diks & Tong (op. cit.), redefined as in ¶3.9, but allowing for the fact that the parameters are unknown. In this section the method used for these tests and the results obtained are set out.

3.11 The problem is that the parameters used to standardise the observed values, and thus (on the null hypothesis) to transform the empirical distribution of the returns from an elliptically symmetric
distribution (in the case of $R$) to a spherically symmetric distribution (in the case of $X$) are not necessarily equal to the estimates $\hat{\mu}$ and $\hat{\Sigma}$ based on the sample values observed.

3.12 To deal with this problem a bootstrapping procedure was used to derive conditional distributions of the p-value given bootstrapped estimates of $\mu$ and $\Sigma$ and hence an unconditional p-value. The approach was similar to that described in ¶3.3. The procedure was as follows:

(1) For $\nu = 1, \ldots, M$:

(a) sample $R_{\nu1}, \ldots, R_{\nu n}$ with replacement from $R_1, \ldots, R_n$ where, as before:

\[
R_i = \begin{pmatrix} R_{i1} \\ \vdots \\ R_{ip} \end{pmatrix}
\]

(b) calculate the bootstrapped parameters:

\[
\hat{\mu}_{\nu}^* = \frac{1}{n} \sum_{t=1}^{n} R_{i \nu t}^* \quad \text{and} \quad \hat{\Sigma}_{\nu}^* = \frac{1}{n} R_{\nu t}^* R_{\nu t}^*
\]

where:

\[
R_{\nu t}^* = \begin{pmatrix} R_{i1}^* - \hat{\mu}_{\nu}^* \\ \vdots \\ R_{ip}^* - \hat{\mu}_{\nu}^* \end{pmatrix}
\]

(c) calculate:

\[
X_{t \nu}^* = \hat{\Sigma}_{\nu}^{-1/2} \left( R_{i \nu t} - \hat{\mu}_{\nu}^* \right) \quad \text{for} \; t = 1, \ldots, n
\]

(note that here we use $R_i$, not $R_{\nu i}$; it is the original observations we use, but they are standardised with the bootstrapped parameters);

(d) calculate the test statistic:

\[
T_{\nu}^* = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{t=1}^{n} K \left( X_{t \nu}^*, X_{i \nu}^* \right)
\]
(e) generate a pseudo-random sample by determining:

\[ Z_{uv}^* = \frac{Y_{uv}}{\|Y_{uv}\|} X_{uv}^* \] for \( \nu = 1, \ldots, N; \ t = 1, \ldots, n; \)

where:

\[ Y_{uv} \sim N(0, I); \]

and hence, for \( \nu = 1, \ldots, N \), calculate:

\[ T = \frac{1}{n(n-1)} \sum_{u \neq v} K(X_u, X_v); \]

(f) calculate the p-value of the test statistic for bootstrap sample \( \nu \) as:

\[ p_{\nu}^* = P\left(T_{\nu}^* \geq T_{uv}^* \mid \mu = \hat{\mu}_{\nu}^*, \Sigma = \hat{\Sigma}_{\nu}^*, \nu = 1, \ldots, N\right) = \frac{k_{\nu}^*}{N+1}; \]

where \( k_{\nu}^* \) is the number of values in \( \bigcup_{\nu=1}^{N} \{T_{\nu}^*\} \bigcup \{T_{uv}^*\} \) that are greater than or equal to \( T_{\nu}^* \).

(2) Determine the p-value for all bootstrap samples combined as:

\[ P\left(T_{\nu}^* \geq T_{uv}^* \mid \mu = \hat{\mu}_{\nu}^*, \Sigma = \hat{\Sigma}_{\nu}^*, \nu = 1, \ldots, N\right) = \frac{1}{M} \sum_{i=1}^{M} p_{\nu}^*. \]

3.13 By substituting \( Y_{uv} \) for \( Z_{uv}^* \) in the calculation of \( T_{uv}^* \) in step (1)(e), the test becomes a test of the null hypothesis that the distribution of the returns is multivariate normal. Since failures of tests based on the assumption that the distribution is multivariate normal may be due to the falsehood of this hypothesis, p-values were also calculated on this basis.

3.14 The formula in step (2) is justified as follows. \( p_{\nu}^* \) is the conditional empirical probability that \( T_{\nu}^* \) is greater than or equal to \( T_{uv}^* \) given that \( \mu = \hat{\mu}_{\nu}^* \) and \( \Sigma = \hat{\Sigma}_{\nu}^* \). The probability that \( \mu = \hat{\mu}_{\nu}^* \) and \( \Sigma = \hat{\Sigma}_{\nu}^* \)
is equal for every $\nu = 1, \ldots, M$, i.e. $\frac{1}{M}$. The p-value of the overall test, i.e. the unconditional probability that $T^{\ast}_{\nu}$ is greater than or equal to $T^{\ast}_{\nu}$, is thus:

$$\sum_{i=1}^{M} P\left(\mu = \hat{\mu}^{\ast}_{\nu}, \Sigma = \hat{\Sigma}^{\ast}_{\nu} \mid \nu = 1, \ldots, M\right) P\left(T^{\ast}_{\nu} \geq T^{\ast}_{\nu} \mid \mu = \hat{\mu}^{\ast}_{\nu}, \Sigma = \hat{\Sigma}^{\ast}_{\nu}, \nu = 1, \ldots, N\right) = \sum_{i=1}^{M} \frac{1}{M} p^{\ast}_{\nu}.$$ 

3.15 For the purposes of determining p-values, programs were written in R; the R code is available from the author free of charge. For hypothetical trial values of observations the results were checked against results obtained for a trial data set in Microsoft Excel.

