Stochastic Simulation of Individual Retirement Accounts in Mexico
(Replacement Rates Comparison)

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   - Model Contribution Density
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Mexican Pension System

Individual Account System

## Mexican Pension System

### Individual Account System

Accounts are managed by Pension Fund Administrators AFORE (for its acronym in Spanish) and every worker may choose one of them.

<table>
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<tbody>
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<td>Afirme Bajío</td>
<td>Metlife</td>
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<td>Azteca</td>
<td>PENSIONISSSTE</td>
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<tr>
<td>Banamex</td>
<td>Principal</td>
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<td>Bancomer</td>
<td>Profuturo GNP</td>
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<td>SURA</td>
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<td>Inbursa</td>
<td>XXI</td>
</tr>
<tr>
<td>Invercap</td>
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Administrators in Operation 2012, Source: Mexican Supervising Institution CONSAR.
According to Worker Age, Administrators invest Retirement Savings in Market Money by using Specialized Investment Societies denominated SIEFOREs. (Portfolios)

For Younger Persons, More Risk.
Benefits, Contributions and Requirements

**Benefits**
- Pension + Survival Insurance = Cumulated Fund
- Minimum Pension Guaranteed (MPG)

**Contributions**
- 6.5% of Salary + Social Quota

**Requirements**
- 65 Years Old
- 1250 Contributions Weeks ~ (24 Years)
Uncertain Benefits

- There are no Historical Replacement Rates to Compare Administrators.
Uncertain Benefits

- There are no Historical Replacement Rates to Compare Administrators.
- Volatility of Investment Returns makes Uncertain Benefits at Future.
Uncertain Benefits

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- Contribution Density affects directly the Pensions.
Uncertain Benefits

- There are no Historical Replacement Rates to Compare Administrators.
- Volatility of Investment Returns makes Uncertain Benefits at Future.
- Contribution Density affects directly the Pensions.
- So...
So, ¿Which Administrator Choose?
Uncertain Benefits

Answer

\[ \Delta\% = \frac{\text{Pension Last Salary}}{\text{Salary}} \]
Answer

- Which gives the **Highest Pension**
Uncertain Benefits

Answer

- Which gives the **Highest Pension**
- Or the **Highest Replacement Rate**
Uncertain Benefits

Answer

- Which gives the **Highest Pension**

- Or the **Highest Replacement Rate**

\[ \Delta \% = \frac{Pension}{Last\ Salary} \]
Uncertain Benefits

In Probability Terms
In Probability Terms

- Which has the **Smallest Probability of Exercising the (MPG) (Not Reaching the GOAL)**.
In Probability Terms

- Which has the **Smallest Probability of Exercising the (MPG) (Not Reaching the GOAL).**

- The Administrator (Portfolio) with the **Highest Probability of Being Better than the Others.**
Section 2

Stochastic Simulation of Individual Retirement Account
Individual Account Factors

- Contribution Density.
- Real Investment Returns.
- Salary History.
CONTRIBUTION DENSITY

Percentage of Contributions Made

CONSAR (2010)

- 70% Males
- 66% Females
Modeling Contribution Density

- Non-Homogeneous Markov Chain Model
- Homogeneous Markov Chain Model
- Bayesian Model
Markov Chain Models

Let $C_n \in \{0, 1\}$ the Stochastic Process of Contributions at time $n$
Let $C_n \in \{0, 1\}$ the Stochastic Process of Contributions at time $n$

- **Non-Homogeneous Transition Matrix**

$$P_x = \begin{pmatrix} p_x & 1 - p_x \\ 1 - q_x & q_x \end{pmatrix}$$

Where $P_x(i, j) = \mathbb{P}[C_{x+1} = j | C_x = i]$ $i, j \in \{0, 1\}$
Markov Chain Models

Let $C_n \in \{0, 1\}$ the Stochastic Process of Contributions at time $n$

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Where $P_x (i, j) = \mathbb{P}[C_{x+1} = j | C_x = i]$, $i, j \in \{0, 1\}$

- **Homogeneous Transition Matrix**

$$\lim_{n \to \infty} \left( \begin{array}{cc} p & 1 - p \\ 1 - q & q \end{array} \right)^n = \left( \begin{array}{cc} \delta & 1 - \delta \\ \delta & 1 - \delta \end{array} \right)$$

Where: $\delta = \text{Contribution Density}$
Bayesian Model

- Contribution $C_n$ given $\delta$ is a $Bernoulli(\delta)$ Random Variable.

\[
f(C_n | \delta) \sim Bernoulli(\delta), \text{ a priori distribution } \pi_0(\delta) \sim Beta(\alpha, \beta) \quad \alpha, \beta > 0
\]

Then, the Predicted Distribution for $C_{n+1}$ given $C_n, \ldots, C_1$ is:

\[
\hat{f}(C_{n+1} | C_n, \ldots, C_1) \sim Bernoulli(\delta_{n+1})
\]

Where

\[
\delta_{n+1} = E[\delta | C_n, \ldots, C_1] = \frac{\alpha + \sum_{j=1}^{n} C_j}{\alpha + \beta + n}
\]
Real Investment Returns are subject to Market Volatility.

Source: Own elaboration with Data published by Regulator Institution CONSAR and Mexican Central Bank
Modeling Real Investment Returns

- Bootstrapping Methods
- Heavy Tailed Distributions
- Mixed Distributions
- Multivariate Distributions
- Time Series Models (ARIMA, VARIMA, GARCH)
- Univariate Distributions
- So on...
Multivariate Normal Distribution

Let $R = (r_1, \ldots, r_n)$ Monthly Investment Returns for $n$ Administrators. $R$ is Normal Distributed if its Probability Density Function is:

$$
\phi(R, \mu, \Sigma) = |2\pi\Sigma|^{-1/2} \exp \left( -\frac{1}{2} (R - \mu)' \Sigma^{-1} (R - \mu) \right)
$$

Where

$\mu = (\mu_1, \ldots, \mu_n)$ Expected Monthly Real Investment Returns.

$\Sigma$ Volatility and Correlation Structure.

Those Parameters are estimated by Maximum Likelihood Method.
Salary History

Salary changes in response of Abilities, Productivity and other Characteristics. The Comparison among Administrators might be Independent of Salary History.

Salary Profiles

- Mexican Average Salary $\mu = 4.2$ Minimum Wages (MW)
- $\mu + \sigma = 8.8$ (MW)

Salary Growth

- 1% Real Annual Growth as Actuarial Valuations from Mexican Social Security Institute
Let a Worker with Age, Salary and Contribution Density at time 0; \( x_0, S_0 \) and \( \delta \). The Individual Account is Simulated as follows:
Simulations

Let a Worker with Age, Salary and Contribution Density at time 0; $x_0$, $S_0$ and $\delta$. The Individual Account is Simulated as follows:

- At time $t$ is determined:

$$SIEFORE_j = \begin{cases} 
1, & x_t \leq 22 \\
2, & 26 < x_t \leq 37 \\
3, & 37 < x_t \leq 45 \\
4, & 45 < x_t \leq 59 \\
5, & 59 < x_t 
\end{cases}$$
Let a Worker with Age, Salary and Contribution Density at time 0; $x_0$, $S_0$ y $\delta$. The Individual Account is Simulated as follows:

- At time $t$ is determined:

$$SIEFORE_j = \begin{cases} 
1, & x_t \leq 22 \\
2, & 26 < x_t \leq 37 \\
3, & 37 < x_t \leq 45 \\
4, & 45 < x_t \leq 59 \\
5, & 59 < x_t 
\end{cases}$$

- Given $SIEFORE_j$ it is generated a Normal Vector of Investment Returns $R_j = (r_1, \ldots, r_n)$ for $n$ Administrators.
Simulation of Individual Account

Simulation

- The Stochastic Process of Contributions $X_t$ is Simulated from the Equilibrium Transition Matrix.

$$
\begin{pmatrix}
\delta & 1 - \delta \\
\delta & 1 - \delta 
\end{pmatrix}
$$
Simulation