3.16 For each period and for all periods combined the p-value was determined for equities and conventional bonds. For period [153, 185] inflation-linked bonds were also included.

3.17 Given information at the start of a quarter, the risk-free force of return for that quarter is known. If the joint ex-ante distribution of the forces of return during the quarter is elliptically symmetrical then the risk-free force of return may be subtracted from them to give the corresponding risk premia during that quarter. The distribution of those risk premia will then also be elliptically symmetrical. For period [153, 185], for which risk-free forces of return were known, the test statistics were also determined for those risk premia. For this purpose, in the definition of $R_{R}$, $r_{i} = R_{i} - R_{i_{i}}$, replaces $R_{i}$ for $i = 1, \ldots, p$.

3.18 It was found that reasonable accuracy to two decimal places was obtained with $M = 100$ and $N = 1000$. Following Diks & Tong (op. cit.) the value of $d$ was taken as 0.25; a range of values was tested, but they made little difference to the p-values. The results of the test are shown in Table 10.
Table 10. Results of the bootstrapped Diks-Tong test

<table>
<thead>
<tr>
<th>Period</th>
<th>p-value</th>
<th>elliptical symmetry</th>
<th>normal</th>
</tr>
</thead>
<tbody>
<tr>
<td>excluding inflation-linked bonds ($p = 4$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[1, 33] ($n = 33$)</td>
<td>0,00</td>
<td>0,00</td>
<td></td>
</tr>
<tr>
<td>[34, 85] ($n = 52$)</td>
<td>0,17</td>
<td>0,02</td>
<td></td>
</tr>
<tr>
<td>[86, 100] ($n = 15$)</td>
<td>0,80</td>
<td>0,91</td>
<td></td>
</tr>
<tr>
<td>[101, 152] ($n = 52$)</td>
<td>0,37</td>
<td>0,08</td>
<td></td>
</tr>
<tr>
<td>[153, 185] ($n = 33$)</td>
<td>0,34</td>
<td>0,29</td>
<td></td>
</tr>
<tr>
<td>all ($n = 185$)</td>
<td>0,00</td>
<td>0,00</td>
<td></td>
</tr>
<tr>
<td>[34, 185] ($n = 152$)</td>
<td>0,20</td>
<td>0,00</td>
<td></td>
</tr>
<tr>
<td>including inflation-linked bonds ($p = 7$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[153, 185] ($n = 33$)</td>
<td>0,83</td>
<td>0,88</td>
<td></td>
</tr>
<tr>
<td>risk premia ($p = 7$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[153, 185] ($n = 33$)</td>
<td>0,76</td>
<td>0,80</td>
<td></td>
</tr>
</tbody>
</table>

3.19 For all periods combined the result for elliptical symmetry was significant at the 5% level. However, the only individual period for which significant results were found at the 5% level was the first period, [1, 33]. This suggests that the significant result for all periods combined arose from that for the first period. The test was therefore also made for the combination of all periods other than the first, i.e. for [34, 185]. The result was not significant. On the basis of these tests it may therefore be concluded that, whereas the distribution of returns was not elliptically symmetric in the period [1, 33], they have been elliptically symmetric since then.

3.20 For periods [86, 100], and for [153, 185] including inflation-linked bonds, where the p-values for the normal distribution are relatively high, they were greater than those for the elliptically symmetric
distribution. The normal distribution being a particular case of the elliptically symmetric distribution, this appears counter-intuitive. However, where (as in the cases noted here) the empirical distribution is closer (in the sense measured by $K(x, y)$) to the normal than to the general elliptically symmetrical distribution, this phenomenon is possible. It makes intuitive sense that this occurs wherever, but only where, the p-value of the normal distribution is high.

3.21 For all periods combined the result for the normal distribution was significant at the 5% level. Here, in addition to the first period, the results were significant for the second period, [34, 85]. This suggests that the significant result for all periods combined arose from that for these periods.

3.22 Another explanation for the significant results over longer periods is that expected returns may change over time. Less change occurs over shorter periods, but over longer periods such changes affect the conditional *ex-ante* distribution of returns and this will result in the rejection of elliptical symmetry or normality over such periods. Thus, for example, whilst neither the elliptical distribution nor the normal distribution can be rejected over the period [101, 152] or [153, 185] where index-linked bonds are excluded, both are rejected over the period [101, 185] (again excluding index-linked bonds). This makes it essential to consider elliptical symmetry over relatively short periods, or to apply the tests to the residuals of the time series being used for stochastic modelling.

4. SUMMARY AND CONCLUSION

4.1 The assumption of normality is not necessary for the CAPM. Furthermore, the question whether the CAPM will apply in the future does not necessarily depend on whether it has explained returns in the past. If market participants’ risk measures are positive-homogeneous and translation-invariant, the CAPM may be used with a suitable elliptically symmetric distribution. At present these properties of risk measures are normative rather than descriptive. However, with increasing awareness amongst market participants of the advantages of such criteria in the choice of risk measures, it is reasonable to
assume that they will become descriptively valid. On the other hand, if it were assumed that they will not become generally applied, the implication would be that the client market participant could outperform the market on a risk-adjusted basis. Whilst that may be useful for strategic investment decision-making, it would be an inappropriate assumption for the purposes of establishing market-consistent prices, market-consistent investment-performance benchmarks or capital adequacy requirements.

4.2 Tests of the elliptical symmetry of the joint distribution of the returns were therefore made. Whilst elliptical symmetry could be rejected for the period \([1, 33]\) (i.e. the period prior to 1973), for the rest of the period it was not possible to reject it. The fact that more recent periods tend to show elliptical symmetry suggests that the distribution of returns has changed in favour of elliptical symmetry. In the absence of a theory explaining such a change it cannot be convincingly asserted that it will not be reversed at some time. However, as the apparent status quo, elliptical symmetry may reasonably be regarded as an appropriate assumption for long-term stochastic modelling.

4.3 Tests of the normality of the joint distribution were also made. Whilst normality could be rejected for the period \([1, 85]\) (i.e. the period prior to 1986), for the rest of the period it was not possible to reject it. This does not necessarily mean that the normal distribution is the best elliptically symmetric distribution to use for stochastic modelling. It may fail to capture the long tails that other elliptically symmetric distributions would capture. But it does suggest that the reason why the CAPM fails to explain returns since 1986 is that:

- the estimates (even the in-period estimates) of beta and of expected values are biased estimates of the \textit{ex-ante} values (i.e. that the REH does not apply); or
- the CAPM fails to explain expected returns in terms of beta (i.e. that market participant’s risk measures are not positive-homogeneous and translation-invariant);
not that the distribution has not been normal.

4.10 Besides the validity of mean–variance analysis, there are other assumptions on which the CAPM rests. The most important of these is the existence of equilibrium. Again, though, for the purposes of establishing market-consistent prices, investment-performance benchmarks or capital adequacy requirements, it would be inappropriate not to make this assumption. For market-consistent prices one is deliberately assuming that the investor’s *ex-ante* assumptions are equal to those of the market. For investment-performance benchmarks departures from market-consistent *ex-ante* assumptions must be left to the investment managers; it is up to them to outperform the market if they can do so within their mandate. Capital-adequacy requirements should not be dependent on the views of the institution; they should be acceptable to the market.

4.11 Other assumptions, such as perfect markets, uniform taxation and homogeneous time horizons and currency, are clearly incorrect, but the questions they raise relate more to second-order effects on the accuracy of the CAPM than to its fundamental validity, particularly for long-term modelling. The validity of these assumptions has been empirically addressed in the literature, but the purpose of this study was to address more fundamental questions.

4.12 All the tests made in this paper presuppose the REH: it is implicitly assumed that *ex-ante* expectations are unbiased and that they can be represented either by in-period data or by prior data. Rejections of the tests may be rejections of the REH rather than rejections of the elliptical symmetry (or normality) of *ex-ante* distributions of returns. Tests of pricing models against true *ex-ante* expectations would require continual monitoring of such expectations, and of prices, over time. On the other hand, where the null hypotheses are accepted, neither the REH nor the null hypothesis can be rejected.
4.13 It may be argued that, despite the evidence in Thomson & Reddy (op. cit.), the CAPM will apply in the future if, as it may be reasonably assumed, the distribution of returns is elliptically symmetric, market participants’ risk measures are positive-homogeneous and translation-invariant, and the REH applies, resulting in uniform, unbiased expectations. Whilst these may be heroic assumptions for an unknown future, particularly in the face of the failure of the CAPM in the past, they are appropriate for the purposes envisaged.

4.14 Where monitoring takes place quarterly, it appears that, for long-term stochastic modelling for the purposes of market-consistent pricing, market-consistent investment-performance benchmarks and capital adequacy requirements, the use of the CAPM with elliptically symmetric distributions of returns may be justified, provided the assumptions used are stated and that it is also made clear that these assumptions have not explained returns in the past.

4.15 The method applied in this paper may be applied to other markets, other periods and other monitoring intervals. That is not a matter for original research. The principal matters for further research arising from this study are the exploration of the use of elliptically symmetric distributions of returns other than the multivariate normal distribution for the purposes of the stochastic modelling of investment returns and the consideration of alternative models to the CAPM for the modelling of prices in equilibrium.

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