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\begin{pmatrix}
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\delta & 1 - \delta
\end{pmatrix}
$$

- The Generated Interest plus Contributions are Cumulated into the Fund for Administrator $i$

$$
F_{i,t} = F_{i,t-1}(1 + r_i)(1 - c_i) + X_t(6.5\% S_t + SQ_t)
$$
Simulation

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\end{pmatrix}
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\]

- After a year, Salary is upgrade one percent and the Process Continue until Retirement Age.
For Administrator $i$ the Pension is:

$$P_i = \Delta\% S_n = \max \left( \frac{F_{i,r}}{\frac{13}{12} (\ddot{a}_x^{(12)} + \ddot{a}_y^{(12)} - \ddot{a}_{xy}^{(12)})}, MPG \right)$$
Replacement Rate Simulation

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Hence, Replacement Rate

$$\Delta \% = \frac{P_i}{S_r}$$
Replacement Rate Simulation

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Hence, Replacement Rate

$$\Delta\% = \frac{P_i}{S_r}$$

It is Constructed $N$ Replacement Rates Scenarios.
Probability of Exercising MPG is estimated for Administrator $i$ as follows:

$$\hat{p}_i = \sum_{i=1}^{N} \frac{I\{P_i = MPG\}}{N}$$
Replacement Rate Simulation

- Probability of Exercising MPG is estimated for Administrator $i$ as follows:

$$
\hat{p}_i = \sum_{i=1}^{N} \frac{I\{P_i = MPG\}}{N}
$$

- Probability of Administrator $i$ is worse than Administrator $j$

$$
\hat{p}_{i,j} = \sum_{i=1}^{N} \frac{I\{\delta_i < \delta_j\}}{N}
$$
Salary Profiles
Salary Profiles

- 20, 30 Years Old
Salary Profiles

- 20, 30 Years Old
- Average Salary $\mu$ 4.2 MW, $\mu + \sigma$ 8.8 MW
Salary Profiles

- **20, 30 Years Old**
- **Average Salary** $\mu$ 4.2 MW, $\mu + \sigma$ 8.8 MW
- **Contribution Density** .70 and .66 for Males and Females
Section 3

Results and Conclusions
Simulation Results

Simulations

- Simulations were elaborated with the statistical package R.
- Data was obtained from the Regulator Institution CONSAR and the Mexican Central Bank webpages at February 2012.
Probabilities of Exercising the MPG

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Probabilities grows up as age as well. Not taking into account Contribution Density Underestimate Actuarial Valuations of MPG Benefits
Probabilities of Exercising the MPG by Administrators, Contribution Density, Salary and Sex

Density Average (Left), Density of 100% (Right)
Replacement Rates are higher for Average Workers due Social Quota. Only Young Workers could reach Replacement Rates above 40%. Differences due Contribution Density go from 4% to 20%
Administrators Expected Replacement Rate

Only two Administrators can give Pensions above the International Labour Organization recommendation of 40%
Administators Expected Replacement Rate

All Administrators among the MPG and ILO Recommendation
Administrators except Invercap and XXI with Probability of .8 will not give Replacement Rates higher than 56% for 8.8 MW Average Workers
### Correlation Matrix

<table>
<thead>
<tr>
<th></th>
<th>Afinme Bajio</th>
<th>Azteca</th>
<th>Banamex</th>
<th>Bancomer</th>
<th>Coppel</th>
<th>Inbursa</th>
<th>Invercap</th>
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<th>Profuturo</th>
<th>GNP</th>
<th>SURA</th>
<th>XXI</th>
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High Correlation due Investment Returns.
**Probability of Being Better than other Administrator**

<table>
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<th>Azteca</th>
<th>Banamex</th>
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XXI And Invercap are more probable to be better than the others Administrator
Conclusions

High Probabilities of Exercising the MPG Replacement Rates System are between MPG and ILO recommendation. XXI and Invercap are more probable to give Higher Replacement Rates. System could be Overvaluated due not taking into account Contribution Density effects. Methodology can be also used for Comparing Investment Strategies, Portfolios, Competitors in terms in final Benefits and Probabilities.
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Merci!!